Nonlinear stage of parametric wave excitation in a plasma

V. E. Zakharov, S. L. Musher, and A. M. Rubenchik

Institute of Automation and Electrometry, Siberian Division, USSR Academy of Sciences

(Submitted December 4, 1973)
ZhETF Pis. Red. 19, 249-253 (March 5, 1974)

The spectra of Langmuir turbulence excited by a high-frequency electric field are investigated. It is shown that at large external-field amplitudes the distribution of the oscillations is essentially nonstationary. This leads to an oscillatory character of the energy absorption in the plasma.

1. When a periodic electric field \( E = E_0 \cos \omega_0 t (\omega_0 \geq \omega_p) \) of sufficient amplitude is applied to a homogeneous isotropic plasma, plasma waves are parametrically excited. If \( T_i < T_e \), the strongest parametric instability leads to a buildup of waves near the \( k \) space surface described by the equation

\[
\omega_n = \omega_p \pm k |\cdot|
\]

(1)

Here \( \omega_p = \omega_p (1 + \frac{3}{2} k^2 \gamma_p^2) \) is the law of dispersion of the Langmuir waves, and \( s = \sqrt{\gamma_p / M} \) is the velocity of the ion sound. At not too large field amplitudes

\[
\left( \frac{E_0^2}{8 \pi n_T} \right) < \sqrt{\frac{k^*}{M}} r_p \left( \frac{\gamma_p}{\sqrt{k^*}} \right)^{\frac{1}{2}},
\]

where \( \gamma_p \) is the damping of the ion sound, \( k_{\text{eff}} \ll k^* \) is the characteristic wave number of the excited waves, \( k_{\text{eff}} = \sqrt{(n/M)(1/r_p)} \), the principal role in the nonlinear limitation of the instability is played by the induced scattering of the Langmuir waves by the plasma ions. This plasma can lead to a wave-energy flux in the region of small wave numbers. At not too large increments of the parametric instability \( \Gamma_0 < \gamma_L (k^*/k_{\text{eff}}) \), the Langmuir-wave energy dissipation is ensured by their linear (collisional) damping \( \gamma_L \), and when \( \Gamma_0 \gg \gamma_L (k^*/k_{\text{eff}}) \) the energy dissipation occurs in regions of small wave numbers and is ensured by a nonlinear dissipation mechanism, namely the collapse of the Langmuir waves. In this case, a region of inertial transfer of the wave energy exists at \( k > k^* \).

The angular anisotropy of the instability increment \( \Gamma_0 \) causes the spectrum of the Langmuir waves to become quasi-one-dimensional, and take the form of symmetrical "jets" elongated in the direction of \( E_0 \) (this was proved in [4] for problems with a characteristic scale \( \Delta k \gg k_{\text{eff}} \)). This enables us to confine ourselves to the consideration of the one-dimensional problem. In the one-dimensional symmetrical case the kinetic equation for the waves is

\[
\frac{\partial \psi}{\partial t} = n_k |\Gamma_k| + \int_{k=k^*}^{\infty} T(k-k') n_k ' \psi' - \gamma_L n_k - \gamma_L n_0.
\]

(2)

Here \( \Gamma_k = [\omega_p E_0^2/8 \pi n T] \phi(\xi) \) is the instability increment, \( T(\xi) = [\omega_0^2/2 n_0 T] \phi(\xi/2) \) is the matrix element of the induced scattering, and \( \phi(\xi) = -\phi(-\xi) \) is a dimensionless structure function such that \( T(k-k') \) has sharp extrema at \( k-k' = \pm k_{\text{eff}} \). At \( T_i / T_e \ll 1 \) we have

\[
\phi(\xi) = \frac{1}{4 \frac{T_i}{T_e} \xi^{-1} + \sqrt{\frac{2 \pi n T}{M n_T} \xi}},
\]

where \( n_0 \) is the amplitude of the thermal noise.

2. Equation (2) was simulated with a computer, using 100 points in the interval from \( k = k^* \) to \( k = 0 \); in the region of the first 10 points, strong linear damping was turned on and guaranteed absorption of the energy condensed in the region of small wave numbers. The numerical experiment has shown that in all cases the one-dimensional spectrum consisted of a chain of narrow (\( \Delta k \ll k_{\text{eff}} \)) peaks located at distances \( k_{\text{eff}} \) from one another. The peak width decreased with decreasing \( T_i / T_e \). It is possible here to separate two cases. At not too large instability increments \( \Gamma / \gamma_L < k^*/k_{\text{eff}} \), a stationary state is established in the form of a sequence of peaks that decreases linearly to zero (Fig. 1). This result agrees with the well known results of Oberman, Valeo, and Perkins. The time required to establish the stationary state is inversely proportional to the noise level \( n_0 \) and is of the order of \((1/\tau) \sim n_0 / n_0 \), where \( n_0 = \Gamma / T \) is the characteristic amplitude of the parametric waves. At large excesses above the instability threshold \((\Gamma / \gamma_L > k/\gamma_{L})\), no stationary state is established, and a relaxation process periodic in time is observed instead; several of the states of this process, following each other in time, are shown in Fig. 2. The energy release in the \( k \sim k^* \) zone then takes the form of pulses that propagate subsequently in the region of small \( k \) along a chain of peaks, in the form of localized excitations of the chain. The maximum peak amplitude is of the order of \( n_L \ln(n_0 / n_0 \Delta k) \), where \( \Delta k \) is the width of the peak, and the time interval between peaks is of the order of \( \sim 1/\gamma_L \ln(n_0 / n_0 \Delta k) \). The pulse propagation velocity is of the order of \( \sim \tau_{\gamma_L} \) and depends little on the noise amplitude.

3. The existence of a discrete chain of peaks makes it possible to replace Eq. (2) by a finite-difference equation that has the following form in terms of the dimensionless variables \( f_n \) (\( f_n \) is the amplitude of the nth peak):

\[
\frac{df_n}{dt} = f_{n+1} - f_{n-1} - \gamma_n + \Gamma_n \frac{\gamma_{L}}{r_{L}} \frac{\gamma_{L}}{r_{L}} + C_n.
\]

(3)
FIG. 2. Distribution of $n_k$ at infinite excess above threshold ($\gamma_k = 0$) for successive instants of time ($t = t_1, t_2, t_3, t_4$) in arbitrary units. The point $k = z$ corresponds to the maximum of the increment.

In the inertial region, neglecting the linear damping and the noise, we have

$$\frac{df_n}{dt} = f_n(f_{n+1} - f_{n-1}).$$

Equation (4) has an exact solution $f_n(t) = f(t-n/s - \tau_0)$, where

$$f(\xi) = f_0 \left(1 + \frac{a}{1 - b + b \cosh \gamma \xi} \right).$$

Here $f_0$, $a$, and $\tau_0$ are arbitrary parameters, while $s$, $b$, and $\gamma$ are functions of $a$ and $f_0$; when $a > 1$ we have

$$\gamma = 2f_0 a; \quad b^2 = \frac{1}{a}; \quad \frac{\gamma}{s} = \ln a.$$