KORTEWEG-DE VRIES EQUATION: A COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM

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The Korteweg-de Vries equation (KdV) arose long ago in an approximate theory of hydrodynamic waves,

$$u_t - 6uu_x + u_{xxx} = 0; \quad u(x, t)|_{t=0} = u(x); \quad -\infty < x < \infty; \ u(x) \to 0, \quad |x| \to \infty; \tag{1}$$

recently it has become the object of intensive study [1-3, 12]. A group of scholars, including Gardner, Green, Zabusky, Kruskal, and Miura, has made the following two important observations:

1. Equation (1) with smooth initial data admits an infinite set of first integrals. These integrals have local densities, i.e., they are representable in the form $I_n[u] = \int_{-\infty}^{\infty} P_n(u, u_x, \ldots)\, dx$, where $P_n(u, u_x, \ldots)$ is a polynomial in u and spatial derivatives of u with orders up to $n-2$, which contains the term u^n. The first three such polynomials have the form $P_1(u) = u$, $P_2(u) = u^2$, $P_3(u, u_x) = u^3 + (u_x^2/2)$. An explicit form for eleven of the P_n is given in [2, 3]; in [3] an explicit procedure for determining them is given. An alternate approach for determining the $P_n(u, u_x, \ldots)$ has been developed by Lax [4].

2. An explicit solution of the KdV equation can be obtained by using the formalism of a scattering problem for the Schroedinger equation,

$$-\psi_{xx} + u(x)\psi = k^2\psi. \tag{2}$$

We clarify this in more detail. If

$$\int_{-\infty}^{\infty} (1 + |x|)\, |u(x)|\, dx < \infty, \tag{3}$$

then Eq. (2) has a two-fold positive continuous spectrum and a finite number of negative characteristic values $-\varkappa_l^2$, $l = 1, \ldots, n$. For proof, see, for example, [5]. Let r(k) be the coefficient of reflection on the left, i.e., a function involving the solution $\psi(x, k)$ of Eq. (2) in the asymptotics for $x \to -\infty$, this function being uniquely defined by the conditions

$$\psi(x, k) = e^{ikx} + r(k)e^{-ikx} + o(1), \ x \to -\infty; \ \psi(x, k) = t(k)e^{ikx} + o(1), \qquad x \to \infty. \tag{4}$$

Further let $\psi_l(x)$ be the characteristic functions of the discrete spectrum, normalized by the condition $\psi_l(x) = e^{\varkappa_l x}(1 + o(1))$, $x \to -\infty$, and c_l, $l = 1, \ldots, m$, the corresponding normalizing factors being $c_l = \left(\int_{-\infty}^{\infty} \psi_l^2(x)\, dx\right)^{-1}$. The set $s = (r(k), \varkappa_l, c_l)$ will be called the scattering data for Eq. (2). The mapping $u(x) \to s$ of potentials u(x) into the scattering data s is uniquely invertible. The corresponding procedure for recovering u(x) from s, which is the inverse scattering problem, was formulated for the first time in [6] and investigated rigorously in [5]. In [5] necessary and sufficient conditions on the scattering data, corresponding to potentials satisfying condition (3), were obtained.

The remarkable result given in [1] consists in the following. In the set of scattering data we consider the action of the one-parameter group

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Institute of Nuclear Physics, Siberian Branch, Academy of Sciences of the USSR. Leningrad Branch, V. A. Steklov Mathematics Institute. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 5, No. 4, pp. 18-27, October-December, 1971. Original article submitted June 14, 1971.

$$r(k) \to e^{8ik^3t} r(k), \qquad \varkappa_l \to \varkappa_l, \qquad c_l \to e^{8\varkappa_l^3 t} c_l. \tag{5}$$

It proves to be the case that the corresponding motion in the set of potentials $u(x) \to u(x, t)$ determines the solution $u(x, t)$ of the KdV equation.

In the present paper we give a new interpretation of this result and a new derivation based on it. This interpretation provides, in our opinion, a simple explanation of the somewhat puzzling conclusions given in [2].

Our interpretation may be formulated in the following way. The KdV equation is a completely integrable Hamiltonian system. The mapping $u \to s$ plays the role of a transformation, transforming the variables $u(x)$ into canonical variables of the type involving angle and action variables (see, for example [7]).

In order to justify these assertions we must:

1) producea simplicial form Ω on the set of potentials $u(x)$ and a Hamiltonian function $H[u]$ on this set, which generate the KdV equation according to the rules of Hamiltonian mechanics (see [8], for example);

2) calculate the preimages of the form Ω and the Hamiltonian $H[u]$ under the mapping $u \to s$ and express the canonical variables in the action-angle form in terms of the scattering data.

The first problem is easily solved. It is not hard to see that the KdV equation may be written in the form

$$u_t = \frac{d}{dx} \frac{\delta I_3[u]}{\delta u(x)}, \tag{6}$$

where the symbol $\delta H[u]/\delta u(x)$ denotes the gradient (Frechet derivative) of the function $H[u]$. It was pointed out in [4] that this result is due to Gardner.

The notation (6) for the KdV equation is clearly Hamiltonian. The corresponding simplicial form

$$\Omega(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\, [\delta_1 u(x)\, \delta_2 u(y) - \delta_1 u(y)\, \delta_2 u(x)] \tag{7}$$

has constant coefficients in the variables u and is therefore closed. We are using here the older but more natural and, for our infinite dimensional case, more suitable coordinate notation for a differential form in terms of the "local coordinates" $u(x)$ and their differentials, the variations $\delta_1 u(x)$ and $\delta_2 u(x)$. The role of the Hamiltonian $H[u]$ is played by the integral of motion

$$H[u] = I_3[u] = \int_{-\infty}^{\infty} \left(u^3(x) + \frac{1}{2} u_x^2\right) dx.$$

We devote the major part of this paper to a solution of the second problem. In §3 we express the Hamiltonian $H[u]$ in terms of the scattering data in the following way:

$$H[u] = -\frac{8}{\pi} \int_{-\infty}^{\infty} k^4 \ln(1 - |r(k)|^2)\, dk - \frac{32}{5} \sum_{l=1}^{m} \varkappa_l^5. \tag{8}$$

We will show that this expression is a special case of the formulas for traces [9, 10, 11]. Simultaneously we obtain explicit formulas for all the first integrals $I_n[u]$, deriving thereby simple recursion relations for the densities $P_n(u, u_x, \ldots)$. In terms of the scattering data these integrals may be expressed by formulas analogous to Eq. (8); i.e., they involve moments of the function $\ln(1 - |r(k)|^2)$ and powers of $\varkappa_l$.

In §2, using the formalism of the inverse scattering problem, we express the form Ω in terms of the scattering data and we find the corresponding canonical variables. We show, in particular, that $P(k) = -(k/\pi) \ln(1 - |r(k)|^2)$, $p_l = \varkappa_l^2$, $l = 1, \ldots, m$, are variables of impulse type, so that the fact of the constancy of the integrals $I_n[u]$ becomes trivial. Solution of the Hamiltonian equations may thus be trivialized with the corresponding answer supplied by the formulas (5).

In §1, as a preliminary, we supply, without going into a detailed derivation, the necessary facts of scattering theory for the one-dimensional Schroedinger equation over the entire axis in a form suitable to our purposes.

The present paper could well be written in the language of the theory of smooth infinite-dimensional manifolds. We shall not pursue this modern tendency in contemporary mathematical physics if only to keep the paper within appropriate bounds. For this reason we omit many of the proofs and concentrate instead on the details of the formal derivations.

§1. Scattering Theory Background Information

The Schroedinger Equation (2), subject to the condition (3), has solutions $f(x, k)$ and $g(x, k)$, uniquely defined for all real k by the conditions

$$f(x,k)=e^{ikx}+o(1),\ x\to\infty;\qquad g(x,k)=e^{-ikx}+o(1),\ x\to-\infty. \tag{9}$$

In addition, $f(x, k) = \overline{f(x, -k)}$, $g(x, k) = \overline{g(x, -k)}$. The pairs $f(x, k)$, $f(x, -k)$ and $g(x, k)$, $g(x, -k)$ form, for $k \neq 0$, two fundamental systems of solutions of Eq. (2). The following relations hold:

$$f(x,k)=b(k)g(x,k)+a(k)g(x,-k),\quad g(x,k)=-b(-k)f(x,k)+a(k)f(x,-k), \tag{10}$$

where the coefficients $a(k)$ and $b(k)$ satisfy the conditions

$$a(-k)=\overline{a(k)};\qquad b(-k)=\overline{b(k)},\qquad |a|^2=1+|b|^2. \tag{11}$$

In addition, $a(k) = (1/2ik)\{f(x, k), g(x, k)\}$, where $\{f, g\} = f_x g - g_x f$.

The solutions $f(x, k)$ and $g(x, k)$ and the coefficient $a(k)$ may be continued analytically into the upper halfplane of the variable k even for large k:

$$a(k)=1+O\left(\frac{1}{|k|}\right),\quad f(x,k)e^{-ikx}=1+O\left(\frac{1}{|k|}\right),\quad g(x,k)e^{ikx}=1+O\left(\frac{1}{|k|}\right).$$

The last two bounds are uniform with respect to x in the intervals (α, ∞) and $(-\infty, \beta)$, respectively, where α, β are arbitrary finite numbers.

For Im $k \neq 0$ the solution $f(x, k)$ decreases exponentially for $x \to \infty$, and $g(x, k)$ does so for $x \to -\infty$. If $a(i\varkappa) = 0$, the solutions $f(x, i\varkappa)$ and $g(x, i\varkappa)$ are linearly dependent, so that $\psi(x) = g(x, i\varkappa)$ decreases exponentially for $|x| \to \infty$,

$$\psi(x)=e^{\varkappa x}(1+o(1)),\ x\to-\infty,\qquad \psi(x)=de^{-\varkappa x}(1+o(1)),\ x\to\infty, \tag{12}$$

and thus defines a characteristic function of Eq. (2). The corresponding characteristic value $(i\varkappa)^2$ must be real since Im$\varkappa = 0$ and the function $\psi(x)$ is real. We denote the derivative of $g(x, k)$ with respect to k for $k = k_0$ by $\dot{\psi}(x)$. This function takes on imaginary values and has the asymptotic behavior

$$\dot{\psi}(x)=O(xe^{\varkappa x}),\ x\to-\infty,\ \dot{\psi}(x)=he^{\varkappa x}(1+o(1)),\ x\to\infty. \tag{13}$$

From the Schroedinger equation we readily find that

$$\frac{1}{c}=\int_{-\infty}^{\infty}\psi^2dx=ihd. \tag{14}$$

We assume that $a(k)$ has a total of m zeros, and we index by l, $l = 1, 2, \ldots, m$, the corresponding functions $\psi_l(x)$ and $\dot{\psi}_l(x)$, normalizing constants c_l, characteristic values $\varkappa_l$, and asymptotic coefficients d_l and h_l.

Consider the solution $\psi(x, k) = f(x, k)/a(k)$. From Eqs. (9) and (10) it is obvious that $\psi(x, k)$ satisfies the asymptotic condition (4), where the coefficient of reflection is representable in the form

$$r(k)=b(k)/a(k). \tag{15}$$

We determine in this way completely the set of scattering data $s = (r(k), \varkappa_l, c_l)$.

We note here that the coefficients $a(k)$ and $b(k)$ may be determined from the scattering data. Indeed, from Eqs. (15) and (11) we obtain

$$1-|r|^2=\frac{|a|^2-|b|^2}{|a|^2}=\frac{1}{|a|^2}, \tag{16}$$

defining $|a(k)|$ with respect to the coefficient of reflection. The condition of analyticity and the knowledge of the zeros of $a(k)$ enables us to write $a(k)$ in the form

$$a(k)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-|r(q)|^2)}{k-q}dq\right\}\prod_{l=1}^{m}\frac{k-i\varkappa_l}{k+i\varkappa_l},\ \mathrm{Im}\,k\neq 0,\quad a(k)=\lim_{\varepsilon\downarrow 0}a(k+i\varepsilon),\ \mathrm{Im}\,k=0. \tag{17}$$

Knowing $a(k)$ we can then find $b(k)$ from Eq. (15).

We proceed now to the inverse problem. Its solution is based on an integral equation, obtained for the first time in [6], which is a particular case of the Gel'fand-Levitan equation. Let $s_1 = (r_1(k), \varkappa_l^{(1)}, c_l^{(1)}, l = 1, \ldots, m_1)$ and $s_2 = (r_2(k), \varkappa_l^{(2)}, c_l^{(2)}, l = 1, \ldots, m_2)$ be the scattering data for the two potentials $u_1(x)$ and $u_2(x)$. We formulate the kernel $F(x, y)$,

$$F(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[r_2(k)-r_1(k)]\,g_1(x,k)\,g_1(y,k)\,dk+\sum_{l=1}^{m_2}c_l^{(2)}g_1(x,i\varkappa_l^{(2)})\,g_1(y,i\varkappa_l^{(2)})-\sum_{l=1}^{m_1}c_l^{(1)}g_1(x,i\varkappa_l^{(1)})\,g_1(y,i\varkappa_l^{(2)}), \tag{18}$$

and consider the equation for the kernel $K(x, y)$,

$$K(x,y)+F(x,y)+\int_{-\infty}^{x}K(x,z)\,F(z,y)\,dz=0,\quad x>y. \tag{19}$$

This equation is uniquely solvable; also

$$u_2(x)-u_1(x)=\frac{d}{dx}K(x,x). \tag{20}$$

For $u_1 = 0$ this equation permits us to reestablish the potential $u_2(x)$ from its corresponding scattering data. In the definition (18) it follows that one should take $g_1(x, k) = e^{-ikx}$ and $c_l^{(1)} = 0$. For $c_l^{(2)}$ and $\varkappa_l^{(2)}$ we may take arbitrary positive numbers, with none of the $\varkappa_l$ being equal. The coefficient $r(k)$ must satisfy the conditions $r(k) = O(1/|k|)$, $|k| \to \infty$; $|(k) \leq 1$, and the Fourier transform of the coefficient $b(k)$, namely

$$B(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(k)\,e^{-ikx}dk,$$

must satisfy the condition

$$\int_{-\infty}^{\infty}(1+|x|)\left|\frac{d}{dx}B(x)\right|dx<\infty$$

(see [5]). Generally, in the matter of smoothness the functions $u(x)$ and $dB(x)/dx$ behave identically.

§2. Calculation of the Form Ω in Terms of the Scattering Data

Consider the potential $u(x)$ and two of its variations $\delta_1 u(x)$ and $\delta_2 u(x)$. Let s, $\delta_1 s$, $\delta_2 s$ be the corresponding scattering data and their variations. We must calculate $\Omega(\delta_1 u, \delta_2 u)$ in terms of s, $\delta_1 s$, and $\delta_2 s$. To do this we need to find an explicit expression for δu in terms of s and δs, and then substitute it into definition (7) of the form Ω.

We now make these calculations. From formulas (18), (19), and (20) we obtain the following expression for the variation $\delta u(x)$:

$$\delta u(x)=-\frac{d}{dx}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta r(k)\,g^2(x,k)\,dk+\sum_{l=1}^{m}(\psi_l^2(x)\,\delta c_l+2ic_l\psi_l(x)\,\dot\psi_l(x)\,\delta\varkappa_l)\right],$$

where the notation $g(x, k)$, $r(k)$, $\psi_l(x)$, $\dot\psi_l(x)$, $\varkappa_l$, and c_l is that introduced above. Substituting these formulas for $\delta_1 u(x)$ and $\delta_2 u(x)$ into Eq. (7) leads to the following expression:

$$\Omega_s(\delta_1 s,\delta_2 s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(k,q)\,[\delta_1 r(k)\,\delta_2 r(q)-\delta_1 r(q)\,\delta_2 r(k)]\,dk\,dq$$

$$+\sum_{l=1}^{m}\int_{-\infty}^{\infty}B_l(k)\,[\delta_1 r(k)\,\delta_2\varkappa_l-\delta_1\varkappa_l\,\delta_2 r(k)]\,dk+\sum_{l,j=1}^{m}[C_{lj}(\delta_1\varkappa_l\delta_2 c_j-\delta_1 c_j\delta_2\varkappa_l)+D_{lj}(\delta_1\varkappa_l\delta_2\varkappa_j-\delta_1\varkappa_j\delta_2\varkappa_l)]$$

$$+\sum_{l=1}^{m}\int_{-\infty}^{\infty}E_l(k)\,[\delta_1 r(k)\,\delta_2 c_l-\delta_1 c_l\delta_2 r(k)]\,dk+\sum_{l,j=1}^{m}F_{lj}\,[\delta_1 c_l\delta_2 c_j-\delta_1 c_j\delta_2 c_l],$$

where

$$A(k,q)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\{g^2(x,k),g^2(x,q)\}\,dx,\quad B_l(k)=\frac{ic_l}{\pi}\int_{-\infty}^{\infty}\{g^2(x,k),\psi_l(x)\dot{\psi}_l(x)\}\,dx;$$

$$C_{lj}=2ic_l\int_{-\infty}^{\infty}\{\psi_l(x)\dot{\psi}_l(x),\psi_j^2(x)\}\,dx;\quad D_{lj}=-4c_lc_j\int_{-\infty}^{\infty}\{\psi_l(x)\dot{\psi}_l(x),\psi_j(x)\dot{\psi}_j(x)\}\,dx,$$

$$E_l(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{g^2(x,k),\psi_l^2(x)\}dx,\quad F_{lj}=\int_{-\infty}^{\infty}\{\psi_l^2(x),\psi_j^2(x)\}\,dx$$

and where we use the usual notation $\{f,g\}=f_xg-g_xf$ for the Wronskian determinant. The integrals appearing in the definitions of $A(k,q)$ and $B_l(k)$ are to be taken in the sense of the theory of generalized functions.

It proves to be the case that all the integrals mentioned can be calculated explicitly and expressed in terms of the scattering data only. We show this in detail in the case of the integral for $A(k,q)$. From the Schroedinger equation it is not difficult to show that

$$g(x,k)\,g(x,q)=\frac{1}{q^2-k^2}\frac{d}{dx}\{g(x,k),g(x,q)\}. \tag{21}$$

On the other hand,

$$\{g^2(x,k),g^2(x,q)\}=2g(x,k)\,g(x,q)\{g(x,k),g(x,q)\},$$

whence, using Eq. (21), we have

$$\{g^2(x,k),g^2(x,q)\}=\frac{1}{q^2-k^2}\frac{d}{dx}(\{g(x,k),g(x,q)\})^2.$$

The latter formula enables us to express the integrals $A(k,q)$, $E_l(k)$, and F_{lj} in terms of the asymptotic behavior of the corresponding Wronskians. It is evident here that $E_l(k)=0$, $F_{lj}=0$, and we obtain for $A(k,q)$ the expression [see (9), (10)]

$$A(k,q)=\lim_{N\to\infty}\frac{1}{q^2-k^2}\cdot\frac{1}{8\pi^2}[(a(k)\,a(q)\,i(q-k)\,e^{-i(k+q)N}$$

$$+a(k)\,b(-q)\,i(k+q)\,e^{i(q-k)N}-a(q)\,b(-k)\,i(k+q)\,e^{i(k-q)N}+b(-k)\,b(-q)\,i(k-q)\,e^{i(k+q)N})^2-(i(q-k)\,e^{i(k+q)N})^2].$$

We use a known relationship from the theory of generalized functions,

$$\lim_{N\to\infty}P\frac{e^{ixN}}{x}=i\pi\delta(x),$$

where the symbol P means that $1/x$ is taken in the principal value sense. The final expression for $A(k,q)$ is then

$$A(k,q)=\frac{ik}{\pi}|a(k)|^2\delta(k+q)+\frac{1}{\pi^2}P\frac{k^2+q^2}{k^2-q^2}a(k)\,a(q)\,b(-k)\,b(-q),$$

where in simplifying the first term we used the fact that by virtue of the relations (11)

$$|a|^4-|b|^4+1=(|a|^2-|b|^2)(|a|^2+|b|^2)+1=|a|^2+|b|^2+1=2|a|^2.$$

The expressions for $B_l(k)$, C_{lj}, and D_{lj} are obtained in an analogous way. In addition we also need the identities

$$\{g^2(x,k),\psi_l(x)\dot{\psi}_l(x)\}=-\frac{1}{k^2+\varkappa_l^2}\cdot\frac{d}{dx}(\{g(x,k),\psi_l(x)\}\{g(x,k),\dot{\psi}_l(x)\})+\ldots,$$

$$\{\psi_l(x)\dot{\psi}_l(x),\psi_j(x)\dot{\psi}_j(x)\}=\frac{1}{2}\frac{1}{\varkappa_l^2-\varkappa_j^2}\frac{d}{dx}(\{\psi_l(x),\dot{\psi}_j(x)\}\{\dot{\psi}_l(x),\psi_j(x)\}+\{\psi_l(x),\psi_j(x)\}\{\dot{\psi}_l(x),\dot{\psi}_j(x)\})+\ldots,$$

where we omit writing the total derivatives of rapidly decreasing terms. We then obtain the result

$$B_l(k)=-\frac{4}{\pi}\frac{k^2-\varkappa_l^2}{k^2+\varkappa_l^2}a(k)\,b(-k),\quad C_{lj}=\delta_{lj}\frac{4\varkappa_l}{c_l},\ D_{lj}=-4\frac{\varkappa_l^2+\varkappa_j^2}{\varkappa_l^2-\varkappa_j^2}(1-\delta_{lj}),$$

where in the calculations we have used the asymptotic relations (12) and (13) and the identity (14).

The resulting formulas enable us to express the form Ω in terms of the scattering data:

$$\Omega_s(\delta_1 s, \delta_2 s) = \int_{-\infty}^{\infty} \frac{ik}{\pi} |a(k)|^2 [\delta_1 r(k)\delta_2 r(-k) - \delta_1 r(-k)\delta_2 r(k)]\, dk$$
$$+\frac{P}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k^2+q^2}{k^2-q^2} a(k)a(q)b(-k)b(-q)[\delta_1 r(k)\delta_2 r(q) - \delta_1 r(q)\delta_2 r(k)]\,dk\,dq$$
$$+\frac{4}{\pi}\sum_{l=1}^{m}\int_{-\infty}^{\infty}\frac{\varkappa_l^2-k^2}{k^2+\varkappa_l^2} a(k)b(-k)[\delta_1 r(k)\delta_2\varkappa_l - \delta_1\varkappa_l\delta_2 r(k)]\,dk$$
$$+2\sum_{l=1}^{m}\frac{\varkappa_l}{c_l}[\delta_1\varkappa_l\delta_2 c_l - \delta_1 c_l\delta_2\varkappa_l] + 2\sum_{\substack{l,j=1\\ l\neq j}}^{m}\frac{\varkappa_l^2+\varkappa_j^2}{\varkappa_j^2-\varkappa_l^2}[\delta_1\varkappa_l\delta_2\varkappa_j - \delta_1\varkappa_j\delta_2\varkappa_l]. \tag{22}$$

We show now that the set of variables

$$P(k) = -\frac{k}{\pi}\ln(1-|r(k)|^2), \qquad Q(k) = \arg b(k),$$
$$p_l = \varkappa_l^2, \quad q_l = 2\ln b_l, \quad b_l = ic_l\frac{d}{dk}a(k)|_{k=i\varkappa_l}, \quad l = 1,\dots,m, \tag{23}$$

is a canonical set, i.e., the form Ω in these variables takes on the following appearance:

$$\Omega_s(\delta_1 s, \delta_2 s) = \int_{-\infty}^{\infty}(\delta_1 P(k)\delta_2 Q(k) - \delta_1 Q(k)\delta_2 P(k))\,dk + \sum_{l=1}^{m}(\delta_1 p_l\delta_2 q_l - \delta_1 q_l\delta_2 p_l). \tag{24}$$

Without dwelling on the motivation for the choice of variables (23) we reassure ourselves of this choice by means of a direct substitution. Noting that arg b (k) = arg r(k) + arg a(k), and using Eqs. (17), we obtain

$$\delta Q(k) = \frac{1}{2i}\left(\frac{\delta r(k)}{r(k)} - \frac{\delta r(-k)}{r(-k)}\right) - \sum_{l=1}^{m}\frac{2k}{k^2+\varkappa_l^2}\delta\varkappa_l - \frac{P}{2\pi}\int_{-\infty}^{\infty}\frac{1}{q-k}\frac{1}{1-|r(q)|^2}(r(q)\delta r(-q) + r(-q)\delta r(q))\,dq.$$

Further, it follows directly from Eq. (22) that

$$\delta P(k) = \frac{k}{\pi}\frac{1}{1-|r(k)|^2}(r(k)\delta r(-k) - r(-k)\delta r(k)), \quad \delta p_l = 2\varkappa_l\delta\varkappa_l.$$

Finally, using Eqs. (17) and deleting terms proportional to $\delta\varkappa_l$, we have

$$\delta q_l = \frac{\delta c_l}{c_l} - 2\sum_{j\neq l}\frac{\varkappa_l}{\varkappa_l^2-\varkappa_j^2}\delta\varkappa_j + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{q-i\varkappa_l}\frac{1}{1-|r(q)|^2}(r(q)\delta r(-q) + r(-q)\delta r(q))\,dq + \dots$$

We substitute these expressions for the variations in Eq. (24). The resulting expression is easily transformed to the form (22) if account be taken of relations (11), (15), and (16).

We have thus expressed, in terms of the scattering data, a set of variables, which is canonical for the simplicial form Ω. In the following section we confirm that this set plays the role of variables of angle-action type with respect to the Hamiltonian H[u].

§3. Identities for Traces and the Behavior of Their Sums

In this section we shall assume that the function u(x) is infinitely differentiable and along with its derivatives decreases rapidly. In this case $\ln a(k)$ admits for $|k| \to \infty$ the asymptotic expansion in inverse powers of k,

$$\ln a(k) = \sum_{n=1}^{\infty}\frac{c_n}{k^n}. \tag{25}$$

We present two methods for calculating the coefficients c_n. Equating the coefficients obtained by these two methods then yields the desired identities expressing the integrals of motion in terms of the scattering data.

To prove the relation (25) we note that by virtue of the aforementioned relationship between the smoothness properties of u(x) and the Fourier transform of the coefficient b(k) the latter coefficient decreases rapidly for $|k| \to \infty$. By virtue of relations (11) and (16) this implies the rapid decrease of $\ln(1-|r(k)|^2)$, i.e., of the integrand in Eq. (17). We can therefore justify the expansion (25), where

$$c_{2j}=0; \qquad c_{2j+1}=\frac{1}{2\pi i}\int_{-\infty}^{\infty} k^{2j}\ln(1-|r(k)|^2)\,dk-\frac{2}{2j+1}\sum_{l=1}^{m}(i\varkappa_l)^{2j+1}.$$

The second method for calculating the coefficients c_n is based on the Schroedinger equation. We consider the function $\chi(x,k)=\ln f(x,k)$, which is defined for all sufficiently large $|k|$, $\operatorname{Im} k>0$. For $\operatorname{Im} k>0$ the function $\chi(x,k)$ exhibits the asymptotic behavior

$$\chi(x,k)=ikx+o(1),\ x\to\infty;\ \chi(x,k)=ikx+\ln a(k)+o(1), \quad x\to-\infty. \tag{26}$$

Consider the function

$$\sigma(x,k)=\frac{d}{dx}\chi(x,k)-ik.$$

It is a solution of an equation of Ricatti type

$$\sigma_x+\sigma^2-u+2ik\sigma=0 \tag{27}$$

and decreases for $x\to\infty$. From the relations (26) we see that

$$\ln a(k)=-\int_{-\infty}^{\infty}\sigma(x,k)\,dx.$$

This equation, obtained for $\operatorname{Im} k>0$, applies in its smoothness even to the real axis. Using the differential equation (27) we may confirm that $\sigma(x,k)$ has the asymptotic representation

$$\sigma(x,k)=\sum_{n=1}^{\infty}\frac{\sigma_n(x)}{(2ik)^n},$$

where the coefficients $\sigma_n(x)$ satisfy the recursion relations

$$\sigma_n(x)=-\frac{d}{dx}\sigma_{n-1}(x)-\sum_{k=1}^{n-1}\sigma_{n-k-1}(x)\,\sigma_k(x),\ n=2,\dots;\quad \sigma_1(x)=u(x).$$

The first several coefficients have the form

$$\sigma_2=-u_x,\quad \sigma_3=-u^2+u_{xx},\quad \sigma_4=-u_{xxx}+4uu_x,\quad \sigma_5=u_{xxxx}-6uu_{xx}-5u_x^2+2u^3.$$

We see that $\sigma_2(x)$ and $\sigma_4(x)$ are total derivatives. This property holds for all the $\sigma_{2j}(x)$. Returning to $\ln a(k)$ we may assure ourselves of the validity of Eq. (25), where

$$c_{2j+1}=-\left(\frac{1}{2i}\right)^{2j+1}\int_{-\infty}^{\infty}\sigma_{2j+1}(x)\,dx,$$

so that

$$c_1=-\frac{1}{2i}\int_{-\infty}^{\infty}u(x)\,dx,\qquad c_3=-\frac{1}{8i}\int_{-\infty}^{\infty}u^2(x)\,dx,\quad c_5=-\frac{1}{32i}\int_{-\infty}^{\infty}(2u^3+u_x^2)\,dx.$$

We thus arrive at the set of relations

$$\int_{-\infty}^{\infty}\sigma_{2j+1}(x)\,dx=(-1)^j\,2^{2j}\int_{-\infty}^{\infty}k^{2j-1}P(k)\,dk-\frac{2^{2(j+1)}}{2j+1}\sum_{l=1}^{m}p_l^{\frac{2j+1}{2}},$$

where in writing the right-hand side we have used the notation introduced in Eqs. (23). In particular,

$$H[u]=\frac{1}{2}\int_{-\infty}^{\infty}\sigma_5(x)\,dx=8\int_{-\infty}^{\infty}k^3P(k)\,dk-\frac{32}{5}\sum_{l=1}^{m}p_l^{\frac{5}{2}},$$

so that the Hamiltonian is actually a function of "impulses" only, which justifies our analogy between the variables $P(k)$, p_l, $Q(k)$, q_l and the variables of angle-action type in Hamiltonian mechanics.

We have now finished the solution of the second problem formulated in the introduction. We note now that in the variables introduced here the KdV equation appears as follows:

$$\frac{d}{dt} P(k) = 0, \quad \frac{d}{dt} p_l = 0 \quad \frac{d}{dt} Q(k) = 8k^3, \quad \frac{d}{dt} q_l = -8\varkappa_l^3;$$

in this form its solution is a trivial matter. The solution is given by the formulas (5).

On the basis of the mechanical analogy developed here we have reproduced all the results obtained in [2-4]. The existence of an infinite set of integrals of motion for the KdV equation has been shown to be a trivial consequence of this analogy. Our method also enables us to give an explicit solution for all the generalized KdV equations

$$u_t = \frac{d}{dx} \sum_{n=1}^{\infty} \varphi_n(t) \frac{\delta I_n[u]}{\delta u(x)}.$$

Our method, however, is most attractive through its systematic point of view: we now have at our disposal a nontrivial model of an infinite-dimensional completely integrable Hamiltonian system.

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