The theory of resonance interaction of wave packets in nonlinear media

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(Submitted June 23, 1975)
Zh. Eksp. Teor. Fiz. 69, 1654-1673 (November 1975)

We use the inverse scattering method to consider the (decay and explosive) resonance interaction of three one-dimensional wave packets in a nondissipative medium. We indicate an algorithm for constructing exact solutions and study in detail the scattering of wave packets due to their decay resonance interaction.

PACS numbers: 03.40.Kf

INTRODUCTION

In various physical situations the problem arises of the interaction, in a nonlinear medium, of three wave packets with characteristic frequencies \( \omega_1, \omega_2, \omega_3 \) and wave vectors \( k_1, k_2, k_3 \), satisfying the resonance conditions

\[
\omega_1 = \omega_2 + \omega_3 \quad \text{and} \quad k_1 = k_2 + k_3.
\]

The physical nature of the packets may be completely different. The interaction characterized by the conditions (1) is the coherent decay of the quanta of wave 1 (we shall call it the pumping wave) into the quanta of waves 2 and 3 (which we shall call the secondary waves) and the reverse process of the fusion of quanta 2 and 3 into a quantum 1; it plays a basic role in induced Raman and Mandel'shtam-Brillouin scattering processes, in various processes of energy exchange between waves in a plasma, and so on (see [1c-2]).

At the present time the theory of type (1) processes (which we shall call decay processes) is developed mainly in the approximation which is linear in the amplitudes of the secondary waves [3-4] and it describes the initial stage of the process: the transfer of energy from the pumping to the secondary waves. However, in many experimental conditions the linear approximation is not satisfied and the reaction of the secondary on the pumping wave plays an important role. To take this effect into account one must construct a consistent nonlinear theory. Such a theory was developed by Armstrong et al. [4] for the strongly idealized case of the interaction of three monochromatic waves. They showed in [4] that when there is no dissipation the development of the decay process (1) leads to a periodic exchange of energy between the pumping and the secondary waves; it is clear from the results of this paper that the problem of the finite intensities and the spectral characteristics of the bounded wave packets which undergo a strong decay interaction is far from trivial.

In stable media which allow only waves with positive energy to propagate the type (1) decay processes are the only three-wave resonance processes. In unstable media in which waves with a negative energy are allowed to propagate (e.g., a plasma with a current) a three-wave interaction is possible which satisfies the resonance conditions:

\[
\omega_1 + \omega_2 + \omega_3 = 0, \quad k_1 + k_2 + k_3 = 0.
\]

Conditions (2) can be satisfied only if at least one of the frequencies \( \omega_i \) is negative and they describe the process of the "creation of three quanta from the vacuum"—an explosive instability of the medium. The explosive instability is a process which is by its very nature nonlinear.

As in the decay situation one can easily construct a theory for it for the case of strictly monochromatic waves. In this case the result of the development of the explosive instability is that after a finite time all three wave amplitudes become infinite. It is clear that when finite wave packets interact one must expect the appearance of singularities of their amplitudes which are localized in space.

We consider in the present paper (both decay and explosive) resonance interactions of three one-dimensional wave packets in a nondissipative medium. We apply to this problem a relatively new mathematical technique: the inverse scattering method. This method was first used in 1967 by Gardner, Greene, Kruskal, and Miura [5] and subsequently improved in [6]; it enables one to study efficiently several nonlinear equations without any assumptions whatever about the nonlinearity level of the form of the initial conditions. The inverse scattering method has been successfully applied to the Korteweg-de Vries equation, [6-9] to the nonlinear "parabolic" equation of stationary self-focusing and defocusing, [10-12] to the problem of the propagation of a pulse through a two-level medium, [13-16] to several systems with discrete degrees of freedom, [17-18] and so on.

When we use the inverse scattering method the solution of the initial problem reduces for the equation studied to the solution of the direct and the inverse spectral problems for some linear operator. In principle the application of this method must enable us to find the solution for any initial conditions and at all times. In reality, however, one has to be satisfied with less: a study of the asymptotic solutions as \( t \to \pm \infty \) and also finding a set of separate exact solutions which enable us, however, to formulate rather complete ideas about the general properties of the solutions. In the present paper the use of the inverse scattering method enables us to study efficiently the scattering of wave packets due to their decay resonance interaction and in the case of "long" (i.e., strongly nonlinear) packets to find explicit expressions for the finite wave amplitudes. It then turns out that the physical picture of the interaction depends in a fundamental way on the ratio of the velocities of the pumping and the secondary waves: if the group velocity of the pumping wave lies between the velocities of the secondary waves a long pumping wave packet is unstable and decays practically completely when colliding with arbitrarily small secondary wave packets. The exact solutions which we have found refer to this situation. If, however, the velocity of the pumping packet is the largest or the smallest, the disintegration of the pumping wave is possible only when it collides with sufficiently strong secondary wave packets.

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1. STATEMENT OF THE PROBLEM AND NECESSARY RESULTS FROM SCATTERING THEORY

The equations which describe the processes of interest to us are well known (see, e.g., [1, 2]). Let the \(u_j(x, t)\) be the complex amplitudes of packets which have the appropriate dispersion laws, and \(v_1\) their group velocities. Then

\[
\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} = i\omega u_1, \quad \frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial x} = i\omega u_2,
\]

in the case of the decay process (1) and

\[
\frac{\partial a_3}{\partial t} + v_3 \frac{\partial a_3}{\partial x} = i\omega a_3, \quad \frac{\partial a_2}{\partial t} + v_2 \frac{\partial a_2}{\partial x} = i\omega a_2,
\]

for the explosive instability. Here \(q\) is the interaction constant which can, without loss of generality, be assumed to be real.

We consider a pair of linear differential operators \(L\) and \(M\):

\[
L = iA \frac{d}{dx} + [A, Q],
\]

\[
M = iB \frac{d}{dx} + [B, Q],
\]

where \(A\) and \(B\) are constant \(N \times N\) diagonal matrices, \(A_{jk} = a_j \delta_{jk}, a_1 > a_2 > \ldots > a_N, a_1 \neq 0, B_{jk} = b_j \delta_{jk}\), and \(Q\) is also a square matrix with elements which are functions of \(x\) and, moreover, depend on the time \(t\) as a parameter. By a direct calculation one verifies easily that if we choose \(Q\) in the form

\[
Q = \begin{pmatrix}
0 & p(a_1-a_3)^{-1}a_{3n} & p(a_1-a_3)^{-1}a_n \\
-p(a_1-a_2)^{-1}a_{1n} & 0 & p(a_1-a_2)^{-1}a_n \\
-p(a_2-a_3)^{-1}a_{2n} & -p(a_2-a_3)^{-1}a_{3n} & 0
\end{pmatrix},
\]

we can write the set (3) (or 4), respectively in the form

\[
\frac{dN}{dt} = (L, M),
\]

which enables us to apply to the sets considered the inverse scattering method. The velocities \(v_1\) are then expressed in terms of the coefficients of the operators \(L\) and \(M\) as follows:

\[
v_1 = a_2 b_1 - a_1 b_2, \quad v_2 = a_3 b_1 - a_1 b_3, \quad v_3 = a_3 b_2 - a_2 b_3.
\]
The inverse scattering method can, however, also be used for another problem: the construction of some exact solutions of the sets (3) or (4) using the given scattering matrix. The analysis of such solutions aids to formulate ideas about the solutions of the systems considered. An algorithm for constructing such solutions and the simplest of them are given in the next section.

The general scheme for applying the inverse scattering method is appreciably simplified if we are interested only in the connection between the asymptotic solutions of the set (3) as \( t \rightarrow \pm \infty \), i.e., in the problem of the scattering of wave packets by one another. If we state it in that way, at \( t \rightarrow -\infty \) the wave packets are spatially separated and, hence, do not interact. It is clear that the order in which these packets are positioned is determined by the ratio of their velocities \( v_1 \). To fix the ideas, let \( v_1 > v_2 > v_3 \). At \( t \rightarrow -\infty \) the packet \( u_1 \) is at the left and then \( u_3 \) and \( u_2 \). The scattering matrix can then be written in the form \( S = S_1 S_2 S_3 \), where the \( S_1 \) are the "partial" scattering matrices, i.e., the scattering matrices of the operators \( L_i \) obtained from the operator \( L \) of (5) if we put in it all \( u \) bar \( u_i \) equal to zero. Each such scattering matrix is appreciably simpler than the complete one as it clearly contains only four nontrivial elements.

We can, further, assume that as \( t \rightarrow \pm \infty \) the wave packets, having undergone a decay interaction, also turn out to be spatially separated. Their positions will then be the inverted one, and the scattering matrix can be written in the form \( S = S_1 ^* S_2 ^* S_3 ^* \), where the \( S_1 ^* \) are again partial scattering matrices. It follows from Eq. (14) that the two preceding relations, obtained in the limiting cases as \( t \rightarrow \pm \infty \) are identically valid at all times. Hence follows a simple way to solve the problem of the connection of the asymptotic states of the wave packets as \( t \rightarrow \pm \infty \). This problem can be solved using the scheme

\[
\psi_1 ^*(x, t) \rightarrow S_1 ^*(\lambda_1, t) \rightarrow S_2 ^*(\lambda_2, t) \rightarrow S_3 ^*(\lambda_3, t) \rightarrow \psi_1 ^*(x, t).
\]

A consistent application of this scheme enables us to obtain directly in terms of the initial partial scattering matrices \( S_1 ^* \) (\( \lambda \)) explicit expressions for the integral intensities, produced as a result of the interaction of the packets \( \psi_1 ^*(x, t) \). This is done in Sec. 3.

2. EXACT SOLUTIONS

In principle the inverse scattering method enables us to find an extremely rich set of exact solutions of the sets (3) and (4) which is everywhere dense. Such solutions are connected with scattering matrices with elements which, apart from an exponential factor, are rational functions of the spectral parameter \( \lambda \). However, in the present paper we restrict ourselves to constructing solutions which are the analogues of the well-known N-soliton solutions of the KdV equation which correspond to "non-reflecting" scattering matrices.

We shall assume that \( a_2 > 0 \), i.e., when applying it to the set (3), we shall consider the case where the velocity of the pumping wave is the intermediate one. If the off-diagonal elements of the scattering matrix are identically equal to zero for real values of \( \lambda \), the above described properties of the elements of the scattering matrix \( S_1 ^*(\lambda) \) and \( S_2 ^*(\lambda) \) enable us to determine these elements unambiguously through their zeroes:

\[
S_{11}(\lambda) = \prod_{\lambda > \lambda_1} \frac{1}{\lambda - \lambda_1} \quad \text{and} \quad S_{12}(\lambda) = \prod_{\lambda < \lambda_1} \frac{1}{\lambda - \mu_1}.
\]

where \( \text{Im} \lambda_1 > 0 \) and \( \text{Im} \mu_1 < 0 \). For the sake of simplicity we shall assume that all zeroes are simple ones and we shall, moreover, assume that none of the zeroes of \( S_1 ^* \) and \( S_3 ^* \) are each other's complex conjugates.

The above determined partially solutions of the problem (11) \( \psi'_{1, 2}(x, \lambda) \) and \( \psi'_{3}(\lambda) \) are then analytical in the whole of the complex \( \lambda \)-plane, except a finite number of points where the elements of these matrices have simple poles. One can, for instance, show (see (12, 18)) that

\[
\psi_{11} ^*(x, \lambda) \exp (i \chi x/2a_1) = F_{\lambda, 11} ^*(x, \lambda) \exp \left( \frac{\lambda_1}{2a_1} x \right),
\]

\[
\psi_{12} ^*(x, \lambda) \exp (i \chi x/2a_2) = F_{\lambda, 12} ^*(x, \lambda) \exp \left( \frac{\lambda_2}{2a_2} x \right),
\]

\[
\psi_{33} ^*(x, \lambda) \exp (i \chi x/2a_3) = F_{\lambda, 33} ^*(x, \lambda) \exp \left( \frac{\lambda_3}{2a_3} x \right),
\]

where \( F_{\lambda, 11} ^*(x, \lambda) \) and \( F_{\lambda, 12} ^*(x, \lambda) \) are then analytical in the whole of the complex \( \lambda \)-plane, except a finite number of points where the elements of these matrices have simple poles. One sees easily that

\[
Q_{11}(x) = \sum_{\lambda} \frac{1}{a_1} \sum_{\mu} F_{\lambda, 11} ^*(x, \lambda) \exp \left( \frac{\mu}{a_1} x \right),
\]

\[
Q_{12}(x) = \sum_{\lambda} \frac{1}{a_1} \sum_{\mu} F_{\lambda, 12} ^*(x, \lambda) \exp \left( \frac{\mu}{a_1} x \right),
\]

where the \( Q_{11} \) are the coefficient functions of the operator \( L \) of (5) which are connected with the solutions of the set (3) or (4), respectively, by Eqs. (7) or (8). We have thus given a prescription to construct exact solutions of the sets (3) and (4). We consider the simplest of such solutions in the case of the decay instability.

Let \( S_1 ^*(\lambda) \) and \( S_2 ^*(\lambda) \) have just a single zero at a point \( \lambda \) in the upper half-plane while \( S_3 ^* = 1 \). Performing the appropriate calculations we find that such a scattering matrix corresponds to a "potential" \( Q \) of the form

\[
Q_{11} = Q_{22} = 0;
\]

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\begin{align*}
Q_{ii} = & \frac{\lambda - \lambda^*}{2i a_i} \exp \left[ \frac{\lambda + \lambda^*}{2} \left( \frac{1}{a_i} - \frac{1}{a_i} \right) (x - \nu t) + 4i \eta \right] \\
& \times \exp \left[ -i \lambda \frac{a_i}{a_i} \left( x - \nu t - \frac{a_i}{a_i} \right) \right],
\end{align*}
\tag{19}

where

\[ q_{ii} = \arg S_{ii}, \quad x_i^* = \frac{2}{1 - (\lambda - \lambda^*)} \left( \frac{1}{a_i} - \frac{1}{a_i} \right) \ln \left( \frac{a_i}{a_i} \right) \frac{1}{|S_{i}|^2}. \]

Using (7) we call solution (19) a wave 2 soliton. Similarly, if \( S_{12} = 1 \) and \( S_{22}(\mu) = 0 \), we have \( Q_{12} = Q_{23} = 0 \),

\[ Q_{12} = \frac{\mu - \mu^*}{2i a_2} \exp \left[ \frac{\mu + \mu^*}{2} \left( \frac{1}{a_2} - \frac{1}{a_2} \right) (x - \nu_2 t) + 4i \eta \right] \\
& \times \exp \left[ -i \mu \frac{a_2}{a_2} \left( x - \nu_2 t - \frac{a_2}{a_2} \right) \right].
\]

- a wave 3 soliton.

From the point of view of the set (3) these solutions are by themselves of no interest as they are trivial (the amplitude of only one of the packets is nonvanishing).

We have, however, given the explicit form of these solutions as the picture of the scattering of such packets by one another looks very simple. The solution describing the collision of the secondary waves by one another can be obtained from the nonreflecting scattering matrix in which the elements \( S_{ii}(\lambda) \) and \( S_{ii}(\lambda) \) have up to a single zero. The above described procedure leads to the potential

\[ Q_{ii} = \frac{1}{a_i} \left( \lambda - \lambda^* \right) S_{ii} \exp \left[ (\lambda - \lambda^*) (x - \nu t) + 4i \eta \right] \left( \Delta(x, t) \right), \tag{21} \]

\[ Q_{12} = \frac{\lambda - \lambda^*}{a_1} \left( S_{12} \exp \left[ (\lambda - \lambda^*) (x - \nu_2 t) + 4i \eta \right] \right) \left( \frac{a_1}{a_2} \right) \left( \Delta(x, t) \right) \]

\[ \times \left[ 1 + \frac{a_2}{a_1} \right] \left( |S_{12}| \right) \exp \left[ i (\lambda - \lambda^*) (x - \nu t) \right] \right], \tag{22} \]

\[ Q_{23} = \frac{\mu - \mu^*}{a_2} \left( S_{23} \exp \left[ (\mu - \mu^*) (x - \nu_1 t) + 4i \eta \right] \right) \left( \frac{a_2}{a_1} \right) \left( \Delta(x, t) \right) \]

\[ \times \left[ 1 + \frac{a_1}{a_2} \right] \left( |S_{23}| \right) \exp \left[ i (\mu - \mu^*) (x - \nu t) \right] \right], \tag{23} \]

where \( \gamma_{ik} = 1/a_i - 1/a_k \),

\[ \Delta(x, t) = \left[ 1 + \frac{a_1}{a_2} \right] \left( |S_{12}| \right) \left( |S_{23}| \right) \left( |S_{ii}| \right) \exp \left[ i (\lambda - \lambda^*) (x - \nu_2 t) + 4i \eta \right] \]

\[ \times \exp \left[ i (\mu - \mu^*) (x - \nu_1 t) + 4i \eta \right] \right]. \tag{24} \]

One sees easily that the solution obtained is, asymptotically as \( t \to \pm \infty \), a superposition of solitons of the secondary waves with amplitudes which as \( t \to -\infty \) are exactly the same as their values when \( t \to -\infty \); the whole collision effect thus reduces to a shift in the phases and the coordinates of the centers of the solitons, expressions for which can easily be found from (21) to (24). If, for instance, \( v_2 > v_3 \), we have

\[ \Delta x_1^* = \Delta x_2^* = 0, \quad \Delta x_3^* = \frac{2}{i (\lambda - \lambda^*)} \gamma_3 \frac{1}{2} \ln \left( \frac{\lambda - \lambda^*}{\lambda - \lambda^*} \right). \]

The solution (21) to (24) depends on four arbitrary complex constants. Of great interest, however, is the case when these constants are somehow interrelated. Let, for instance, \( \lambda - \mu^* \to 0 \), \( S_{12} \to 0 \), so that \( S_{12} / (\lambda - \mu^*) \to \text{const} = T \). Then

\[ \Delta(x, t) = e^{i \lambda \frac{a_1}{a_1} \left( x - \nu t \right) + 4i \eta \left( \Delta(x, t) \right)} \left( \frac{a_1}{a_1} \right) \left( \Delta(x, t) \right) \]

\[ \times \exp \left[ i (\lambda - \lambda^*) (x - \nu_1 t) + 4i \eta \left( \Delta(x, t) \right) \right]. \tag{25} \]

If then \( v_3 > v_2 \), we see easily from (21), (22), and (25) that as \( t \to -\infty \) we have \( Q_{12}, Q_{23} \to 0 \), and

\[ Q_{13} = -\frac{1}{\lambda - \lambda^*} \left( S_{13} \exp \left[ (\lambda - \lambda^*) (x - \nu_3 t) + 4i \eta \right] \right) \left( \Delta(x, t) \right) \]

The expression obtained for \( Q_{13} \) is completely analogous to (19) and (20); for that reason we shall call that kind of solution a pumping wave soliton. However, as \( t \to -\infty \) the solution again consists of secondary wave solitons; the solution found thus describes the total decay of the pumping soliton into secondary wave solitons. Completely analogously, taking in (21) to (24) the limits as \( \lambda - \mu^* \to 0 \), \( S_{12} / (\lambda - \mu^*) \to \text{const} \), we get a solution describing the fusion of secondary wave solitons into a pumping soliton, if \( v_2 > v_3 \) and the decay of the pumping soliton when the velocity ratios are the opposite. If, however, we put in (21) to (24) \( \lambda - \mu^* \to 0 \), \( S_{12} \to 0 \), \( S_{23} \to 0 \), \( S_{12} \to 0 \), \( S_{12} S_{23} / (\lambda - \mu^*) \to \text{const} \), we get a "pure" pumping soliton.

We can similarly also consider the analogous situation when \( S_{1i}(\lambda) \) and \( S_{ii}(\lambda) \) have an arbitrary number of zeroes. It then turns out that the solutions obtained contain, asymptotically as \( t \to \pm \infty \), only secondary waves; if, however, we take in the solutions obtained the limit as \( \lambda_i - \mu_i \to 0 \), \( S_{ii} / (\lambda_i - \mu_i) \to \text{const} \), the expressions obtained, which are also exact solutions of the set (3), will, as \( t \to -\infty \), contain, in general, all three packets, including the pumping packet, which vanishes as \( t \to -\infty \). The solutions corresponding to scattering matrices in which \( S_{1i}(\lambda) = S_{ii}(\lambda) \) describe in the general case the spontaneous decay of the pumping packet into secondary wave packets or the inverse process of the total fusion of the latter into the pumping packet, or even both at the same time. If, on the other hand, some of the \( \lambda_i \) are "close" to some \( \mu_i \), in the scattering of the secondary wave packets there arises a situation when after a sufficiently long time there exists a pumping packet which is spatially separated from the main bulk of the secondary waves which have amplitudes which are small in the region where the pumping wave is localized. The period during which such an "intermediate asymptotic behavior" exists can be very long; it depends logarithmically on \( \gamma_{ii} - \mu_i \).

We now consider the case of the explosive instability. We must then put \( G = 1 \) in Equs. (15) to (17). It is clear that we can use the formulae which we have already obtained, replacing in them \( S_{1i}(\lambda) \) and \( S_{ii}(\mu) \), respectively, by \( -S_{1i}(\lambda) \) and \( -S_{ii}(\mu) \). Amongst the solutions occurring then only the solution obtained from (25) is regular as \( t \to -\infty \). It takes the form

\[ Q_{13} = \frac{\lambda - \lambda^*}{a_1} \left( S_{13} \exp \left[ (\lambda - \lambda^*) (x - \nu_3 t) + 4i \eta \right] \right) \left( \Delta(x, t) \right) \]

\[ \times \exp \left[ i \lambda \frac{a_1}{a_1} \left( x - \nu t \right) + 4i \eta \left( \Delta(x, t) \right) \right]. \tag{26} \]

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Here \( S_{31}(\lambda) \) and \( S_{32}(\lambda) \) are arbitrary constants. One sees easily that \( \Delta(x, t) \), starting at some time \( t \geq t^* \), vanishes for several values of \( x \). Simple calculations give
\[
\begin{align*}
\tau &= \frac{i}{(k-\lambda)(\nu-v_1)} \left\{ \nu \tau \gamma_{11} \frac{a_{11}a_{21}a_{31}}{a_{11}a_{21}a_{31}} \right\} \\
&\quad - \frac{i}{(k-\lambda)(\nu-v_1)} \left\{ \nu \tau \gamma_{11} \frac{a_{11}a_{21}a_{31}}{a_{11}a_{21}a_{31}} \right\} \\
\end{align*}
\]
i.e., at \( t = t^* \) there occurs in the solution (26) a singularity at the point
\[
x^* = \frac{i}{(k-\lambda)(\nu-v_1)} \left\{ \nu \tau \gamma_{11} \frac{a_{11}a_{21}a_{31}}{a_{11}a_{21}a_{31}} \right\}
\]
The amplitudes of the packets then behave as \( (x - x^*)^2 \). The solution obtained describes the spontaneous explosion of a soliton of the wave which has the intermediate velocity.

In the general case given initial conditions lead as the result of the development of the explosive instability to the formation of a singularity if the elements \( S_{31}(\lambda) \) and \( S_{32}(\lambda) \) have zeroes. If, for instance, initially the amplitudes of the waves with the largest and smallest velocities are small compared to the amplitude of the wave with the intermediate velocity the condition for the occurrence of zeroes in \( S_{11}(\lambda) \) and \( S_{33}(\lambda) \) is of the form
\[
g_{1i}L_i^2|v_{i-1}|v_i|v_{i+1}| \geq c,
\]
where \( u_i \) and \( L_i \) are the characteristic amplitude and length of the packet of wave 1, while the number \( c \sim 1 \); the exact value of \( c \) depends on the shape of packet 1 and cannot be evaluated for a packet of arbitrary shape. In accordance with what has been said (27) is the criterion for the formation of a singularity from the initial conditions of the type indicated.

### 3. SCATTERING OF WAVE PACKETS

In the present section we consider the problem of the connection between the asymptotic solutions of the set (3), i.e., the problem of the resonance scattering of wave packets. We obtain explicit expressions for the integral intensities of the wave packets arising as \( t \rightarrow -\infty \), directly in terms referring to \( t \rightarrow +\infty \). First, however, we must study the integrals of motion of the set (3). To do this we write \( \psi'(x, \lambda) \) in the form
\[
\psi'(x, \lambda) = \delta_0 \exp \left( -\frac{i}{\lambda} x + \frac{1}{2} \int \mathcal{H}(x, \lambda) \, dx \right)
\]
and, eliminating \( \chi_j(x, \lambda) \), we get for \( A_{ij}(x, \lambda) \)
\[
\begin{align*}
\frac{\partial A_{ij}(x, \lambda)}{\partial x} &= a_{ij} \sum_{\nu} (a_{\nu} - a_{\nu}) Q_{i\nu} A_{\nu} + a_{ij} (a_{\nu} - a_{\nu}) Q_{i\nu} \\
&\quad + a_i \sum_{\nu} (a_{\nu} - a_{\nu}) Q_{i\nu} A_{\nu} - a_{ij} Q_{i\nu} A_{\nu} \\
\end{align*}
\]
From this equation there follows a recurrence relation for the coefficients of the asymptotic expansion of \( A_{ij}(x, \lambda) \) in powers of \( 1/\lambda \). Putting
\[
A_{ij}(x, \lambda) = \sum_{n=-1}^{\infty} A_{ij}^{(n)}(x),
\]
we find
\[
A_{ij}^{(n+1)}(x) = -\frac{1}{a_i - a_j} \left\{ \frac{\partial A_{ij}^{(n)}}{\partial x} + a_i \sum_{\nu} (a_{\nu} - a_{\nu}) Q_{i\nu} A_{\nu}^{(n)} \\
- a_j \sum_{\nu} (a_{\nu} - a_{\nu}) Q_{i\nu} A_{\nu}^{(n)} \right\}, \, n = 1, 2, \ldots
\]
\[
A_{ij}^{(0)}(x) = -a_i Q_{i\nu}(x).
\]
Using now the expression for \( \chi_j(x, \lambda) \) in terms of the \( A_{ij}(x, \lambda) \):
\[
\chi_j(x, \lambda) = \sum_{n=-1}^{\infty} (a_{i} - a_{j}) Q_{i\nu}(x) A_{ji}^{(n)}(x),
\]
we get an expansion for the quantity \( \chi_j(x, \lambda) \):
\[
\chi_j(x, \lambda) = \sum_{n=-1}^{\infty} \chi_j^{(n)}(x, \lambda).
\]
We note further that we get directly from (28) the relation
\[
\int_{-\infty}^{\infty} \chi_j^{(n)}(x) \, dx = L_j S_j(\lambda),
\]
whence by virtue of the conservation of \( S_{jj}(\lambda) \) it follows that the quantities
\[
I_s^{(n)}(x) = \int_{-\infty}^{\infty} \chi_j^{(n)}(x) \, dx
\]
are integrals of the set (3). Equations (29) give us the possibility to evaluate all \( I_s^{(n)}(\lambda) \). As applied to the set (3) the \( I_s^{(n)} \) give the well-known Manley-Rowe relations. We give also for the set (3) the second integrals:
\[
H = -p^2 \left( \frac{\partial x}{\partial t} + \frac{\partial x_j^{(n)}}{\partial x} \right) + \frac{\partial x_j^{(n)}}{\partial x}.
\]
where \( p \) is the canonical momentum:
\[
P = \frac{\partial x_j^{(n)}}{\partial x} + \frac{\partial x_j^{(n)}}{\partial x}.
\]
There is no simple physical interpretation for the other integrals \( I_s^{(n)}(\lambda) \) (\( n \geq 3 \)).

We now turn directly to the problem of the scattering of the packets. We first of all consider the case where the velocity of the pumping wave is the intermediate one. To fix the ideas we shall assume that \( v_3 < v_1 < v_3 \). It is clear that as \( t \rightarrow \pm \infty \) any solution decays into non-interacting packets. As \( t \rightarrow -\infty \) the packets are then positioned along the \( x \)-axis in the order 3, 1, 2. As \( t \rightarrow +\infty \) the sequence of the packets is the reverse. The transition matrix (i.e., the scattering matrix) can be written in the form
\[
S(t) = S_t^{(1)} S_t^{(2)} S_t^{(3)}, \quad t \rightarrow -\infty,
\]
\[
S(t) = S_t^{(1)} S_t^{(2)} S_t^{(3)}, \quad t \rightarrow +\infty.
\]
where the \( S_t^{(i)} \) are the partial transition matrices, described below. It is clear that the asymptotic behavior (31) is by virtue of (14) valid at any time. Referring both representations (31) to the time \( t = 0 \) we get
\[
S_t^{(i)} S_t^{(i)} S_t^{(i)} = -S_t^{(i)} S_t^{(i)} S_t^{(i)}.
\]
Relation (32) and the analytical properties of the diagonal elements of the \( S_i \) enable us, as we shall show below, to

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determine unambiguously the $S_i^\pm$ for given $S_i^\sigma$. Re-establishing then the amplitudes of the packets $u_{k}^{\pm} (x, t)$ in terms of the $S_i^\pm$ we completely determine the characteristics of the wave packets which occur as a result of the scattering.

First of all we study in detail the properties of the partial transition matrices. Putting in the operator $L$ of (5) all $Q_{ij}$ = 0 except $Q_{a\beta}^\sigma = Q_{\beta a}^\sigma = (-Q_{a\beta}^\sigma)$ we get instead of (11)

$$\frac{\partial}{\partial x} u_{a\sigma}^\pm = (a_{a\sigma} - a_{a\alpha}) Q_{\sigma\beta}^\alpha u_{\beta\sigma}^\pm + \delta_{\sigma\alpha} u_{a\sigma}^\pm,$$

$$\frac{\partial}{\partial x} u_{a\sigma}^\pm = (a_{a\sigma} - a_{a\alpha}) Q_{\sigma\beta}^\alpha u_{\beta\sigma}^\pm + \delta_{\sigma\alpha} u_{a\sigma}^\pm,$$

$$\frac{\partial}{\partial x} u_{a\sigma}^\pm = (a_{a\sigma} - a_{a\alpha}) Q_{\sigma\beta}^\alpha u_{\beta\sigma}^\pm + \delta_{\sigma\alpha} u_{a\sigma}^\pm.$$ (33)

The set (33) is a particular case of (11). It is clear that $S_{a\alpha} = \delta_{a\alpha}$, if $i \neq \alpha, \beta$. There are thus in this case only four non-trivial elements in the transition matrix:

$S_{a\alpha}, S_{a\beta}, S_{\beta a}, S_{\beta\beta}$. We put $a < \beta$. In that case, as $a_3 > 0$, the quantities $S_{a\alpha}$ are analytical in the upper $\lambda$-half-plane, and the $S_{\beta\beta}$ in the lower one. By virtue of (12) and (13) we then have

$$S_{a\alpha} - S_{a\alpha}^* = -\frac{s_a}{s_a} S_{a\alpha}^*.$$ (34)

It will be more convenient to use in what follows instead of the transition matrix $S$ the matrix $s$ with the elements

$$s_{a\alpha} = |s_{a\alpha}|^* s_{a\alpha}.$$ (35)

which is clearly unitary and unimodular. In terms of $s$ (34) can be rewritten as

$$s_{a\alpha} = s_{a\alpha}^*.$$ (36)

We note that the diagonal elements of the matrices $S$ and $s$ are the same.

We now introduce the following notation:

$$s_{\pm} = \left( \begin{array}{cc} \sigma_{\pm}^T & 0 \\ 0 & \eta_{\pm}^T \end{array} \right), \quad s_{\pm}^* = \left( \begin{array}{cc} \sigma_{\pm}^T & 0 \\ 0 & \eta_{\pm}^T \end{array} \right), \quad s_{\pm} = \left( \begin{array}{cc} 1 & 0 \\ 0 & \eta_{\pm}^T \end{array} \right)$$

where the $s_{\pm}$ are matrices connected through Eq. (35) with the partial transition matrices $S_{i\pm}$. The $s_{i\pm}$ are then analytical in the upper $\lambda$-half-plane and

$$|s_{1\pm}(\lambda)| + |s_{1\pm}(\lambda)| = 1.$$ (37)

We shall in what follows consider only the scattering of finite wave packets. In that case all elements of the transition matrix are analytical in the whole $\lambda$-plane.

Equation (32) which for the $s_{\pm}$ matrices has, as before, the form

$$s_{\pm} s_{\pm}^* s_{\pm}^* s_{\pm} = s(0),$$

enables us to find at once $|s_{1\pm}(\lambda)|$ and $\eta_{1\pm}(\lambda)$ for given $s_{1\pm}(\lambda)$. Using (36) we have

$$|s_{1\pm}|^2 = |s(0)|^2, \quad |s_{1\pm}|^2 = |s(0)|^2,$$

$$\eta_{\pm} = \mp \eta_{\pm}^*, \quad \eta_{\pm} = \mp \eta_{\pm}^*. $$ (38)

Expressing the elements of the matrix $s(0)$ in terms of $s_{1\pm}$ and $s_{1\pm}^*$ we get

$$s_{1\pm} = -s_{1\pm}^* \eta_{1\pm}^* \eta_{1\pm}^* - s_{1\pm}^* \eta_{1\pm}^* \eta_{1\pm}^*.$$

It is clear from (37) that for a complete determination of the $s_{1\pm}$ matrices we must still know the arguments of $s_{1\pm}^*$. In view of the analytical properties of $s_{1\pm}$ it is sufficient for this to determine the position of the zeroes of $s_{1\pm}$ for $\lambda > 0$. Equating the matrix elements $(s_{1\pm} s_{1\pm}^*)_{ij}$, $(s_{1\pm} s_{1\pm}^*)_{ij}$ we find

$$s_{1\pm}^* s_{1\pm}^* s_{1\pm} = s_{1\pm}^* s_{1\pm}.$$

From this it follows at once that if $s_{1\pm}(\lambda) = 0$, $s_{1\pm}^* \neq 0$, also $s_{1\pm}(\lambda) = 0$, $s_{1\pm}(\lambda) = 0$ and if $s_{1\pm}(\lambda) = 0$, $s_{1\pm}(\lambda) = 0$, then $s_{1\pm}(\lambda) = 0$, $s_{1\pm}(\lambda) = 0$.

We restrict here the considerations to the situation when $S_{1\pm}(\lambda)$ and $S_{3\pm}(\lambda)$ have only simple zeroes, i.e., the case when the zeroes of $s_{1\pm}$ and $s_{3\pm}$ or of $s_{1\pm}$ and $s_{3\pm}$ are not the same. The general case can be obtained by taking the limit. Bearing this restriction in mind we see that if $s_{1\pm}(\lambda) = 0$, it follows from (39) that either $s_{1\pm}(\lambda) = 0$, or $s_{3\pm}(\lambda) = 0$, or $s_{1\pm}(\lambda) = 0$, $s_{3\pm}(\lambda) = 0$, the zeroes of $s_{1\pm}$ (or of $s_{3\pm}$) coincide. We denote the zeroes of $s_{1\pm}$ by $\lambda_{n}$, $\lambda_{n} = \lambda_{n}^*$. The latter occurs only when $s_{1\pm}(\lambda) = 0$, $s_{3\pm}(\lambda) = 0$, 0, $s_{1\pm}(\lambda) = 0$, or remain with $s_{1\pm}(\lambda)$. The latter occurs only when $s_{1\pm}(\lambda) = 0$, $s_{3\pm}(\lambda) = 0$, 0, $s_{1\pm}(\lambda) = 0$, or remain with $s_{1\pm}(\lambda)$. The latter occurs only when $s_{1\pm}(\lambda) = 0$, $s_{3\pm}(\lambda) = 0$, 0, $s_{1\pm}(\lambda) = 0$, or remain with $s_{1\pm}(\lambda)$.

Equation (32) which for the $s_{1\pm}$ matrices has, as before, the form

$$s_{1\pm} s_{1\pm}^* s_{1\pm}^* s_{1\pm} = s(0),$$

enables us to find at once $|s_{1\pm}(\lambda)|$ and $\eta_{1\pm}(\lambda)$ for given $s_{1\pm}(\lambda)$. Using (36) we have

$$|s_{1\pm}|^2 = |s(0)|^2, \quad |s_{1\pm}|^2 = |s(0)|^2,$$

$$\eta_{\pm} = \mp \eta_{\pm}^*, \quad \eta_{\pm} = \mp \eta_{\pm}^*. $$ (38)

It is clear from (37) that for a complete determination of the $s_{1\pm}$ matrices we must still know the arguments of $s_{1\pm}^*$. In view of the analytical properties of $s_{1\pm}$ it is sufficient for this to determine the position of the zeroes of $s_{1\pm}$ for $\lambda > 0$. Equating the matrix elements $(s_{1\pm} s_{1\pm}^*)_{ij}$, $(s_{1\pm} s_{1\pm}^*)_{ij}$ we find

$$s_{1\pm}^* s_{1\pm}^* s_{1\pm} = s_{1\pm}^* s_{1\pm}.$$ (39)

We have thus completely determined the partial transition matrices $s_{1\pm}$. Using the inverse method equations (see [16, 20]) we can now in principle completely determine the asymptotic behavior of the solution as $t \to \infty$. However, we can find an answer to the most interesting physical problems also without solving the inverse method equations. We find directly from (30), by calculating the first term in the expansion of $|s_{1\pm}^*|$ in powers of $1/\lambda$:
where the $u_l^s(x,t)$ are the asymptotic expressions of the $u_l(x,t)$ as $t \to \pm \infty$, and the $\lambda_l^{(0)}$ are the zeroes of the $\sigma_1^2(A)$. Equations (40) together with (37) give expressions for the integral intensities of the packets which occur as $t \to \infty$.

We consider the most interesting special cases. When a strong ($qL_2 \gg 1$) pumping packet collides with small secondary wave packets ($qL_2 \ll 1$) for which the $\sigma_2^3(A)$ have no zeroes, we can neglect in (40) the integral terms. As a result we get

\[
\int |u_l^s(x,t)|^2 \, dx = \frac{1}{2\pi} \int |1 - |\eta_l^2| \eta_l^2 - \eta_l^3| |^2 \, dA.
\]

The initial intensity of the pumping wave is then

\[
\int |u_l^s(x,t)|^2 \, dx \approx \frac{1}{2\pi} \sum \Im \lambda_l^{(0)}.
\]

We see thus that in the case considered the initial intensity of the pumping wave decays practically completely.

In general the partial problems of (33) for the intensities of the packets simplify considerably. Indeed, putting in (33)

\[
\psi_{qs} = \psi_{sa} \exp \left(-\frac{a_s + a_q}{2a}$ $\lambda_{sa} \right) \psi_{aq},
\]

and neglecting terms $\sim \partial \varphi_{\alpha}/\partial t$ we get for $\varphi_{\alpha}$:

\[
-\frac{\partial}{\partial t} \psi_{aq} = \frac{\partial}{\partial t} \varphi_{\alpha} = \frac{1}{2} \frac{a_a - a_q}{a_a} \varphi_{aq},
\]

where

\[
V_{aq} = \binom{a_s - a_q}{a_s} \varphi_{aq}, \quad \xi = \frac{1}{2} \left( \frac{a_a - a_q}{a_a} \right).
\]

The potential $V_{\alpha}(\xi)$ in the one-dimensional Schrödinger equation (41) satisfies then in the case of strong packets the quasi-classical condition $V_{\alpha}(\xi) \gg 1$ and the quantities $\eta_l(\lambda)$ turn out to be exponentially small in the parameter $V_{\alpha}(\xi)$. From the point of view of the inverse scattering method such potentials are completely determined by the zeroes of the $\sigma_2^3(A)$, i.e., the $\lambda_l^{(0)}$ and the corresponding quantities $\eta_l(\lambda_l^{(0)})$. The zeroes of the $\sigma_2^3$ lie then at the points $\lambda_l^{(0)} = \left( \frac{a_q}{a_q} \varphi_{aq} [a_a - a_q] \right)$ where the $-\xi^2$ are the eigenvalues of (41) which are determined by the quasi-classical quantization rule:

\[
\int \frac{V_{aq}(\xi)}{\xi^2} \sim 2 \pi n + \xi^2 \, dx = 2 \pi n (n + 1/2).
\]

Their total number $N \sim V_{\alpha}(\xi)$ is large. Therefore, in the general case of the collision of large packets the intensities of the waves which occur are described by Eqs. (40) in which we must neglect the contributions from the continuous spectrum.

We note also that in the case of a collision of strong secondary wave packets there occurs, as a rule, no pumping wave in the asymptotic behavior as $t \to \infty$, as the coincidence of the zeroes of $\sigma_1^2(3)$ and $\sigma_2^3(3)$ is a very improbable event. By a special "preparation" of the packets 2 and 3 one can, however, achieve the coincidence of the zeroes of $\sigma_1^2(3)$ and $\sigma_2^3(3)$. If, in particular,

\[
u_l^s(3) \sim \nu_l^s(3) = \nu_l^s(3) \sim \nu_l^s(3),
\]

one sees easily that the partial transition matrices $s_1^2$ and $s_1^3$ are exactly the same. If the packets $u_1^2$ and $u_1^3$ satisfy the quasi-classical condition, there then occurs a practically complete fusion of the secondary waves into a pumping wave. However, as the quasi-classical condition is very sensitive to small changes in $V_{\alpha}(\lambda)$, the appearance of a strong pumping wave occurs clearly only in an intermediate stage; asymptotically as $t \to \infty$ it decays. A similar situation occurs if some of the zeroes of $\sigma_1^2(3)$ and $\sigma_2^3(3)$ are very close to one another. In that case we can state that the scattering of the secondary wave packets takes place in stages. As a result of the interaction in the first stage there appear three packets with distances between them which can be large. The intensity of the pumping wave is then

\[
\int |u_l^s(x,t)|^2 \, dx \approx \frac{1}{2\pi} \sum \Im \lambda_l^{(0)}
\]

where the summation is over the zeroes of $\sigma_1^2(3)$ which are close to zeroes of $\sigma_2^3(3)$ while the amplitudes of the packets 2 and 3 are small, but nevertheless non-zero, in the region where the pumping packet is localized. The time during which such an intermediate state exists may be very long. Finally, in the last stage the pumping packet decays practically completely. Simple estimates show that the lifetime of the intermediate state depends on the distance $|\Delta \lambda |$ between the close zeroes of $\sigma_1^2(3)$ and $\sigma_2^3(3)$ as $\ln (1/|\Delta \lambda |)$.

A similar effect of the formation of an "intermediate asymptotic behavior" also occurs in the case where a pumping packet collides with secondary wave packets, if the quantity $\epsilon(\lambda) = \sigma_2^3(3) \sigma_1^2(3) + \sigma_2^3(3) \sigma_1^2(3)$ is small at the zero of $\sigma_2^3(3)$. We shall show below that, if $\epsilon = 0$, a zero of $\sigma_2^3(3)$ does not lead to zeroes of $\sigma_2^3(3)$ or $\sigma_2^3(3)$, but to a zero of $\sigma_1^2(3)$ which, according to (40), means the conservation of the corresponding contribution to the pumping intensity as $t \to \infty$. One understands easily, however, that the case of a small $\epsilon$ means the occurrence of an intermediate state of the kind just described. The lifetime of such a state depends logarithmically on $|\epsilon |$.

Equations (40) and (37) show also that when "small" packets, for which the $\sigma_1^2(3)$ do not have zeroes, scatter, the integral intensities of the waves are of the same order as $t \to \infty$. For instance, when small secondary wave packets collide, a pumping wave is produced with intensity

\[
\int |u_l^s(x,t)|^2 \, dx \approx \frac{1}{2\pi} \Im \lambda_l^{(0)}
\]

We emphasize, however, that the absence of zeroes of the $\sigma_1^2(3)$ does not at all mean that the $u_1^s(3)$ are small. The amplitude of the packet (and its integral intensity) can then be arbitrarily large, provided the spectral width of the packet is sufficiently large. However, in this case a strong (but in $k$-space sufficiently wide) pumping packet when colliding with small (in the sense of having a small amplitude) secondary wave packets does practically not decay and even suffers little change in its shape. Only such a situation can be considered in the approximation of a given pumping field. Indeed, one can see from (37) that the partial transition matrix $s_1^2(3)$ is close to $s_1^3(3)$ provided $\sigma_1^2(3)$ has no zeroes and $|\eta_l(3)| \eta_l(3) \ll |\eta_l(3)|$. This is, strictly speaking, also the criterion for the applicability for the approximation of a given pumping field in the case when the pumping velocity is neither the largest nor the smallest.
We summarize what has been said. A sufficiently strong pumping wave which satisfies the quasi-classical condition decays completely when it collides with any secondary wave packets. The interaction process can then be accompanied by the appearance of "intermediate asymptotic behavior" and the residual pumping intensity decreases when its amplitude in the "input" increases.

We studied the scattering of packets in the case when the pumping velocity is the intermediate one. In most physical situations the group velocity of the pumping wave is larger (in absolute magnitude) than the velocities of the secondary waves which appear when it decays. It is clear from Eq. (10) that this is possible only when $a_3 < 0$ (if $a_1 > a_3 > 0$). To fix the ideas we put $v_1 > v_3 > v_2$. For such velocity ratios we must write instead of (32)

$$S^{(+)}_1 S^{(-)}_1 S^{(+)}_1 S^{(-)}_1 S^{(+)}_1 S^{(-)}_1 = S(0).$$

As above, we can easily find $S^+_1$ for given $S^-_1$ after which we can without difficulty get explicit expressions for the integral intensities of the packets which are formed. However, we must then bear in mind that as $a_3 < 0$, the regions of analyticity of the diagonal elements of the $S^+_1$ are, in general, not the same as in the case considered above.

We introduce the following notation for the elements of the partial transition matrices:

$$s^{+}_1 = \begin{pmatrix} \sigma^+ & 0 \\ 0 & \eta_1^{+} \end{pmatrix}, s^{+}_2 = \begin{pmatrix} \sigma^+ & 0 \\ 0 & \eta_2^{+} \end{pmatrix}, s^{+}_3 = \begin{pmatrix} \sigma^+ & 0 \\ 0 & \eta_3^{+} \end{pmatrix}$$

(they are connected with the $S^+_1$ by Eqs. (35)). The functions $\sigma^+_i(\lambda)$ defined in this way are analytical in the upper half-plane while, in view of the unimodularity of the $s^+_i$, $\sigma^+_1$ and $\sigma^+_2$ do not have zeroes there. However, one sees easily from (42) that the zeroes of $\sigma^+_3$ are the same as the zeroes of $\sigma^+_i(\lambda)$.

Performing the appropriate calculations we find

$$|\sigma^+_i|^2 = |\sigma^+_1|^2 = |\sigma^+_2|^2 = |\sigma^+_3|^2 = |\sigma^+_1|^2 = |\sigma^+_2|^2 = |\sigma^+_3|^2 = |F^+|^2 = |F^-|^2 = 1,$$

$$|\sigma^+_i|^2 = |\sigma^+_1|^2 = |\sigma^+_2|^2 = |\sigma^+_3|^2 = |F^+|^2 = |F^-|^2 = 1.$$

Equation (42) enables us to evaluate also all other elements of the partial transition matrices; we do not give the expressions for them.

The integral intensities of the packets which are formed can be expressed directly in terms of the $\sigma^+_i(\lambda)$. For $i = 2$ we have, as before, (40) and for $i = 1, 3$ we have

$$\int |u_1(\lambda, \lambda, t)|^2 dx = \frac{1}{2\eta^+_1} \frac{\int |\sigma^+_1|^2 d\lambda}{\int |\sigma^+_1|^2 d\lambda}.$$  

We consider concrete situations. A small change in the pumping intensity (and in its shape) occurs also when the pumping wave collides with small secondary wave packets. The expressions for the amplitudes of these waves can then be obtained in the approximation of a given pumping field.

A nontrivial dynamics occurs when strong packets of the pumping wave and of wave 3 collide. Assuming that the conditions $q_1 v_1 > v_3 > 1$, $q_2 v_2 > v_3 > 1$ are satisfied, we are again led from problem (33) to (41); however, now the potential $V_{\Omega_3}(\lambda)$ in Eq. (41) is positive. Using the well-known formulae of the quasi-classical approximation we easily find expressions for $\sigma^+_i(\lambda)$ and $\sigma^+_3(\lambda)$. Substituting then the answers obtained into (43) and (44) we see that if

$$\max |u_1(\lambda, t)| > \frac{a_2}{a_1 - a_3} \max |u_3(\lambda, t)|,$$

the integral $\int |u_1(\lambda, t)|^2 dx$ is exponentially small, i.e., the pumping wave decays almost completely. (45) is thus the condition for the annihilation of the pumping wave when strong packets of waves 1 and 3 collide. Using Eq. (10) to eliminate the quantities $a_1$ from (45) we find that the condition for the total decay of the pumping wave is

$$\max |u_1(\lambda, t)| = \frac{|v_1 - v_3|}{|v_1 - v_2|} \max |u_3(\lambda, t)|.$$

If, however, this condition is not satisfied, we have

$$\int |u_1(\lambda, t)|^2 dx = \int \left[ 1 - \frac{2}{\pi^2} \ln \frac{|v_1 - v_3|}{|v_1 - v_2|} \right] \frac{d\lambda}{\sigma^+_1(\lambda)}.$$

The integration in (46) is over the region where $|u_1(\lambda)| \geq \max |u_3(\lambda)|$. The expression given here is valid provided one maximum of $|u_1(\lambda)|$ is larger than $\max \gamma |u_3(\lambda)|$. The intensities of the secondary waves which are formed as a result of the collision are obtained from (46) and the Manley-Rowe relations.

We note the following facts. It follows from (46) that the integral intensities of the waves which occur are independent of the shape of the packet of wave 3 and are determined solely by the maximum value of its amplitude. The result of the interaction is also independent of the magnitude of the interaction constant $q$.

We note, finally, that (48) does not contain the phases of the colliding packets which is the result of assuming that the functions $Q_{\Omega_2}(\lambda) = 0$ to be smooth, as we did when deriving Eq. (41): for this it is necessary that the phases of the packets vary sufficiently slowly with $x$. 

**APPENDIX (JULY 10, 1975)**

**THE EQUATIONS OF THE INVERSE PROBLEM ($a_3 > 0$)**

We give here the equations of the inverse problem for the systems considered; through lack of space we omit their derivation which can be found in [18, 26].

When studying the inverse problem a fundamental role is played by the analytical properties of the solutions of the direct spectral problem. One can show that the functions $\phi^{(+)}(\lambda, \lambda)$ and $\phi^{(-)}(\lambda, \lambda)$, where by definition

$$\phi^{(+)}(\lambda, \lambda) = \frac{S^{(+)}_2(\lambda)}{S^{(+)}_1(\lambda)} \phi^{(+)}(\lambda, \lambda),$$

for real $\lambda$ allow an analytical continuation into the...
upper $\lambda$-half-plane and the functions $\psi_{13}(x, \lambda)$ and

\[
\psi_{13}^{-1}(x, \lambda) - \psi_{12}^{-1}(x, \lambda) = \frac{S_{11}(\lambda)}{S_{10}(\lambda)} \psi_{21}(x, \lambda)
\]

(for real $\lambda$) one into the lower one; however, the functions $\psi_{12}(x, \lambda)$ have then simple poles, respectively, at the zeroes of $S_{11}(\lambda)$ and $S_{13}(\lambda)$ if the latter zeroes are simple ones (as we shall assume in what follows; we shall also assume that none of the zeroes of $S_{11}(\lambda)$ and $S_{13}(\lambda)$ are coincident). The functions enumerated here can be written in the form (cf. (15) to (18))

\[
\psi_{n}^{-1}(x, \lambda) \exp \left( i \frac{\lambda}{\alpha_{1}} x \right) = \delta_{n1} + \sum_{n=2}^{\infty} \frac{G_{n}}{a_{n}} \sum_{k=1}^{\infty} \alpha_{k} \psi_{k}^{(n)}(x, \lambda) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right)
\]

(A.1)

\[
\psi_{n}^{-1}(x, \lambda) \exp \left( i \frac{\lambda}{\alpha_{1}} x \right) = \delta_{n1} + \sum_{n=2}^{\infty} \frac{G_{n}}{a_{n}} \sum_{k=1}^{\infty} \alpha_{k} \psi_{k}^{(n)}(x, \lambda) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right)
\]

(A.2)

These expressions allow us in an obvious way to obtain a closed set of equations for the functions $\psi_{13}(x, \lambda)$, $\psi_{12}(x, \lambda)$, $\psi_{11}(x, \mu_{0})$, $\psi_{12}(x, \mu_{0})$, and $\psi_{13}(x, \lambda^{*})$ which is just the set of equations of the inverse problem. The potential $Q_{13}(x)$ is determined by the solutions of the latter through the formula

\[
Q_{13}(x) = - \frac{i}{2\pi a_{1}} \sum_{n=1}^{\infty} \frac{S_{1n}}{S_{10}} \psi_{n}^{*}(x, \lambda) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right) \delta_{n1} + \sum_{n=2}^{\infty} \frac{G_{n}}{a_{n}} \sum_{k=1}^{\infty} \alpha_{k} \psi_{k}^{(n)}(x, \mu_{0}) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right)
\]

\[
Q_{12}(x) = - \frac{i}{2\pi a_{1}} \sum_{n=1}^{\infty} \frac{S_{1n}}{S_{10}} \psi_{n}^{*}(x, \lambda) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right) \delta_{n1} + \sum_{n=2}^{\infty} \frac{G_{n}}{a_{n}} \sum_{k=1}^{\infty} \alpha_{k} \psi_{k}^{(n)}(x, \mu_{0}) \exp \left( \frac{i \mu_{n}}{a_{1}} x \right)
\]

Translated by D. ter Haar