ON THE COMPLETE INTEGRABILITY OF A NONLINEAR SCHRÖDINGER EQUATION

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It is shown that a nonlinear Schrödinger equation, regarded as the Hamiltonian of a system, is completely integrable. A transition to angle and action variables is made by means of the S matrix of the one-dimensional Dirac operator.

INTRODUCTION

In recent time, the theory of classical nonlinear fields has attracted the attention of ever more investigators, largely because of the extraordinary variety of physical applications. Particular attention is devoted to the problem of statistical description of wave fields. Fields are regarded in this case as conservative Hamiltonian systems with infinitely many degrees of freedom, and their statistical description is based on the hypothesis of ergodicity of this system.

On the other hand, in recent years considerable progress has been achieved in the investigation of certain classes of one-dimensional nonlinear fields. This progress is related to the use of "quantum-mechanical" methods for the study of nonlinear systems. The essence of the new approach, which has been called the "method of the inverse scattering problem" is the following. One associates with a considered classical field a certain differential operator with coefficients from this field whose spectral characteristics (S matrix, spectrum) change in time in a known manner. The Cauchy problem for the nonlinear field equations is thus reduced to the study of the direct and the inverse spectral problem for a linear operator (direct and inverse scattering problem).

The method of the inverse scattering problem was first applied by Kruskal et al. [1] to the Korteweg-de Vries equation:

$$u_t+6uu_x+u_{xxx}=0, \tag{1}$$

which arose already in the last century in connection with problems of waves on the surface of a liquid. A one-dimensional Schrödinger operator is associated with Eq. (1).

Subsequently this approach was applied by Shabat and one of the present authors (V. Z.) to the nonlinear Schrödinger equation [2, 3]

$$i\psi_t+\psi_{xx}+\varkappa|\psi|^2\psi=0. \tag{2}$$

Equation (2) arose in a study of different physical problems; it arises, for example, in the theory of a weakly nonideal Bose gas at T = 0 [4, 5]. The same equation describes the two-dimensional selffocusing of a strong light beam in a nonlinear medium and other effects [6, 7]. The nonlinear equation (2) can be investigated by means of the one-dimensional Dirac operator, which we write in the form

$$L=i\begin{pmatrix}1+p & 0\\ 0 & 1-p\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}0 & \psi^*\\ \psi & 0\end{pmatrix},\quad \varkappa=\frac{2}{1-p^2}. \tag{3}$$

Systems described by Eqs. (1) and (2) are Hamilton systems. They have an exceptional property, being completely integrable, i.e., there exist canonical variables that are single-valued "functions" of the field variables (angle and action variables) in which the equations of motion (1)-(2) take the form

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$$\dot{S}_k=0, \quad \dot{\Phi}_k=\frac{\delta H}{\delta S_k}; \quad H=H\{S_k\}. \tag{4}$$

For Eq. (1) this fact was established by Zakharov and Faddeev [8]. The variables S and Φ are related simply to the scattering matrix of the associated operator. The proof of the complete integrability of the nonlinear Schrödinger equation (2) is the main content of the present paper.

With regard to the problem of a statistical description, complete integrability of considered systems means that there is no stochastization in the system; nonlinear interaction does not lead to a redistribution of energy between the different "modes." The time of "phase mixing" of a certain system is thus determined not by the magnitude of the nonlinearity but by the "departure" from the "nearest" completely integrable system.

Great interest also attaches to the problem of describing a whole class of fields for whose study one can apply the method of the inverse scattering problem. For equations that are integrable by means of a Schrödinger operator this problem was solved in [8], where it was shown that all systems whose Hamiltonians are traces of polynomials of the associated operator with time dependent coefficients are completely integrable. There exists simple recursive formulas for the calculation of these Hamiltonians. A similar result is obtained below for the operator (3).

In connection with the fact that the method we use differs from that employed in [8], we give the scheme of the new proof of complete integrability of the Korteweg-de Vries equation. We note also that for one special case of the problem we consider ($\psi \to 0$, $\varkappa < 0$ for $|x| \to \infty$) the theorem of complete integrability of Eq. (2) was proved by Takhtadzhyan.

1. Angle and Action Variables for the Nonlinear Schrödinger Equation ($\varkappa > 0$)

We consider Eq. (2) on the infinite interval $-\infty < x < \infty$ for $\varkappa > 0$. A physically sensible statement of the problem in this case requires vanishing of the field at infinity.

We write Eq. (2) in the Hamilton form

$$i\psi_t=\frac{\delta H}{\delta\psi^*}; \quad i\psi_t^*=-\frac{\delta H}{\delta\psi}, \tag{5}$$

where the Hamiltonian H is

$$H=\int\limits_{-\infty}^{\infty}\left(|\psi_x|^2-\frac{\varkappa}{2}|\psi|^4\right)dx. \tag{6}$$

We shall denote the naturally defined Poisson brackets of the two functionals α and β by $\{\alpha, \beta\}$:

$$\{\alpha,\beta\}=i\int\limits_{-\infty}^{\infty}dx\left\{\frac{\delta\alpha}{\delta\psi}\frac{\delta\beta}{\delta\psi^*}-\frac{\delta\alpha}{\delta\psi^*}\frac{\delta\beta}{\delta\psi}\right\}. \tag{7}$$

Further, we consider the eigenvalue problem for the operator $\hat{L}$ in (3):

$$\hat{L}\varphi=\lambda\varphi, \quad \varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{8}$$

and make the substitution

$$\varphi_1=\sqrt{1-p}\exp\left(-i\frac{\lambda}{1-p^2}x\right)u_2, \quad \varphi_2=\sqrt{1+p}\exp\left(-i\frac{\lambda}{1-p^2}x\right)u_1.$$

Equation (8) takes the form

$$\frac{\partial u_1}{\partial x}+i\xi u_1=q(x)u_2, \quad \frac{\partial u_2}{\partial x}-i\xi u_2=-q^*(x)u_1, \tag{9}$$

where $\xi = \lambda p(1-p^2)^{-1}$, $q(x) = i(1-p^2)^{-1/2}\psi(x)$. If q(x) decreases sufficiently rapidly at infinity, each solution of the system (9) for real ξ is uniquely determined by one of its asymptotic behaviors as $x \to +\infty$ or $x \to -\infty$. The scattering problem for the operator $\hat{L}$ consists of determining one of these asymptotic behaviors given the other. By the inverse scattering problem we shall understand the problem of recovering q(x) from the data of the scattering problem (i.e., from the S matrix). The direct and the inverse problem were studied in detail in [2] and we shall therefore not dwell on the justification of the assertions made below.

The Jost functions $\varphi(x, \xi)$ and $\psi(x, \xi)$, defined as the solutions of (9) with the asymptotic behaviors

$$\varphi(x,\xi) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \quad \text{as} \quad x \to +\infty,$$

$$\psi(x,\xi) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \quad \text{as} \quad x \to -\infty,$$

allow analytic continuation into the upper half-plane of ξ for every x.

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the solution of (9) for real ξ, then $\tilde{\mathbf{u}} \overset{\text{def}}{=} \begin{pmatrix} u_2^* \\ -u_1^* \end{pmatrix}$ is also a solution of the system (9). The functions $\psi(x, \xi)$ and $\tilde{\psi}(x, \xi)$ form a complete set for the system (9), and therefore

$$\varphi(x, \xi) = a(\xi)\tilde{\psi}(x, \xi) + b(\xi)\psi(x, \xi). \tag{10}$$

The elements $a(\xi)$ and $b(\xi)$ of the S matrix can be expressed as follows in terms of the Jost functions:

$$a(\xi) = (\varphi_1\psi_2 - \varphi_2\psi_1)(x, \xi), \quad b(\xi) = (\tilde{\psi}_1\varphi_2 - \tilde{\psi}_2\varphi_1)(x, \xi), \tag{11}$$

where $a(\xi)$ is analytic in the upper half-plane of ξ; $a(\xi) \to 1$ as $\xi \to \infty$, $\operatorname{Im}\xi \geq 0$. The unitarity relation for the S matrix in the considered case has the form

$$|a(\xi)|^2 + |b(\xi)|^2 = 1. \tag{12}$$

The zeros of $a(\xi)$ in the upper half-plane correspond to the eigenvalues of the problem (9); at the same time

$$\varphi(x, \xi) = c\psi(x, \xi) \quad (a(\xi) = 0). \tag{13}$$

The "potential" $q(x)$ can be recovered from the scattering matrix $a(\xi)$, $b(\xi)$ and the set of quantities c at each zero of $a(\xi)$, $\operatorname{Im}\xi \geq 0$. A necessary and sufficient condition of solvability of the inverse problem is the relation (12) and analyticity of $a(\xi)$ for $\operatorname{Im}\xi \geq 0$.

Let us calculate the Poisson brackets between the elements of the S matrix. To determine the variational derivatives $\delta a(\xi)/\delta q(x)$, $\delta a/\delta q^*$, $\delta b/\delta q$, $\delta b/\delta q^*$ we use the representation (11):

$$\frac{\delta a(\xi)}{\delta q(x)} = \frac{\delta}{\delta q(x)}[\varphi_1(y,\xi)\psi_2(y,\xi) - \varphi_2(y,\xi)\psi_1(y,\xi)].$$

The expression in the brackets does not depend on y. One can therefore set $y = x + 0$. Since $\psi(x, \xi)$ is determined by the asymptotic behavior as $x \to +\infty$, it follows that $\delta\psi(y, \xi)/\delta q(x) = 0$ if $y > x$. The expressions for $\lim\limits_{y\to x+0} \delta\varphi_{1,2}(y, \xi)/\delta q(x)$ are readily found from (9):

$$\lim_{y\to x+0} \frac{\delta\varphi_1}{\delta q} = \varphi_2(x,\xi), \quad \lim_{y\to x+0} \frac{\delta\varphi_2}{\delta q} = 0,$$

and thus

$$\frac{\delta a(\xi)}{\delta q(x)} = \varphi_2(x,\xi)\psi_2(x,\xi). \tag{14a}$$

All the remaining variational derivatives are calculated similarly:

$$\begin{gathered} \frac{\delta a(\xi)}{\delta q^*(x)} = \varphi_1(x,\xi)\psi_1(x,\xi), \\ \frac{\delta b(\xi)}{\delta q(x)} = -\tilde{\psi}_2(x,\xi)\varphi_2(x,\xi), \quad \frac{\delta b(\xi)}{\delta q^*(x)} = -\tilde{\psi}_1(x,\xi)\varphi_1(x,\xi). \end{gathered} \tag{14b}$$

To calculate the integrals that determine the Poisson brackets, we note that if $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are two arbitrary solutions of the system (9) for $\xi = \xi_1$, and $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are solutions for $\xi = \xi_2$, then

$$\{u_1^{(1)}u_1^{(2)}v_2^{(1)}v_2^{(2)} - u_2^{(1)}u_2^{(2)}v_1^{(1)}v_1^{(2)}\} = \frac{i}{2(\xi_1-\xi_2)}\frac{d}{dx}[(u_1^{(1)}v_2^{(1)} - u_2^{(1)}v_1^{(1)})(u_1^{(2)}v_2^{(2)} - u_2^{(2)}v_1^{(2)})]. \tag{15}$$

The relation (15) follows directly from (9). Using (15), we readily see that all the integrands that arise are total derivatives. Calculating, for example, $\{a(\xi), b(\xi')\}$, we find that

$$\{a(\xi), b(\xi')\} = -\frac{\varkappa}{4(\xi-\xi')} a(\xi) b(\xi') + \frac{\varkappa}{4} \lim_{x\to\infty} \frac{e^{2i(\xi-\xi')x}}{\xi'-\xi} a(\xi') b(\xi).$$

Using the well known relation $P\left(\lim_{x\to\infty} \frac{e^{i\xi x}}{\xi}\right) = \pi i \delta(\xi)$, we obtain

$$\{a(\xi), b(\xi')\} = -\frac{\varkappa}{4(\xi-\xi')} a(\xi) b(\xi') - \frac{\varkappa}{4} \pi i a(\xi) b(\xi) \delta(\xi-\xi').$$

Proceeding similarly, we can find the Poisson brackets between all possible pairs of elements of the S matrix, after which it is readily seen that the quantities

$$P_\xi = \sqrt{\frac{2}{\varkappa}} \ln \frac{1}{|a(\xi)|^2}, \quad Q_\xi = \frac{1}{\pi} \sqrt{\frac{2}{\varkappa}} \arg b(\xi) \tag{16}$$

have canonical "commutation relations"

$$\{P_\xi, P_{\xi'}\} = \{Q_\xi, Q_{\xi'}\} = 0, \quad \{P_\xi, Q_{\xi'}\} = \delta(\xi-\xi').$$

If $a(\xi)$ does not have zeros in the upper half-plane of ξ, then the set P_ξ, Q_ξ $(-\infty < \xi < \infty)$ is complete, i.e., $q(x)$ can be uniquely recovered from the given set P, Q, since it completely determines the S matrix.

In the case when $a(\xi)$ has N zeros in the upper half-plane, which we shall assume are simple, the variables P_ξ and Q_ξ must be augmented with a discrete set of canonical variables associated with the zeros of $a(\xi)$. Suppose $a(\zeta_n) = 0$, $\operatorname{Im} \zeta_n > 0$, $n = 1, 2, \ldots, N$. At the point $\xi = \zeta_n$ we have $\varphi(x, \zeta_n) = c_n \psi(x, \zeta_n)$. For a finite-range potential $q(x)$, the functions $a(\xi)$ and $b(\xi)$ are analytic in the whole of the complex plane, and the variational derivatives $\delta c_n/\delta q(x)$ and $\delta c_n/\delta q^*$ can be obtained by analytic continuation of $\delta b/\delta q$ and $\delta b/\delta q^*$, which gives

$$\frac{\delta c_n}{\delta q(x)} = -\bar{\psi}_2(x, \zeta_n) \varphi_2(x, \zeta_n), \quad \frac{\delta c_n}{\delta q^*} = -\bar{\psi}_1(x, \zeta_n) \varphi_1(x, \zeta_n).$$

We also find $\delta\varphi_n/\delta q(x)$, using standard perturbation theory for the system (9):

$$\frac{\partial \psi_1'}{\partial x} + i(\zeta+\delta\zeta)\psi_1' = q(x)\psi_2' + \delta q \delta(x-z)\psi_2',$$
$$\frac{\partial \psi_2'}{\partial x} - i(\zeta+\delta\zeta)\psi_2' = -q^*(x)\psi_1', \quad \psi_{1,2}'(\pm\infty) = 0.$$

The solution of these equations has the form

$$\psi' = \begin{cases} c_1\psi(x, \zeta+\delta\zeta), & x>z, \\ c_2\varphi(x, \zeta+\delta\zeta), & x<z. \end{cases}$$

Fitting of the solutions at the point x = z gives

$$c_1\psi_1 - c_2\varphi_1 = \delta q c_1 \psi_2, \quad c_1\psi_2 - c_2\varphi_2 = 0.$$

A nontrivial solution of this system exists if

$$\det \begin{vmatrix} \psi_1 - \delta q \psi_2, & -\varphi_1 \\ \psi_2, & -\varphi_2 \end{vmatrix} = 0,$$

but

$$\det \begin{vmatrix} \psi_1, & -\varphi_1 \\ \psi_2, & -\varphi_2 \end{vmatrix} = -a(\zeta+\delta\zeta) = -a'(\xi)\delta\zeta,$$

from which we find

$$\delta\zeta_n/\delta q(x) = -(a'(\zeta_n))^{-1}\psi_1(x, \zeta_n)\varphi_1(x, \zeta).$$

Similarly,

$$\delta\zeta_n/\delta q^*(x) = -(a'(\zeta_n))^{-1}\psi_2(x, \zeta_n)\varphi_2(x, \zeta_n).$$

It is now easy to see that

$$\{\zeta_n, P_\xi\} = \{\zeta_n, Q_\xi\} = \{c_n, P_\xi\} = \{c_n, Q_\xi\} = 0,$$
$$\{\zeta_n, \zeta_{n'}\} = \{c_n, c_{n'}\} = 0.$$

In addition, $\{\zeta_n, c_{n'}\} = 0$, where $n \neq n'$.

We calculate $\{\zeta_n, \ln c_n\}$:

$$\{\zeta_n, \ln c_n\} = \frac{\varkappa}{2}\frac{i}{a_n' c_n}\int (\varphi_1\varphi_2 + \varphi_2\varphi_1)\,dx. \tag{17}$$

Generally,

$$\varphi_1(x,\zeta)\varphi_2(x,\zeta') + \varphi_2(x,\zeta)\varphi_1(x,\zeta') = i(\zeta-\zeta')^{-1}\cdot\frac{d}{dx}(\varphi_1(x,\zeta)\varphi_2(x,\zeta') - \varphi_2(x,\zeta)\varphi_1(x,\zeta'));$$

as $\zeta' \to \zeta$

$$2\varphi_1(x,\zeta)\varphi_2(x,\zeta) = i\frac{\partial}{\partial(\zeta-\zeta')}\left[\frac{\partial}{\partial x}(\varphi_1(\zeta)\varphi_2(\zeta') - \varphi_2(\zeta)\varphi_1(\zeta'))\right]_{\zeta=\zeta'}.$$

We now integrate (17) in finite limits, differentiate with respect to $\zeta - \zeta'$, and make the passage to the limit $\zeta' \to \zeta$, after which we let the limits of integration tend to infinity, remembering that $a(\zeta) = 0$. As a result, we obtain $\{\zeta_n, \ln c_n\} = \varkappa/4$. Thus, the discrete set of canonical variables has the form

$$P_n = \sqrt{\frac{2}{\varkappa}}\zeta_n, \quad Q_n = \sqrt{\frac{2}{\varkappa}}\ln\frac{1}{c_n^2}, \quad n = 1, 2, \ldots, n. \tag{18}$$

We note that the variables (18) are complex.

The set of variables (16) and (18) is complete; for suppose $\zeta_1, \ldots, \zeta_N$, $\operatorname{Im}\zeta_n > 0$ are zeros of $a(\xi)$. Then the function $a_1(\xi) = a(\xi)\prod_{n=1}^{N}(\xi - \zeta_n^*)/(\xi - \zeta_n)$ is analytic in the upper half-plane and does not vanish in the region $\operatorname{Im}\xi \geq 0$; therefore $\ln a_1(\xi)$ is analytic in this region. On the real axis, $\ln a_1(\xi) = \ln|a(\xi)| + i\arg a_1(\xi)$, and analyticity of $\ln a_1(\xi)$ gives

$$\arg a_1(\xi) = -\frac{1}{\pi}P\int\frac{\ln|a(\xi')|\,d\xi'}{\xi'-\xi},$$

but

$$\arg a_1(\xi) = \arg a(\xi) + \frac{1}{i}\sum_{n=1}^{N}\ln\frac{\xi-\zeta_n^*}{\xi-\zeta_n}.$$

Thus, $a(\xi)$ can be completely recovered from the set P_ξ $(-\infty < \xi < \infty)$ and P_n, $n = 1, 2, \ldots, N$. The quantity $b(\xi)$ can be determined trivially from the unitarity relation and Q_ξ:

$$b(\xi) = (1 - |a(\xi)|^2)^{1/2}\exp(i\pi(\varkappa/2)^{1/2}Q_\xi).$$

It remains to find an expression for the Hamiltonian H in terms of the new canonical variables. For this, we represent u_1 from (9) in the form $u_1 = \exp(-i\xi x + \Phi(x))$ and, eliminating u_2 (assuming at the same time $u_2(-\infty) = 0$), we obtain an equation for $\Phi(x)$:

$$2i\xi\Phi' = |q|^2 + \Phi'^2 + q\frac{d}{dx}\left(\frac{1}{q}\Phi'\right), \tag{19}$$

which enables one to calculate in a regular manner the coefficients of the asymptotic expansion of $\Phi(x, \xi)$ in powers of $1/\xi$: $\Phi(x, \xi) = \sum_{n=1}^{\infty} f_n(x)/(2i\xi)^n$. To determine $f_n(x)$ we have recursion relations that follow from (19):

$$f_{n+1} = q(x)\frac{d}{dx}\left(\frac{1}{q}f_n\right) + \sum_{j+k=n} f_j f_k. \tag{20}$$

The function $\ln a(\xi)$ also allows asymptotic expansion in powers of $1/\xi$ and $\xi \to \infty$, $\operatorname{Im}\xi \geq 0$:

$$\ln a(\xi) = \sum_{k=1}^{\infty} C_k/\xi^k.$$

It is obvious that $C_k = (2i)^{-k}\int_{-\infty}^{\infty} f_k(x)\,dx$. On the other hand, the C_k can be readily expressed in terms of P_ξ in (16) and P_n in (18):

$$C_k = \frac{1}{2\pi i}\left(\frac{\varkappa}{2}\right)^{1/2}\int_{-\infty}^{\infty} P_\xi \xi^{k-1}\,d\xi - \sum_{n=1}^{N}\left(\frac{\varkappa}{2}\right)^{k}\frac{P_n^{\,k} - P_n^{*k}}{k}. \tag{21}$$

By virtue of the recursion formula (20), the C_k are polynomials in ψ and ψ^* and their derivatives with respect to x. We give the first four C_k:

$$C_1 = \frac{1}{2i}\frac{\varkappa}{2}\int |\psi|^2\,dx,$$

$$C_2 = -\frac{1}{(2i)^2}\frac{\varkappa}{4}\int (\psi^*\psi_x - \psi\psi_x^*)\,dx,$$

$$C_3 = -\frac{1}{(2i)^3}\frac{\varkappa}{2}\int \left(|\psi_x|^2 - \frac{\varkappa}{2}|\psi|^4\right)dx,$$

$$C_4 = \frac{1}{(2i)^4}\frac{\varkappa}{2}\int \left(\psi\psi^*_{xxx} + \frac{3}{2}\varkappa\psi\psi_x^*|\psi|^2\right)dx.$$

Note that C_3 is to within a factor the Hamiltonian (6) for Eq. (2).

Since all the C_k (including C_3) can be expressed solely in terms of P_ξ and P_n and the transformation from ψ and ψ^* to the variables P and Q in (16) and (18) is obviously canonical, the nonlinear equation (2) can be written in the variables P and Q in the form (4), i.e., these variables are angle and action variables for the considered system.

Together with the Eq. (2), all the nonlinear fields whose Hamiltonians can be represented in the form of linear combinations of C_k with coefficients that are in general time dependent are completely integrable. For example, the equation generated by Hamiltonian C_4 is the Korteweg-de Vries equation with cubic nonlinearity.

To conclude this section we note that, assuming that the zeros of $a(\xi)$ are simple, we did not at all reduce the generality of the treatment: as is readily seen, the potentials q(x) that lead to multiple zeros of $a(\xi)$ can be approximated with arbitrary accuracy by potentials with close but simple zeros of $a(\xi)$.

2. Angle and Action Variables for a Nonlinear Schrödinger Equation ($\varkappa < 0$)

The case $\varkappa < 0$ requires special treatment, since the physically interesting solutions of Eq. (2) with $\varkappa < 0$ belong to the class $|\psi| \to$ const as $|x| \to \infty$. The formalism of the direct and inverse scattering problems for such potentials developed in [3] differs significantly from that described in the foregoing section.

Let us consider the eigenvalue problem for the operator $\hat{L}$ in (3): $\hat{L}\varphi = E\varphi$, and make the substitution $v_1 = (p-1)^{-1/2}\exp(-iEx/(p^2-1))\varphi_1$, $v_2 = (p+1)^{-1/2}\exp(-eEx/(p^2-1))\varphi_2$. The system $\hat{L}\varphi = E\varphi$ is reduced by it to the form

$$i\frac{\partial v_1}{\partial x} + q^* v_2 = \lambda v_1,\quad \lambda = \frac{pE}{p^2-1},\quad -i\frac{\partial v_2}{\partial x} + qv_1 = \lambda v_2,\quad q = \psi(p^2-1)^{-1/2}. \tag{22}$$

We shall assume that $|q| \to 1$ as $x \to \pm\infty$. Without loss of generality one can also assume that $q \to 1$ as $x \to +\infty$; then, generally speaking, $q(-\infty) = e^{i\alpha}$.

The role of the Jost functions for the system (22) is played by the solutions of (22) with asymptotic behaviors

$$\varphi \to e^{-i\zeta x}\begin{pmatrix} 1 \\ e^{i\alpha}(\zeta-\lambda)\end{pmatrix},\quad x \to -\infty,$$
$$\psi \to e^{-i\zeta x}\begin{pmatrix} 1 \\ \zeta-\lambda\end{pmatrix},\quad x \to +\infty. \tag{23}$$

where $\zeta(\lambda) = \sqrt{\lambda^2 - 1}$ is a two-valued function of λ. We note that if $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2\end{pmatrix}$ is a solution of (22), then $\tilde{\mathbf{v}} \overset{\text{def}}{=} \begin{pmatrix} v_2^* \\ v_1^*\end{pmatrix}$ is also a solution of this system. The functions ψ and $\tilde{\psi}$ are a complete set of solutions of (22), and therefore $\varphi(x, \lambda) = a(\lambda, \zeta)\psi(x, \lambda) + b(\lambda, \zeta)\tilde{\psi}(x, \lambda)$. And, as follows directly from the system (22),

$$a(\lambda) = \frac{W\{\varphi, \tilde{\psi}\}}{2\zeta(\lambda-\zeta)},\qquad b(\lambda) = -\frac{W\{\varphi, \psi\}}{2\zeta(\lambda-\zeta)}, \tag{24}$$

where $W\{\mathbf{u}, \mathbf{v}\}$ is the Wronskian of the two solutions (22): $W(\mathbf{u}, \mathbf{v}) = u_1 v_2 - u_2 v_1$, and it obviously does not depend on x.

The function $\zeta(\lambda)$ is defined on a two-sheeted Riemann surface with cuts $(-\infty, -1)$, $(1 + \infty)$; on the upper sheet, $\operatorname{Im} \zeta > 0$. As is shown in [3], the Jost functions φ and $\tilde{\psi}$ are analytic on the upper sheet of the Riemann surface; (24) then guarantees analyticity of $a(\lambda)$ on this sheet. The zeros of $a(\lambda)$ correspond to the eigenvalues of the system (22). Since the system is selfadjoint, the zeros are distributed on the real axis $-1 < \lambda < 1$ and are simple. If $a(\lambda_n) = 0$, then

$$\varphi(x, \lambda_n) = b_n \tilde{\psi}(x, \lambda_n). \tag{25}$$

Note also that $a(\lambda, -\zeta) = a^*(\lambda, \zeta) e^{i\alpha}$, $b(\lambda, -\zeta) = b^*(\lambda, \zeta) e^{i\alpha}$ for real λ and ζ. Here, α is a phase of $q(x)$ as $x \to -\infty$, which, as can be shown (see [3]), is completely determined by the zeros of $a(\lambda)$:

$$e^{i\alpha} = \prod_{j=1}^{N} \frac{\lambda_j + i\sqrt{1-\lambda_j^2}}{\lambda_j - i\sqrt{1-\lambda_j^2}} \qquad (a(\lambda_j) = 0).$$

Study of the inverse scattering problem for the system (22) shows that $q(x)$ can be uniquely recovered from the sets $a(\lambda, \zeta)$ and $b(\lambda, \zeta)$, the eigenvalues λ of the system (22), and the corresponding quantities b_n (see (25)).

The introduction of canonical variables associated with the S matrix of the system (22) is performed as in the foregoing section. The variational derivatives of the elements of the S matrix are calculated on the basis of the representation (24). The details of the calculations are almost identical to those given above; we therefore omit them.

It can be shown that the Poisson brackets (7) of the quantities

$$\begin{aligned} P_\lambda &= \frac{1}{\varkappa} \ln |a(\lambda, \zeta)|^2, \qquad Q_\lambda = -\frac{2}{\pi} \arg b_\lambda, \\ P_n &= \lambda_n, \qquad Q_n = \frac{1}{\varkappa} \ln \frac{1}{b_n} \end{aligned} \tag{26}$$

are $\{P_\lambda, P_{\lambda'}\} = \{Q_\lambda Q_{\lambda'}\} = 0$, $\{P_\lambda, Q_{\lambda'}\} = \delta_{\lambda-\lambda'}$. Here $\delta_{\lambda-\lambda'}$ is $\delta(\lambda - \lambda')$ for the continuous spectrum and the Kronecker delta for the discrete spectrum. The values of $a(\lambda, \zeta)$ and $b(\lambda, \zeta)$ in (26) are taken on the boundaries of the cut in the λ plane on which the sign of $\zeta(\lambda)$ is the same as the sign of λ. The complete S matrix can be recovered in an obvious manner from the set P and Q.

The integrals of motion of the system (2) can be obtained, as in the case $\psi \to 0$ as $|x| \to \infty$, by expanding $\ln a(\lambda)$ in powers of $1/\lambda$, $\operatorname{Im} \lambda \geq 0$. They have the form $I_n = \int_{-\infty}^{\infty} [f_n(x) - f_n(\infty)] dx$, where f_n are determined from the recursion relations (20).

3. Complete Integrability of the Korteweg-de Vries Equation

We apply the above approach to Eq. (1). This equation can be represented in the form

$$u_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)},$$

where $H = \int (u_x^2/2 - u^3) dx$. The Poisson brackets therefore have the form

$$\{\alpha, \beta\} = -\frac{1}{2} \int \left[\frac{\delta\alpha}{\delta u} \frac{\partial}{\partial x} \frac{\delta\beta}{\delta u} - \frac{\delta\beta}{\delta u} \frac{\partial}{\partial x} \frac{\delta\alpha}{\delta u} \right] dx.$$

We consider further the scattering problem for the one-dimensional Schrödinger operator

$$\frac{d^2\psi}{dx^2} + u(x)\psi + k^2\psi = 0. \tag{27}$$

All the facts mentioned below relating to the direct and inverse spectral problems for the operator (27) can be found in [10]. The solutions of (27) with the asymptotic behaviors

$$\begin{aligned} \varphi &\simeq e^{-ikx} \qquad \text{as } x \to -\infty, \\ \psi &\simeq e^{ikx} \qquad \text{as } x \to +\infty \end{aligned}$$

(Jost functions) are analytic in the upper half-plane of the complex variable k. On the real axis,

$$\psi = a_k\varphi^* + b_k\varphi,$$
$$a_k = \frac{\varphi\psi_x - \psi\varphi_x}{2ik}, \qquad b_k = \frac{\varphi^*\psi_x - \psi\varphi_x^*}{2ik}, \tag{28}$$

from which it can be seen that $a(k)$ is analytic in the upper half-plane of k. The unitary relation for the S matrix gives $|a_k|^2 - |b_k|^2 = 1$. The zeros of $a(k)$ are situated on the imaginary axis, and they correspond to eigenvalues of the problem (27). At them $\psi(x, i\varkappa_n) = b_n\varphi(x, i\varkappa_n)$. The role of the relation (15) is played in this case by the identity

$$(u_1v_1)\frac{\partial}{\partial x}(u_2v_2) - (u_2v_2)\frac{\partial}{\partial x}(u_1v_1) = \frac{1}{k_1^2 - k_2^2}\frac{\partial}{\partial x}\{(u_1u_{2x} - u_{1x}u_2)(v_1v_{2x} - v_2v_{1x})\},$$

where u_1, v_1 and u_2, v_2 are arbitrary pairs of solutions of (27) with $k^2 = k_1^2$ and k_2^2, respectively.

The variational derivatives of a_k and b_k can be calculated on the basis of the representation (28) in the same way as was done in the first section. The integrals that determine the Poisson brackets are, as above, integrals of total derivatives, which enables one to find readily the brackets between all elements of the S matrix. By direct calculation one can show further that the quantities

$$P_k = \frac{k}{\pi}\ln|a_k|^2, \qquad Q_k = \arg b_k,$$
$$P_n = \varkappa_n^2, \quad Q_n = ib_n\frac{\partial}{\partial k}a(k)|_{k=i\varkappa_n}, \qquad n = 1, 2, \ldots, N, \tag{29}$$

satisfy canonical commutation relations.

Asymptotic expansion of $\ln a_k$ in powers of $1/k$ has as its coefficients integrals of certain polynomials in u(x) and its derivatives, which can be found directly from (27). On the other hand, these coefficients can obviously be expressed solely in terms of P_k and P_n, i.e., one has a situation identical to that considered in the first section. One of these polynomials is equal to the Hamiltonian for the equation (1), and this expresses the complete integrability of the latter. Since the arguments of this paragraph differ in no way from those given in [8], we omit the corresponding calculations.

CONCLUSIONS

The above results give the solution of the problem of integrability of Eqs. (1) and (2) considered on the infinite interval $-\infty < x < +\infty$, though they are not applicable on finite intervals with any boundary conditions. In this case, the question of the integrability of the systems (1) and (2) remains open. One can, nevertheless, give some arguments that indicate integrability of the considered fields on finite intervals.

Let us consider, for example, the nonlinear Schrödinger equation on the interval $[0, l]$ with periodic boundary conditions. It is not difficult to see (see [2]) that this equation is identical to the following operator relation:

$$\frac{\partial\hat{L}}{\partial t} = i[\hat{L}, \hat{A}], \tag{30}$$

where $\hat{L}$ is the one-dimensional Dirac operator (3), and the operator $\hat{A}$ has the form

$$\hat{A} = -p\frac{\partial^2}{\partial x^2} + \begin{pmatrix} |\psi|^2/(1+p), & i\psi_x^* \\ -i\psi_x, & -|\psi|^2/(1-p) \end{pmatrix}.$$

The relation (30) and the Hermiticity of the operator $\hat{L}$ in (3) then guarantee conservation in time of the eigenvalues of the eigenvalue problem for the operator $\hat{L}$:

$$\hat{L}\varphi_n = \lambda_n\varphi_n, \quad \varphi_n(0) = \varphi_n(l), \quad n = 1, 2, \ldots \tag{31}$$

Thus, we have a countable set of integrals of motion of the system (2). We show that all the λ_n "commute." Expressions for $\delta\lambda_n/\delta\psi$ and $\delta\lambda_n/\delta\psi^*$ are found directly from (31):

$$\delta\lambda_n = \int_0^l \{(\varphi_n)_1^*\delta\psi^*(\varphi_n)_2 + (\varphi_n)_2^*\delta\psi(\varphi_n)_1\}dx$$

(it is assumed that $\|\varphi\| = \int_0^l(\varphi^+\varphi dx)^{1/2} = 1$; it is easy to achieve conservation in time of such a normalization).

Thus, $\delta\lambda_n/\delta\psi = (\varphi_n)_2^*(\varphi_n)_1$ and $\delta\lambda_n/\delta\psi^* = (\varphi_n)_1^*(\varphi_n)_2$. Now, calculating the Poisson brackets (7) of the quantities λ_n and λ_m and using the analog of the relation (15), we see that for the periodic problem (31) $\{\lambda_n, \lambda_m\} = 0$.

For a dynamical system with finitely many degrees of freedom the existence of commuting integrals of the motion in a number equal to the number of degrees of freedom is equivalent to complete integrability of this system (Liouville's theorem). In the considered case the number of degrees of freedom is countable; the existence of a countable set of conservation laws is at least a necessary condition of integrability.

Similar assertions can also be justified for the Korteweg-de Vries equation (1).

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