THE HAMILTONIAN FORMALISM FOR WAVES IN NONLINEAR MEDIA HAVING DISPERSION*

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UDC 530.18

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1. The Hamiltonian Formalism

In studying wave phenomena in various nonlinear media their similarity is striking. Such processes as, for example, parametric instabilities, self-focusing, and generation of higher harmonics may occur for the passage of powerful light pulses through transparent dielectrics, for propagation of finite-amplitude waves in a plasma, and for propagation of spin waves in ferromagnets or gravitational waves along the surface of a fluid.

Therefore a compelling necessity arises for formulating a general theory of waves in nonlinear media which could treat all of these processes from a unified point of view while avoiding the specifics of the medium. A sample of such a theory may be found in the classical mechanics formulated in the language of canonical variables. The reference to classical mechanics also delineates the framework within which one should construct the general theory of nonlinear waves; it is clear that one may hope for serious success only by assuming that the dissipative effects are small (absent entirely in the first approximation) – i.e., one should limit the analysis to conservative systems.

From the fundamental point of view the formulation of a general theory of waves in nonlinear conservative media is no more than the generalization of classical mechanics for the case of systems with a continuous number of degrees of freedom.

Such a generalization is made in classical field theory which preceded, for example, quantum electrodynamics. Under these conditions it is usually assumed that the field has a local Lagrangian density which depends on a finite number of derivatives with respect to the space variables. Attempts at formulating a general theory of waves in nonlinear media according to the prototype of classical field theory have been made recently by Whigham [1], Lighthill [2], and a series of other authors (viz., for example, [3]).

However, the theory starting from a local Lagrangian does not have a degree of generality sufficient to encompass the examples given above. In the majority of practical cases the Lagrangian is nonlocal (or it is local but requires fulfillment of additional conditions which greatly complicate the problem). Therefore it is reasonable to start from the most general of the formalisms of classical mechanics – the Hamiltonian formalism.

*Here we present the contents of just the first part of the lecture course under the general heading "The Hamiltonian Formalism and Exact Methods in the Physics of Waves in Nonlinear Media Having Dispersion." The exposition of the second part, which is devoted to exact methods, will be published later in the form of a separate survey.

Novosibirsk State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 17, No. 4, pp. 431-453, April, 1974.

The present paper contains a consistent (and therefore to a considerable degree trivial) carryover of the ideas of classical Hamiltonian mechanics to the case of a continuous number of degrees of freedom and a translationally-invariant Hamiltonian in the general form.

The most difficult feature in all of this procedure is to write the equations for the various specific media in the canonical variables (Section 5) – these variables are often related in a very bizarre way to the "natural" physical variables which describe the nonlinear medium.

As the first example of the use of the Hamiltonian formalism for continuous media let us consider the equation for potential flow of an ideal compressible fluid in which the pressure is a unique function of the density $p(\rho)$. These equations may be written in the form

$$\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + w(\rho) = 0, \quad \rho_t + \operatorname{div}\rho\nabla\Phi = 0. \tag{1}$$

Here Φ is the velocity potential; $w(\rho) = \int(1/\rho)(dp/d\rho)d$/is the specific energy of the fluid. The system of Eqs. (1) conserves the energy of the fluid

$$H = \int\left(\frac{1}{2}\rho(\nabla\Phi)^2 + \varepsilon(\rho)\right)dr, \quad \varepsilon(\rho) = \int w(\rho)\,d\rho. \tag{2}$$

It is not difficult to check the fact that Eqs. (1) may be written in the form

$$\frac{\partial\rho}{\partial t} = \frac{\delta H}{\delta\Phi}, \quad \frac{\partial\Phi}{\partial t} = -\frac{\delta H}{\delta\rho}. \tag{3}$$

Here the symbols $\delta/\delta\Phi$ and $\delta/\delta\rho$ denote variation of derivatives.

Equations (3) constitute a direct analog of the Hamiltonian equations of classical mechanics – a pair of canonically conjugate quantities (the generalized coordinate $\rho(\mathbf{r}, t)$ and the generalized momentum $\Phi(\mathbf{r}, t)$) are stipulated at each point in space.

A direct generalization of the considered example is the case in which the medium can be described by two functions of the coordinates and time – the generalized coordinates $q(\mathbf{r}, t)$ and the generalized momentum $p(\mathbf{r}, t)$ which are governed by the equations

$$\frac{\partial q}{\partial t} = \frac{\delta H}{\delta p}, \quad \frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}, \tag{4}$$

where H is a certain functional of $p(\mathbf{r}, t)$, $q(\mathbf{r}, t)$ (usually it has the meaning of the energy of the medium). Equations (4) in general are not differential in the space coordinate.

Let us consider the expansion of the Hamiltonian H in powers of p and q. If the medium is not subject to external forces, then this expansion begins with terms that are quadratic in p, q:

$$H = H_0 + H_1 + \ldots .$$

In a spatially homogeneous medium the first term of the expansion has the form

$$H_0 = \frac{1}{2}\int\{A(r-r')p_r p_{r'} + 2B(r-r')p_r q_{r'} + C(r-r') \times q_r q_{r'}\}\,dr\,dr', \tag{5}$$

where $A(\mathbf{r})$, $B(\mathbf{r})$ and $C(\mathbf{r})$ are certain structural functions. It is obvious that $A(\mathbf{r}) = A(-\mathbf{r})$, $C(\mathbf{r}) = C(-\mathbf{r})$. Performing a Fourier transformation in the coordinates

$$p_r = \frac{1}{(2\pi)^{3/2}}\int p_k e^{ikr}\,dk, \quad p_{-k} = p_k^*,$$

$$q_r = \frac{1}{(2\pi)^{3/2}}\int q_r e^{ikr}\,dk, \quad q_{-k} = q_k^*,$$

we reduce H_0 to the form

$$H_0 = \frac{1}{2}\int\{A_k p_k p_k^* + 2B_k p_k q_k^* + C_k q_k q_k^*\}\,dk. \tag{6}$$

Here $A_k = A(\mathbf{r})e^{ikr}dr$, and B_k and C_k are defined analogously. We have

$$\begin{gathered} B_k = B_{1k} + iB_{2k}, \\ A_{-k} = A_k = A_k^*, \quad C_{-k} = C_k = C_k^*, \\ B_{-k} = B_{-k}^*, \quad B_{1-k} = B_{1k}, \quad B_{2-k} = -B_{2k}. \end{gathered} \tag{7}$$

The Hamiltonian Eqs. (4) in the variables p_k, q_k have the form

$$\frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*}, \quad \frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*}. \tag{8}$$

Substituting $H = H_0$ from Eq. (5) into (8), we have

$$\frac{\partial p_k}{\partial t} = -C_k q_k - B_k p_k, \quad \frac{\partial q_k}{\partial t} = B_k^* q_k + A_k p_k. \tag{9}$$

Assuming p, q $\sim e^{\lambda t}$ in the system (9) we find

$$\lambda_{1,2} = i\left\{-B_{2k} \pm \sqrt{A_k C_k - B_{1k}^2}\right\}$$

for λ.

The medium is stable relative to buildup of small perturbations if

$$A_k C_k > B_{1k}^2. \tag{10}$$

The stability condition (10) will henceforth be assumed fulfilled, whence it follows that A_k and C_k have identical signs. If the medium is invariant relative to reflection of the coordinates, then $B(-\mathbf{r}) = B(\mathbf{r})$ and $B_{2k} = 0$. In such a medium $-\lambda^2 = A_k C_k - B_{1k}^2$. From the variables p_k and q_k we perform a conversion to the new variables a_k, a_k^* according to the formulas

$$\begin{aligned} a_k &= \alpha_k p_k + \beta_k q_k, \quad \alpha_{-k} = \alpha_k, \\ a_k^* &= \alpha_k^* p_k^* + \beta_k^* q_k^*, \quad \beta_{-k} = \beta_k \end{aligned} \tag{11}$$

and require fulfillment of the condition

$$\alpha_k \beta_k^* - \alpha_k^* \beta_k = i. \tag{12}$$

Condition (12) leads to a situation in which the equations for a_k have the form

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0. \tag{13}$$

We further require that in the new variables the Hamiltonian H_0 have the form

$$H_0 = \int \omega_k a_k a_k^* dk, \tag{13a}$$

where $-i\omega_k$ is one of the functions $\lambda_{1,2}$. Expressing p_k and q_k from (11), we make use of the condition (11) to obtain the following result after performing substitution into (6):

$$\begin{gathered} |\alpha_k|^2 = \frac{A_k}{2\omega_0(k)}, \quad |\beta_k|^2 = \frac{C_k}{2\omega_0(k)}, \\ \alpha_k \beta_k^* + \alpha_k^* \beta_k = \frac{B_{1k}}{\omega_0(k)}, \end{gathered} \tag{14}$$

where $\omega_0(k) = \sqrt{A_k C_k - B_{1k}^2}$; the sign in front of the radical must coincide with the sign of A_k. The general solution of the system of Eqs. (14) has the form

$$\alpha_k = \sqrt{\frac{A_k}{2\omega_0(k)}} \exp(i\Phi_k), \quad \beta_k = \sqrt{\frac{1}{2A_k \omega_0(k)}} (B_{1k} + i\omega_0(k)) \exp(i\Phi_k),$$

where Φ_k is an arbitrary phase multiplier. Under these conditions

$$\omega_k = -B_2(k) + (\operatorname{sign} A_k)\sqrt{A_k C_k - B_{1k}^2}. \tag{15}$$

The sign of ω_k determined from Eq. (15) (the wave dispersion law) coincides with the sign of the energy of the waves in a nonlinear medium.

Assuming for simplicity that $\Phi_k = 0$, we express p_k and q_k in terms of a_k, a_k^*:

$$\begin{gathered} p_k = -i\sqrt{\frac{1}{2A_k \omega_0(k)}}\{(B_{ik} - i\omega_0(k))a_k - (B_{1k} + i\omega_0(k))a_k^*\}, \\ q_k = -\sqrt{\frac{A_k}{2\omega_0(k)}}(a_k - a_{-k}^*). \end{gathered} \tag{16}$$

Let us substitute (16) into the cubic term of the expansion of the Hamiltonian H in powers of a_k and a_k^*. We have

$$H_1 = \int \{ V_{k,k_1k_2} a_k^* a_{k_1} a_{k_2} + V_{k,k_1k_2}^* a_k a_{k_1}^* a_{k_2}^* \} \delta_{k-k_1-k_2} \prod_i dk_i$$

$$+ \frac{1}{3} \int \{ U_{kk_1k_2} a_k a_{k_1} a_{k_2} + U_{kk_1k_2}^* a_k^* a_{k_1}^* a_{k_2}^* \} \delta_{k+k_1+k_2} \prod_i dk_i + \dots . \tag{17}$$

The functions V and U have symmetry properties:

$$V_{k,k_1k_2} = V_{k,k_2k_1}, \quad U_{kk_1k_2} = U_{kk_2k_1} = U_{k_1kk_2}. \tag{18}$$

In the case when the medium can be described by several pairs of canonical variables the problem of diagonalizing the quadratic part of the Hamiltonian is less trivial.

This problem may be solved if, for example, H_0 has the form

$$H_0 = \sum \int \{ V_{ij}(r-r')p_i(r)p_j(r') + Q_{ij}(r-r')q_j(r)q_j(r') \} \, dr \, dr'$$

and one of the matrices V_{ij} and Q_{ij} is positive-definite. In this case the diagonalization of the Hamiltonian is equivalent to the problem of finding the normal variables in an oscillatory system with N degrees of freedom. In such a medium there are N wave modes having the dispersion laws $\omega_i(\mathbf{k})$ and the amplitudes $a_i(\mathbf{k})$.

All of the preceding reasoning has rested on the assumption that the equations of the medium are written in canonical variables. As a rule, the "natural" physical variables in which the equations of continuous media are written (the components of the velocities, displacements, electric and magnetic fields) are not canonical. Note, however, that the equations of a medium that allow the introduction of canonical variables must (if they are equations of the first order in time) have the following form in the arbitrary coordinates $A_n(\mathbf{r}, t)$:

$$\sum_m \int G_{n,m}(r, r') \frac{\partial A_m(r')}{\partial t} dr' = \frac{\delta H}{\delta A_n(r)}, \tag{19}$$

where $G_{n,m}(\mathbf{r}, \mathbf{r}')$ is the kernel of a nondegenerate linear operator (with coefficients that in general depend on $A_n(\mathbf{r})$) which satisfies the conditions

$$G_{n,m}(r, r') = -G_{m,m}(r, r') \quad \text{(the asymmetry condition);}$$

$$G_{n,m}(r, r') = \frac{\delta p_n(r)}{\delta A_m(r')} - \frac{\delta p_m(r')}{\delta A_n(r)} \quad \text{(the closure condition).}$$

Under these conditions the functional H is a Hamiltonian and is conserved, while $G_{n,m}(\mathbf{r}, \mathbf{r}')$ is governed by the relationship

$$\frac{\delta G_{n,m}(r, r')}{\delta A_k(r'')} + \frac{\delta G_{k,n}(r'' r)}{\delta A_m(r')} + \frac{\delta G_{m,k}(r' r'')}{\delta A_n(r)} = 0.$$

Equations (19) minimize the action

$$S = \sum_n \int dt + dr \{ p_n(r) \dot{A}_n(r) \} + \int H dt, \tag{20}$$

which constitutes a linear function of the first derivative of the coordinates with respect to time. It may be proved that if the medium allows the variational principle and the action depends solely on the finite order of the time derivatives of the variables describing the medium, then canonical variables may also be introduced in it [4].

2. Canonical Transformations

Let us consider the transformation from the variables a_k to the new variables b_k, and let us require that the equations of these variables have the form (13) with the same Hamiltonian H. It is easy to check the fact that for this purpose the desired transformations must satisfy the following conditions:

$$\int \left\{ \frac{\delta a_k}{\delta b_{k'}} \frac{\delta a_{k''}}{\delta b_{k'}^*} - \frac{\delta a_k}{\delta b_{k'}^*} \frac{\delta a_{k''}}{\delta b_{k'}} \right\} dk' = 0,$$

$$\int \left\{ \frac{\delta a_k}{\delta b_{k'}} \frac{\delta a_{k''}^*}{\delta b_{k'}^*} - \frac{\delta a_k}{\delta b_{k'}^*} \frac{\delta a_{k''}^*}{\delta b_k} \right\} dk' = \delta_{k-k'}. \tag{21}$$

Conditions (21) are the conditions for canonicity of the Poisson brackets which ensure canonicity of the transformations. Let us represent the canonical transformation from a_k to b_k in the form of an integro-power series

$$a_k = b_k + \int \{ V^{(1)}_{k k_1 k_2} b_{k_1} b_{k_2} + V^{(2)}_{k k_1 k_2} b_{k_1} b^*_{k_2} + V^{(3)}_{k k_1 k_2} b^*_{k_1} b^*_{k_2} \} dk_1 dk_2 + \int \{ W^{(1)}_{k k_1 k_2 k_3} b_k b_{k_1} b_{k_2} b_{k_3} + \ldots + W^{(4)}_{k k_1 k_2 k_3} b^*_{k_1} b^*_{k_2} b^*_{k_3} \} dk_1 dk_2 dk_3. \tag{22}$$

Equations (21) impose specific constraints on the coefficients of the series (22). Thus, the functions $V^{(1)}$, $V^{(2)}$, $V^{(3)}$ must satisfy the conditions

$$V^{(2)}_{k, k'', k_2} = -2 V^{*(1)}_{k'', k k_2}, \quad V^{(3)}_{k k_1 k_2} = V^{(3)}_{k_1 k_2 k} = V^{(3)}_{k k_2 k_1}, \tag{23}$$

while the functions $W^{(i)}_{kk_1k_2k_3}$ must satisfy the conditions

$$3 W^{(1)}_{k k'', k_1 k_2} + 4 \int \left(V^{*(1)}_{k_1, k'' k'} V^{*(3)}_{k, k' k_2} - V^{(1)}_{k'' k' k_1} V^{(1)}_{k' k k_2} \right) dk' = W^{*(3)}_{k'' k, k_1 k_2},$$

$$W^{(2)}_{k k'', k_1 k_2} + 2 \int \{ V^{(1)}_{k'' k' k_2} V^{(1)*}_{k k' k_2} + V^{(1)}_{k' k k_1} V^{*(1)}_{k' k'' k_2} - V^{*(1)}_{k, k'' k'} V^{(1)}_{k, k_1, k'} - V^{(3)}_{k'' k' k_2} V^{(3)*}_{k k' k_1} \} dk' = W^{*(2)}_{k'' k k_2 k_1}, \tag{24}$$

$$W^{(4)}_{kk'', k_1 k_2} = W^{(4)}_{k'' k k_1 k_2} + 4 \int \left(V^{(3)}_{k'' k' k_1} V^{*(1)}_{k' k k_2} - V^{(3)}_{k k' k_1} V^{*(1)}_{k' k'' k_2} \right) dk'.$$

The canonical transformations written in the form of the series (22) allow simplification of the wave-interaction Hamiltonians by excluding "nonessential" terms from them. Thus, the transformation

$$V^{(1)} = V^{(2)} = 0, \quad V^{(3)}_{k k_1 k_2} = -\frac{1}{3} \frac{U^*_{kk_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}} \delta_{k+k_1+k_2} \tag{25}$$

excludes the second two terms in the Hamiltonian (17), whereas the transformation

$$V^{(1)}_{k k_1 k_2} = \frac{V_{k k_1 k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \delta_{k-k_1-k_2} \tag{26}$$

excludes the first pair of terms in the Hamiltonian (17). The procedure of successive exclusion of terms of the Hamiltonian by means of canonical transformations is called the classical perturbation theory (viz., for example, [5]). A characteristic difficulty of the classical perturbation theory and problems with a finite number of degrees of freedom is the appearance of "small denominators." In the physics of nonlinear waves this difficulty is manifested in the form of the appearance of nonintegrable singularities in the coefficients of the canonical transformations. Thus, for fulfillment of the conditions

$$k + k_1 + k_2 = 0, \quad \omega_k + \omega_{k_1} + \omega_{k_2} = 0 \tag{27}$$

a singularity arises in the coefficient $V^{(3)}_{kk_1k_2}$, while when the conditions

$$k = k_1 + k_2, \quad \omega_k = \omega_{k_1} + \omega_{k_2} \tag{28}$$

are fulfilled a singularity appears in the coefficient $V^{(1)}_{kk_1k_2}$.

Fulfillment of conditions (27) is possible only in a medium where ω_k changes sign (i.e., if waves with negative energy may exist in the medium). In media which allow only waves with a positive energy a Hamiltonian of the type $\int U a^* a^* a^* + \ldots$ may always be excluded and in this sense is "nonessential."

The possibility of the existence of solutions of the system (28) depends on the form of the function ω_k. If $\omega_0 = 0$ and $\omega'_k > 0$, then the system (28) has solutions for $\omega''_k > 0$ and has no solutions if $\omega''_k < 0$; for $\omega_0 \neq 0$ this problem is more complicated. If the system (28) has solutions, then the principal term of the wave-interaction Hamiltonian has the form

$$H_{\text{int}} = \int \{ V_{k k_1 k_2} a^*_k a_{k_1} a_{k_2} + V^*_{k k_1 k_2} a_k a^*_{k_1} a^*_{k_2} \} \delta_{k-k_1-k_2} dk\, dk_1 dk_2. \tag{29}$$

Under these conditions the function $V_{kk_1k_2}$ is rigorously definite only on the surface (28); off this surface it may be changed by an appropriate transformation of the form (22) which adds the term $-(\omega_k - \omega_{k_1} - \omega_{k_2}) \cdot V^{(1)}_{kk_1k_2}$ to $V_{kk_1k_2}$. The form of $V^{(1)}_{kk_1k_2}$ may be chosen on the basis of convenience considerations.

If the system (27) has no solutions, then the cubic terms may be excluded from the interaction Hamiltonian.

Under these conditions, however, fourth order terms appear among which likewise not all of them are essential. The Hamiltonian

$$H_{\text{int}} = \frac{1}{2} \int T_{k k_1 k_2 k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk\, dk_1\, dk_2\, dk_3 \tag{30}$$

is essential. In an attempt to exclude this Hamiltonian a singularity develops which is concentrated on the manifold defined by the system of equations

$$\begin{gathered} k + k_1 = k_2 + k_3, \\ \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}. \end{gathered} \tag{31}$$

The system (30) has solutions regardless of dependence on the form of $T_{kk_1k_1k_3}$.

Let us represent an explicit expression for $T_{kk_1k_2k_3}$, which is caused by the cubic terms in the Hamiltonian:

$$\begin{aligned} T_{k k_1 k_2 k_3} = & -2 \frac{U_{-k_2-k_3, k_2, k_3} U^*_{-k-k_1, k k_1}}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} - 2 \frac{V^*_{k_2+k_3, k_2 k_3} V_{k+k_1, k, k_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} \\ & -2 \frac{V_{k k_2, k-k_2} V^*_{k_1 k_1, k_3-k_1}}{\omega_{k_3-k_1} + \omega_{k_1} - \omega_{k_3}} - 2 \frac{V_{k_1 k_3, k_1-k_3} V^*_{k_2 k, k_2-k}}{\omega_{k_2-k} + \omega_k - \omega_{k_2}} \\ & -2 \frac{V_{k_1 k_2, k_1-k_2} V^*_{k_3 k, k_3-k}}{\omega_{k_3-k} + \omega_k - \omega_{k_3}} - 2 \frac{V_{k k_3, k-k_3} V_{k_2 k_1, k_2-k_1}}{\omega_{k_2-k_1} + \omega_{k_1} - \omega_{k_2}} \end{aligned} \tag{32}$$

for $\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}$. In deriving Eq. (30) it was assumed that the waves have a low amplitude

$$|T|\,|a|^2 \ll \omega_k' k. \tag{33}$$

The use of canonical transformations allows the entire diversity of Hamiltonians for nonlinear media to be reduced to a small number of "standard" or "essential" Hamiltonians. Thus, in a medium with waves having positive energy the standard interaction Hamiltonian has the form (29) if the law for the dispersion of ω_k has decay (i.e., processes of the type (28) are allowed, or the form (30) holds if these processes are forbidden). Under these conditions the coefficients of the functions V and T, which enter into the Hamiltonian, are rigorously definite only on the "resonance surfaces" (28) and (31), and may be chosen arbitrarily far away from these surfaces. Analogous standard forms of the Hamiltonian may also exist for other more complex cases. Let us present the standard Hamiltonian of the medium for the important problem of the interaction of high-frequency waves having an amplitude a_k with low-frequency waves having an amplitude b_k [6]:

$$H_{\text{int}} = \int \{ V_{k k_1 k_2} b_k a_{k_1} a_{k_2}^* + V^*_{k k_1 k_2} b_k^* a_{k_1}^* a_{k_2} \} \delta_{k+k_1-k_2} dk\, dk_1\, dk_2. \tag{34}$$

The Hamiltonian of the type (34) describes, for example, interaction of light and sound in dielectrics, the interaction of Langmuir and ion-sound waves in a plasma, of spin and acoustic waves in a ferromagnet, etc.

3. The Instability of Stationary Low-Amplitude Waves

The calculation of the first several coefficients in the expansion of the Hamiltonian H of the medium in the powers of the canonical variables a_k and a_k^* automatically allows the solution of a series of important problems associated with the nonlinear interaction of waves. First of all, this applies to the problem of the stability of low-amplitude stationary waves. Waves are usually called stationary in those particular cases of motion of a nonlinear medium when all quantities describing the medium depend solely on the combination x−vt. Under these conditions it is obvious that $a_k(t) = c(k)e^{-ikvt}$.

The equations of motion of the medium in the linear approximation has the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = 0,$$

for a stationary wave they yield

$$(\omega_k - kv) a_k = 0. \tag{35}$$

If the dispersion law is not linear, $\omega_k \neq ck$, then $c_k = a\delta(k-k_0)$, is the sole solution of Eq. (35); here k_0 is the zero of the expression $w_k - kv$. In a linear medium with dispersion all stationary quantities are monochromatic.

For a fairly low amplitude a this is also valid in a nonlinear medium. Since neither Eqs. (27) nor Eqs. (28) for $\omega_k \neq ck$ can be satisfied if two of the vectors (k, k_1, k_2) are equal to the same vector k_0, one should use the Hamiltonian (30) to describe a monochromatic wave.

The equations for the medium within the framework of this Hamiltonian have the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k + i\int T_{kk_1,k_2k_3} a^*_{k_1} a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk_1\, dk_2\, dk_3 \tag{36}$$

and allow the solution $a_k(t) = a\delta(k-k_0)e^{ik_0vt}$, where

$$k_0 v = \omega_k + T|a|^2 = \tilde{\omega}_{k_0}. \tag{36a}$$

The nonlinearity of the medium is manifested first of all in the appearance of a quadratic dependence of the velocity of the monochromatic wave on amplitude. Equation (36a) is true if the effects of nonlinearity are smaller than the effects of dispersion (i.e., for $T|a|^2 \ll k^2\omega_{k''}$). Otherwise the shape of the stationary wave is far from sinusoidal.

Let us consider the problem of the development of perturbations in a medium having a low (but finite) amplitude. For this purpose we go over to a frame of reference in which the stationary wave is at rest after performing the substitution of variables $a_k = c_k e^{-ikvt}$ and the Hamiltonian transformation

$$H \to \tilde{H} - \int (kv)\, c_k c^*_k\, dk.$$

In the variables c_k we have

$$\frac{\partial c_k}{\partial t} + i\frac{\delta H}{\delta c^*_k} + \gamma_k c_k = 0. \tag{37}$$

In Eq. (37) we introduce damping γ_k of the phenomenological waves.

Let us now place

$$c_k = a\delta(k - k_0) + \alpha_k \quad (\alpha_k k^3 \ll a) \tag{38}$$

and let us retain only the terms which are quadratic in α_k in the Hamiltonian H. We have

$$\tilde{H} = \int (\omega_k - kv)\,\alpha_k \alpha^*_k\, dk + \tilde{H}_1, \tag{38a}$$

where the Hamiltonian $\tilde{H}_1$ vanishes along with the amplitude a of the stationary wave.

In the Hamiltonian $\tilde{H}_1$ one should retain only the "essential" terms which are not excluded by the canonical transformation. In a medium in which three-wave processes (28) are allowed one should use the Hamiltonian (29) and H_{int}, and for $\tilde{H}_1$ we have

$$\tilde{H}_1 = a\int \left(V_{k_0k_1k_2}\alpha_{k_1}\alpha_{k_2} + V^*_{k_0k_1k_2}\alpha^*_{k_1}\alpha^*_{k_2}\right)\delta_{k_0-k_1-k_2}\, dk_1\, dk_2 + 2a\int\left(V_{k_1k_0k_2}\alpha^*_{k_1}\alpha_{k_2} + V^*_{kk_0k_2}\alpha_{k_1}\alpha^*_{k_2}\right)\delta_{k_1-k_0-k_2}\, dk_1\, dk_2. \tag{39}$$

The first term of the Hamiltonian (39) is essential near the surface

$$k_0 = k_1 + k_2, \qquad \omega_{k_0} = \omega_{k_1} + \omega_{k_2}. \tag{40}$$

The second is essential near the surface

$$k_0 = k_1 - k_2, \qquad \omega_{k_0} = \omega_{k_1} - \omega_{k_2}. \tag{41}$$

If these surfaces are spaced sufficiently far apart, only one of the terms of the Hamiltonian (39) is essential.

Limiting ourselves to the first term, we have

$$\frac{\partial\alpha_k}{\partial t} + i(\omega_k - kv)\alpha_k + 2iaV^*_{k_0,k_0-k,k}\alpha^*_{k_0-k} + \gamma_k\alpha_k = 0. \tag{42}$$

Considering the solution of this equation in the form $\alpha_k = e^{\lambda t}$, we arrive at a well-known formula for the growth rate γ of parametric instability [7, 8]:

$$\gamma = \operatorname{Re}\lambda = -\frac{1}{2}(\gamma_1 + \gamma_2) + \sqrt{-\frac{1}{4}\delta^2 + \gamma_{max}^2 + \frac{1}{4}(\gamma_1 - \gamma_2)^2}, \tag{43}$$

where

$$\gamma_{max} = 2\,|V_{k_0, k, k_0-k}|\,a, \quad \gamma_1 = \gamma_k, \quad \gamma_2 = \gamma_{k_0-k}.$$

Equation (43) demonstrates the existence of instability if

$$\gamma_{max}^2 > \gamma_1\gamma_2;$$

with a maximum growth rate on the surface (27) ($\delta = 0$). It is important to note that the growth rate of parametric decay instability of the first order can be expressed directly in terms of the coefficient $V_{kk_1k_1}$ of the Hamiltonian (29).

An analysis of the Hamiltonian which is essential near the surface (41) shows that this Hamiltonian does not lead to instability. A simple mnemonic rule exists which allows separation of the terms in the interaction Hamiltonian which lead to instability. Note that this canonical variables a_k may be treated as the classical analog of the quantum annihilation operators of the quanta of a Bose field which develops as a result of quantization of a classical field with the Hamiltonian H, while the variables a_k^* may be treated as the classical analogs of the reduction operators of the same quanta. Since instability means growth of coupled pairs of waves, the term "responsible" for this instability in the Hamiltonian is the term containing the products $\alpha_{k_1}^*\alpha_{k_2}^*$.

If decay processes are forbidden in the medium, then it is necessary to use the Hamiltonian (30) to study instability. In this case decay instability of the second order occurs in the medium [9]. Instability holds for waves a_k whose vectors are concentrated near the surface:

$$\begin{gathered} 2\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2, \\ 2\omega_{k_0} = \omega_{k_1} + \omega_{k_2}, \end{gathered} \tag{44}$$

while the growth rate is given as previously by Eq. (43) where, however, it is necessary to assume

$$\gamma_{max} = |T_{k_0, k_0, k, 2k_0-k}|\,a, \quad \gamma_1 = \gamma_k, \quad \gamma_2 = \gamma_{2k_0-k}.$$

In the expression for $\delta = 2\tilde{\omega}_{k_0} - \tilde{\omega}_k - \tilde{\omega}_{2k_0-k}$ one should now take into account the quadratic corrections to the frequencies of the waves. The correction to $\tilde{\omega}_{k_0}$ is given by Eq. (36), while the corrections to $\tilde{\omega}_k$ and $\tilde{\omega}_{2k_0-k}$ can be calculated directly from the Hamiltonian H_1:

$$\tilde{\omega}_k = \omega_k + 2T_{k, k_0, k, k_0}|a|^2.$$

For $\mathbf{k}_1$ and k_2 which are not too close to $\mathbf{k}_0$ one may achieve $\delta = 0$ by means of a shift from the surface (44) by the amount $\delta k \sim T|a|^2/\omega' \mathbf{k}$; then in the absence of damping $\gamma = \gamma_{max}$; for $k \to \mathbf{k}_0$ one should take account of δ. In the simplest case, when the coefficients of the Hamiltonian T_{kk_0,kk_0} and $T_{k_0,k_0,k,2k_0,-k}$ are continuous for $k \to \mathbf{k}_0$ Eq. (48) yields the following result in the absence of damping:

$$\operatorname{Re}\lambda = \sqrt{T\Delta - \frac{1}{4}\Delta^2}. \tag{45}$$

Here $\Delta = \partial^2\omega/\partial k_\alpha \partial k_\beta|_{k=k_0}\,\delta k_\alpha \delta k_\beta$, $\delta\mathbf{k} = \mathbf{k} - \mathbf{k}_0$. In an isotropic medium $\Delta = q(\theta)|\delta k|^2$, where θ is the angle between $\delta\mathbf{k}$ and $\mathbf{k}_0$,

$$q(\theta) = \omega_k'' \cos^2\theta + \frac{1}{k}\omega_k' \sin^2\theta.$$

Equation (45) describes modulation or self-focusing instability [10, 15]. In more complicated cases the limits of the quantities T_{k,k_0,k,k_0} for $\mathbf{k} \to \mathbf{k}_0$ depend (as can easily be verified from Eq. (32)) on the direction δk. In this case (as shown in [6]) Eq. (45) remains in force, but the coefficient T becomes a function of the angle θ. In all cases the instability growth rates can easily be calculated from the coefficients of the third- and fourth-order terms in the expansion of the Hamiltonian in a_k and a_k^*.

Let us once more mention the case of the decay of a high-frequency wave into high-frequency and low-frequency waves within the framework of the Hamiltonian (34) (if $\omega_0 = 0$, then the role of the low-frequency wave will be played by a wave of the same kind as the high-frequency wave if this low-frequency wave has a small wave vector). In this case it follows that if the amplitude of the high-frequency wave is sufficiently small, conventional decay instability with a maximum growth rate takes place:

$$\gamma_{max} = |V_{k_0, k, k_0-k}| a \quad (\gamma_{max} \ll \Omega_k).$$

Here Ω_k is the dispersion law for low-frequency waves. In the opposite limiting case $k\omega'_k \gg \gamma_{max} \gg \Omega_k$ it is impossible to separate the surfaces (40) and (41), and it is necessary to retain both terms in the Hamiltonian (39). Under these conditions the growth rate is determined from a fourth-order equation, and its analysis leads to the detection of instability with a growth rate

$$\gamma \sim (\gamma_{max}^2 \Omega_k)^{1/3} \tag{46}$$

(modified decay instability). This result was obtained by many authors (viz., for example, [11, 6]) [26].

4. The "Abridged" Equations

Using the Hamiltonian formalism, it is easy to derive the "abridged" equations that describe the simplified models of nonlinear media in various approximations. Let us consider, for example, the interaction of three spectrally narrow wave packets having the characteristic wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ in a nonlinear medium. In order for this interaction to be substantial it is necessary that these vectors lie close to the resonance surface (28). Assuming that the relationship $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$, $\omega_1 = \omega_2 + \omega_3$ is fulfilled, we represent $a(\mathbf{k})$ in the form

$$a(\boldsymbol{k}) = a_1(\boldsymbol{k}_1 + \varkappa) + a_2(\boldsymbol{k}_2 + \varkappa) + a_3(\boldsymbol{k}_3 + \varkappa). \tag{47}$$

Substituting (47) into the Hamiltonian (29) and excluding nonessential terms, we obtain

$$H_{int} = 2V \int \{ a^*(\varkappa_1) a(\varkappa_2) a(\varkappa_3) + a(\varkappa_1) a^*(\varkappa_2) a^*(\varkappa_3) \} \delta_{\varkappa_1 - \varkappa_2 - \varkappa_3} d\varkappa_1 d\varkappa_2 d\varkappa_3. \tag{48}$$

Further, using the narrowness of the packets, we use the following expansion in the quadratic Hamiltonian:

$$\omega(\boldsymbol{k}_i + \varkappa) = \omega(\boldsymbol{k}_i) + \boldsymbol{v}_i \varkappa, \quad \boldsymbol{v}_i = \frac{\partial \omega}{\partial \boldsymbol{k}_i} \quad (i = 1, 2, 3)$$

and go over to variables $a_i = c_i \exp[(-i\omega(k_i)t]$. In the variables c_i

$$\tilde{H} = H - \sum_i \omega(\boldsymbol{k}_i) \int c_i c_i^* d\boldsymbol{k}$$

$$= \sum_i \int (\varkappa \boldsymbol{v}_i) c_i(\varkappa) c_i^*(\varkappa) d\varkappa + V \int [c_1(\varkappa_1) c_2^*(\varkappa_2) c_3^*(\varkappa_3) + c_1^*(\varkappa_1) c_2(\varkappa_2) c_3(\varkappa_3)] \delta_{\varkappa_1 - \varkappa_2 - \varkappa_3} d\varkappa_1 d\varkappa_2 d\varkappa_3. \tag{49}$$

Now it is convenient to perform the inverse Fourier transformation of the coordinates. Assuming $\psi_i(\mathbf{r}) = (2 \cdot \pi)^{-3/2} \int c_i(\mathbf{k}) e^{-i\mathbf{kr}} d\mathbf{k}$ and operating according to the rule

$$\frac{\partial \psi_i}{\partial t} + i \frac{\delta \tilde{H}}{\delta \psi_i^*} = 0, \tag{50}$$

we obtain the well-known equation for resonance interaction [13]:

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} + (\boldsymbol{v}_1 \nabla) \psi_1 &= -\frac{iV}{2\pi} \psi_2 \psi_3, \\ \frac{\partial \psi_2}{\partial t} + (\boldsymbol{v}_2 \nabla) \psi_2 &= -\frac{iV}{2\pi} \psi_1 \psi_3^*, \\ \frac{\partial \psi_3}{\partial t} + (\boldsymbol{v}_3 \nabla) \psi_3 &= -\frac{iV}{2\pi} \psi \psi_2^*. \end{aligned} \tag{51}$$

Assuming $\psi_1 \gg \psi_2, \psi_3$ in (51), it is easy to obtain Eq. (43) which was already derived earlier for the growth rate of decay instability.

If waves having a negative energy may exist in the medium, then resonance interaction between three wave packets whose characteristic wave numbers satisfy condition (27) is possible. Proceeding as previously, we reveal that the principal role is now played by the Hamiltonian

$$\frac{1}{3} \int \{ U_{k k_1 k_2} a_k a_{k_1} a_{k_2} + \text{c.c.} \} \delta_{k + k_1 + k_2} dk\, dk_1 dk_2,$$

while the equations for the amplitudes of the wave packets have the form

$$\frac{\partial \psi_1}{\partial t} + (\boldsymbol{v}_1 \nabla) \psi_1 = -\frac{iU^*}{2\pi} \psi_2^* \psi_3^*,$$

$$\frac{\partial \psi_2}{\partial t} + (\boldsymbol{v}_2 \nabla) \psi_2 = -\frac{iU^*}{2\pi} \psi_1^* \psi_3^*,$$

$$\frac{\partial \psi_3}{\partial t} + (\boldsymbol{v}_3 \nabla) \psi_3 = -\frac{iU^*}{2\pi} \psi_2^* \psi_1^*. \tag{52}$$

Equations (51) have the solution $\psi_1 = \psi_2 = \psi_3 = Ae^{-i\Phi}$,

$$\Phi = \frac{\pi}{6} + \frac{1}{3} \arg U, \quad A(t) = \frac{2\pi}{|U|(t_0 - t)}. \tag{53}$$

This solution goes to infinity at a finite time $t = t_0$, and Eqs. (52) describe "explosion" instability [12]. The "growth rate" of this instability $2\pi/U$ can likewise be expressed directly in terms of the coefficient of the cubic Hamiltonian. From the quantum-mechanics point of view the term in the Hamiltonian

$$\int U^*_{k k_1 k_2} a^*_{k_1} a^*_{k_1} a^*_{k_2} \delta_{k+k_1+k_2} dk\, dk_1\, dk_2,$$

which is responsible for explosion instability describes the simultaneous production of three quanta of the wave field from the vacuum. By means of the Hamiltonian formalism it is likewise easy to describe the "higher-rank" explosion instability (terms of the type $a^*a^*a^*a^*$), explosion instabilities of finite-amplitude waves (terms of the type $aa^*a^*a^*$), etc.

The next example of an "abridged" equation may be obtained by starting from the Hamiltonian (30) and assuming that the wave field constitutes a single spectrally-narrow wave packet. Let the average wave vector of the packet be $\mathbf{k}_0$; then, assuming that

$$a(\boldsymbol{k}) = c(\boldsymbol{k}_0 + \varkappa) \exp(-i\omega(\boldsymbol{k}_0)t),$$

$$H \to \widetilde{H} = H - \omega(\boldsymbol{k}_0) \int |c|(\boldsymbol{k})|^2 d\boldsymbol{k},$$

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_0 + \varkappa) = \omega(\boldsymbol{k}_0) + \varkappa \boldsymbol{v} + \frac{1}{2} \frac{\partial^2 \omega}{\partial k_\alpha \partial k_\beta} \varkappa_\alpha \varkappa_\beta,$$

we obtain

$$\frac{\partial \psi}{\partial t} + (\boldsymbol{v}\nabla)\psi + i \frac{\partial^2 \omega}{\partial k_\alpha \partial k_\beta} \frac{\partial^2 \psi}{\partial x_\alpha \partial x_\beta} + \frac{iT}{(2\pi)^3} |\psi|^2 \psi = 0$$

for $\psi(\mathbf{r})$. Equation (53) describes the "self-action" of spectrally-narrow wave packets in a nonlinear medium; Eq. (45) for modulation instability may be derived directly from (53).

A very interesting example of a simple model of a nonlinear medium consists of the equations for the interaction of a spectrally-narrow high-frequency wave packet of an arbitrary kind with sound. The nature of this interaction resides in the fact that in the presence of variations of the velocity and density of the medium created by the sound waves the law for the dispersion of ω_k for the high-frequency wave varies. This fact allows the Hamiltonian for the interaction of the high-frequency waves with sound to be written immediately:

$$H_{int} = \int |\psi|^2 (\alpha \delta\rho + \beta \nabla \Phi)\, dr,$$
$$\alpha = \left.\frac{\partial \omega}{\partial \rho}\right|_{k=k_0}, \quad \beta = \left.\frac{\partial \omega}{\partial \boldsymbol{v}}\right|_{k=k_0}. \tag{54}$$

In this formula the first term describes the variation of the frequency of the high-frequency wave due to the variation of the density of the medium; the second term describes the "entrainment effect" of the high-frequency wave by the moving medium. Remembering (Section 1) that $\delta\rho$ and Φ are canonical variables for a compressible fluid, we obtain the equation [6]

$$i(\psi_t + \boldsymbol{v}\nabla\psi) + \frac{\partial^2 \omega}{\partial k_\alpha \partial k_\beta} \frac{\partial^2 \psi}{\partial x_\alpha \partial x_\beta} - (\alpha\delta\rho + \beta\nabla\Phi) = 0,$$
$$\frac{\partial}{\partial t} \delta\rho + \rho_0 \Delta\Phi + \beta\nabla |\psi|^2 = 0, \tag{55}$$
$$\frac{\partial \Phi}{\partial t} = -s^2 \delta\rho - \alpha |\psi|^2,$$

which was analyzed in detail in [6].

Equations (55) describe, for example, interacting Langmuir and ion-sound waves in a plasma (according to the terminology developed by V. I. Karpman these are "electroacoustic waves") [14].

One may assign the universal equations which record nonlinear sound waves in media having dispersion to this same category. In order to obtain these equations we note that Eq. (2) for the energy of a compressible fluid during its potential flow may be rewritten in the form

$$H = \frac{1}{2}\int \rho(\nabla\Phi)^2\,dr + \varepsilon_{in},$$

where ε_{in} is the internal energy of the fluid and is a functional of its density. This functional may be represented in the form of a series in powers of $\Delta\rho$:

$$\varepsilon_{in} = \int \left\{ \varepsilon(\rho) + \frac{\nu}{2}(\nabla\rho)^2 \right\} dr, \tag{56}$$

"classical" gas-dynamics corresponds to the retention of only the first term of the series (56), while for consideration of the second term we obtain the Boussinesq system of equations:

$$\begin{gathered} \frac{\partial\rho}{\partial t} + \operatorname{div}\rho\nabla\Phi = 0, \\ \frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + w(\rho) = \nu\Delta\rho \end{gathered} \tag{57}$$

(for a detailed derivation of this system see [16]).

Equations (57) are applicable for a wave amplitude that is not too large; therefore one should place

$$\rho = \rho_0 + \delta\rho,$$

$$w(\rho_0 + \delta\rho) - w(\rho_0) = \delta^2\frac{\delta\rho}{\rho_0}\left(1 + \alpha\frac{\delta\rho}{\rho_0}\right).$$

In the unidimensional case Eqs. (57) are conveniently written in Lagrange coordinates. Introducing x = z + ξ and writing the continuity equation in the form

$$\rho_0 = (\rho_0 + \delta\rho)\left(1 + \frac{\partial\xi}{\partial z}\right), \tag{58}$$

and substituting the quantity $\delta\rho$ obtained from (58) into the equation

$$\xi_{tt} = -\frac{\partial}{\partial z}\left\{\frac{s^2\delta\rho}{\rho_0}\left(1 + \alpha\frac{\delta\rho}{\rho_0}\right) - \nu\rho_{zz}\right\},$$

we find

$$\xi_{tt} - s^2\xi_{zz} + s^2(1+\alpha)\frac{\partial}{\partial z}\xi_z^2 + \nu\xi_{zzzz} = 0 \tag{59}$$

within small terms.

Equation (59) is likewise a Hamiltonian equation, and it may be rewritten in the form

$$u_t = \Phi_{zz} = -\frac{\delta H}{\delta\Phi}, \tag{60}$$

$$\Phi_t = s^2 u - s^2(1+\alpha)u^2 - \nu u_{zz} = \frac{\delta H}{\delta u},$$

$$H = \frac{1}{2}\int\left(\Phi_z^2 + s^2u^2 - \frac{s^2}{3}(1+\alpha)u^3 + \nu u_z^2\right)dz.$$

We shall call Eq. (59) the equation for a nonlinear string.

Assuming in Eq. (59) that

$$u_{tt} - s^2u_{xx} = \left(\frac{\partial}{\partial t} + s\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial z}\right)u \approx 2s\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t} - s\frac{\partial}{\partial z}\right)u, \tag{61}$$

which corresponds to consideration of waves moving only in one direction, we obtain the well-known Korteweg–de Vries equation

$$2s(u_t - su_z) + s^2(1+\alpha)\frac{\partial}{\partial z}u^2 + \nu u_{zzz} = 0,$$

which it will be convenient for us to write in dimensionless form in the frame of reference moving to the right at a velocity s:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{62}$$

The Korteweg–de Vries (KdV) equation is likewise a Hamiltonian system. In order to verify this we write the KdV equation in the form

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}, \quad H = \int_{-\infty}^{\infty}\left(u^3 + \frac{1}{2}u_x^2\right)dx$$

or in the equivalent form

$$\int_{-\infty}^{\infty}\eta(x-x')\frac{\partial u(x')}{\partial t}dx' = \frac{\delta H}{\delta u}. \tag{63}$$

The notation of the KdV Eqs. (63) according to Eqs. (20)-(22) is manifestly a Hamiltonian notation.

5. The Kinetic Equations

If the ensemble of interacting waves has random phases for a low nonlinearity level, then this ensemble may be described statistically by introducing the correlation function

$$\langle a_k a_{k'}^* \rangle = n_k \delta_{k-k'}.$$

The quantity n_k is "the number of quanta" corresponding to a Bose field (accurate to within 1/h) and is governed by the kinetic equation. The Hamiltonian description of the medium is also very convenient in deriving this kinetic equation. Let us begin by considering the case of a decay spectrum having the Hamiltonian (29). The equations for the medium have the following form in this case:

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k + i\int\{V_{kk_1k_2}a_{k_1}a_{k_2}\delta_{k-k_1-k_2} + 2V^*_{k_1kk_2}a_{k_1}a_{k_2}\delta_{k-k_1+k_2}\}dk_1dk_2. \tag{64}$$

Assume that the waves have infinitely small damping. Multiplying the equation by a_k^*, adding it to the complex-conjugate expression, and averaging over the ensemble of waves, we obtain

$$\frac{\partial n_k}{\partial t} + 2\,\mathrm{Im}\int\{V_{kk_1k_2}I_{kk_1k_2}\delta_{k-k_1-k_2} + 2V^*_{k_1kk_2}I_{k_1kk_2}\delta_{k-k_1+k_2}\}dk_1dk_2. \tag{65}$$

Here $\delta_{k-k_1-k_2} I_{kk_1k_2} = \langle a_k^* a_{k_1} a_{k_2}\rangle$ is the third-order correlation function.

Assuming that the odd correlation functions decrease rapidly with an increase in the order, while the even correlation functions are unlinked with improving accuracy via the binary functions (this corresponds to the hypothesis of phase randomness), we assume that the fifth order correlation function is absent and that among the correlation functions the principal role is played by

$$\langle a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}\rangle \approx n_{k_1}n_{k_2}(\delta_{k_1-k_3}\delta_{k_2-k_4} + \delta_{k_1-k_4}\delta_{k_2-k_3}), \tag{66}$$

for the ternary correlation function we find

$$\frac{\partial I_{kk_1k_2}}{\partial t} - i(\omega_k - \omega_{k_1} - \omega_{k_2})I_{kk_1k_2} = -2iV_{kk_1k_2}n_{k_1}n_{k_2} + 2iV_{k_2kk_1}n_k n_{k_1} + 2V_{kk_1k_2}n_k n_{k_1}.$$

Hence, neglecting the slow variation of $I_{kk_1k_2}$ with time and making use of the equation $\mathrm{Im}(x+i\varepsilon)^{-1} = \pi\delta(x)$, we obtain

$$\mathrm{Im}\, I_{kk_1k_2} = 2\pi\delta(\omega_k - \omega_{k_1} - \omega_{k_2})(V_{kk_1k_2}n_{k_1}n_{k_2} - V_{k_2kk_1}n_k n_{k_1} - V_{k_1kk_2}n_k n_{k_2}) \tag{67}$$

for I. Substituting (67) into (65), we finally obtain

$$\frac{\partial n_k}{\partial t} = 4\pi\int[\{|V_{kk_1k_2}|^2\delta_{k-k_1-k_2}\delta_{\omega_k-\omega_{k_1}-\omega_{k_2}}(n_{k_1}n_{k_2} - n_k n_{k_1} - n_k n_{k_2})\}$$

$$+ 2|V_{k_1kk_2}|^2\delta_{k_1-k-k_2}\delta_{\omega_{k_1}-\omega_k-\omega_{k_2}}(n_{k_1}n_{k_2} + n_k n_{k_1} - n_k n_{k_2})]\,dk_1dk_2. \tag{68}$$

The kernel of the kinetic equation can be expressed simply in terms of the coefficient function $V_{kk_1k_2}$ of the Hamiltonian. Analogously, in a medium in which ternary processes are forbidden it is easy to obtain the kinetic equation from the Hamiltonian (30):

$$\frac{\partial n_k}{\partial t} = 2\pi\int|T_{kk_1k_2k_3}|^2\delta_{k+k_1-k_2-k_3}\delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}}$$

$$\times(n_{k_1}n_{k_2}n_{k_3} + n_k n_{k_2}n_{k_3} - n_k n_{k_1}n_{k_3} - n_k n_{k_1}n_{k_2})\,dk_1dk_2dk_3. \tag{69}$$

An analysis of the kinetic equations shows [17] that they may be used only for studying the interactions of wave distributions that are fairly wide in k-space. The characteristic distribution width n_k must satisfy the conditions

$$\frac{\Delta k}{k} \gg \left(\frac{T}{k^2 \omega''_k} \int n_k dk\right)^{1/2}. \tag{70}$$

Within the framework of the Hamiltonian formalism it is also convenient to construct a more exact theory of analytic description of wave fields which is based, for example, on a diagram technique of the Wild type.

6. The Canonical Variables

Let us present a brief description of the canonical variables for several of the most important models of continuous media. Let us begin with the generalized equations of ideal hydrodynamics:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0, \\ &\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \nabla) \boldsymbol{v} + \nabla \frac{\delta \varepsilon}{\delta \rho} = 0. \end{aligned} \tag{71}$$

Here ε is a certain functional of ρ. In the particular case when $\varepsilon = \int \varepsilon(\rho) dr$, Eqs. (71) describes a barotropic fluid.

For Eqs. (71) the pairs of variables (λ, μ) and (ρ, Φ) are canonical – these are the generalized Clebsch variables [18] which are defined from the condition

$$\boldsymbol{v} = \frac{\lambda}{\rho} \nabla \mu + \nabla \Phi. \tag{72}$$

In these variables the Euler equation decays into three equations:

$$\frac{\partial \lambda}{\partial t} = -\operatorname{div} \lambda \boldsymbol{v} = \frac{\delta H}{\delta \mu},$$

$$\frac{\partial \mu}{\partial t} = -(\boldsymbol{v} \nabla) \mu = -\frac{\delta H}{\delta \lambda},$$

$$\frac{\partial \Phi}{\partial t} = -\frac{1}{2} v^2 + \frac{\lambda}{\rho} (\boldsymbol{v} \nabla) \mu - \frac{\delta \varepsilon}{\delta \rho} = -\frac{\delta H}{\delta \rho},$$

while the continuity equation may be written in the form

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \Phi},$$

where

$$H = \frac{1}{2} \int \rho v^2 dr + \varepsilon$$

is the total energy of the fluid.

The variables (72) constitute the prototype according to which the Hamiltonian formalism is introduced into various equations of the hydrodynamics type – first of all into hydrodynamic models for the description of a plasma [17]. The two simplest plasma models – the hydrodynamics of an electron gas and the hydrodynamics of ion sound (viz., for example, [18]) belong directly to the type (72). For potential flows $\lambda = \mu = 0$ we arrive at the pair (ρ, Φ) which has already been encountered in Section 1 in connection with the Boussinesq Eqs. (55).

The variables (72) can easily be generalized for the case of relativistic hydrodynamics, for which it is sufficient to replace v by $\mathbf{p}/m = v/\sqrt{1-v^2/c^2}$ in (72), and also for the case of the hydrodynamics of a charged fluid which interacts with an electron field. In this case

$$\frac{\boldsymbol{p}}{m} = \frac{\lambda}{\rho} \nabla \mu + \nabla \Phi - \frac{e}{mc} \boldsymbol{A}, \tag{73}$$

where **A** is a new vector canonic variable – the vector-potential of the electromagnetic field taken in the Coulomb gauge. The quantity that is the canonical conjugate of **B** is the vector

$$B = -\frac{E}{4\pi c},$$

where E is the electric field intensity.

The variables (73) allow immediate computation of the expansion coefficients of the Hamiltonian of the plasma energy in powers of the amplitudes of the Langmuir waves a_k, the electromagnetic waves S_k, and the ion-sound waves b_k, and the finding of the principal characteristics of their interaction in an isotropic plasma [17]. It is curious that in the nonrelativistic limit the Hamiltonian contains only quadratic and cubic terms.

If a constant magnetic field H_0, is present in the plasma, then its corresponding vector potential is a linear function of the coordinates, and direct diagonalization of the quadratic Hamiltonian, which was described in Section 1, is impossible. In this case it is necessary to perform a canonical transformation which excludes the constant part of the vector potential $A_0 = 2^{-1}(-iy + jx)H_0$. After this transformation the canonical substitution has the form

$$\frac{p}{m} = \frac{\omega_H^{1/2}}{\rho^{1/2}}(-i\lambda - j\mu) + \nabla\Phi - \frac{e}{mc}\tilde{A} + \frac{1}{2\rho}(\lambda\nabla\mu - \mu\nabla\lambda). \tag{74}$$

Here (λ, μ), (ρ, Φ) and $(\tilde{A} = A - A_0, B)$ are new pairs of canonical variables. The use of the variables (74) allows successful computation [19, 20] of the instability growth rate and of the kernels of the kinetic equations for the interaction of waves in a magnetically active plasma (their calculation by other methods is extremely cumbersome).

For the magnetohydrodynamics equations

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \operatorname{div}\rho v = 0,\\ &\frac{\partial v}{\partial t} + (v\nabla)v = -\nabla w(\rho) + \frac{1}{4\pi\rho}[\operatorname{rot} H, H],\\ &\frac{\partial H}{\partial t} = \operatorname{rot}[v, H] \end{aligned} \tag{75}$$

the transition to canonical variables is accomplished by means of the substitution

$$v = \frac{1}{\rho}[H, \operatorname{rot} S] + \nabla\Phi, \tag{76}$$

in which the pairs (ρ, Φ) and (H, S) are canonically conjugate.

The magnetohydrodynamics equations in these variables have the form

$$\begin{aligned} &\frac{\partial\rho}{\partial t} = -\operatorname{div}\rho v = \frac{\delta \mathrm{H}}{\delta\Phi},\\ &\frac{\partial H}{\partial t} = \operatorname{rot}[vH] = \frac{\delta \mathrm{H}}{\delta S},\\ &\frac{\partial S}{\partial t} = -\frac{H}{4\pi} + [v, \operatorname{rot} S] - \nabla\psi = -\frac{\delta \mathrm{H}}{\delta H},\\ &\frac{\partial\Phi}{\partial t} = \frac{v^2}{2} - w(\rho) - (v\nabla)\Phi = -\frac{\delta \mathrm{H}}{\delta\rho}. \end{aligned} \tag{77}$$

Here $\mathrm{H} = \int\{\rho v^2/2 + \varepsilon(\rho) + H^2/8\pi - \psi \operatorname{div} H\} d\mathbf{r}$ is the energy of the medium; $\psi = (H_0, \mathbf{r})/4\pi$ is the gauge function chosen on the basis of convenience. For the natural conditions $\operatorname{div} S = 0$

$$\psi = \frac{1}{\Delta}\operatorname{div}[v, \operatorname{rot} S] + \psi_0, \quad \Delta\psi_0 = 0.$$

For diagonalization of the Hamiltonian against the background of a constant magnetic field H_0 is it convenient to choose

$$\psi = \frac{1}{4\pi}(H_0, r).$$

The variables (76) allow simple computation of the matrix elements in the interaction of magnetohydrodynamic waves and the growth rate of decay instability of the Alfven wave among the magnetohydrodynamic waves.

For potential oscillations of the surface of a fluid which is situated in a uniform gravitational field g directed downward along the z axis the canonical pair is $\eta(x, y, t)$ (the deviation of the surface of the fluid from the equilibrium value) and $\psi(x, y, t)$ (the hydrodynamic potential on the surface) (see [22]).

The equation for the oscillation of the surface in these variables has the form

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{78}$$

Here

$$H = \frac{9}{2} \int \eta^2 dr + \int G(s, s') \psi(s) \psi(s') ds\, ds' \tag{79}$$

is the total energy of the fluid; G(s, s') is the Green's function of the Laplace equation in the domain bounded by the surface of the fluid and the bottom. Assuming that the surface of the fluid deviates little from the plane, one may expand G in a series in powers of η (see [22]), and then by diagonalizing the quadratic part of the Hamiltonian one can calculate the first coefficients of the expansion of the Hamiltonian H in powers of the complex amplitudes a_k of the surface waves. This method allows us to take into account the effects of surface tension and is the most economical method of calculating the instability growth rates, the coefficients of the abridged equations, and the kernels of the kinetic equations for the surface waves.

For spin waves in a ferromagnet, which are described by the phenomenological Landau–Lifshits equation for the density of the magnetic moment

$$\frac{\partial \boldsymbol{M}}{\partial t} = g\left[\boldsymbol{M}, \frac{\delta H}{\delta \boldsymbol{M}}\right], \quad |\boldsymbol{M}| = M_0, \tag{80}$$

where g is the gyromagnetic ratio, H is the total energy of the ferromagnet; the well-known Holstain–Primakoff variables [23] are the canonical variables:

$$M^+ = M_x + iM_y = \sqrt{9 M_0}\, a(\boldsymbol{r}) \sqrt{1 - \frac{g|a|^2}{2M_0}}.$$

In these variables Eqs. (80) take the form

$$\frac{\partial a}{\partial t} + i\frac{\delta H}{\delta a^*(\boldsymbol{r})} = 0.$$

The coefficients of expansion of the Hamiltonian of the ferromagnet in powers of the normal amplitude a_k of the spin wave have been calculated (for cubic crystals) in [24]. The values of these coefficients allows great progress to be made in the theory of nonlinear interaction of spin waves.

The equations of nonlinear electrodynamics in media having dispersion do not allow the exact introduction of canonical variables, since they are not differential with respect to time. For them, however, an approximate introduction of canonical variables is possible in the form of series in powers of the "natural" variables $\mathbf{E}(\mathbf{k}, \omega)$ with coefficients calculated by means of the nonlinear susceptibility tensors. Such a calculation, which was performed in [25], allows Hamiltonian wave dynamics to be incorporated in this scheme along with nonlinear electrodynamics, provided only that the amplitude of the electromagnetic waves is not too great. As a whole, the use of canonical variables leads to a cardinal simplification of the computations and to a clarification of the essential features in studying the processes of interaction of waves in various nonlinear media. The unification of these calculations allows results obtained for one medium to be given a general physical meaning easily.

7. The Quasilinear States

If the Hamiltonian of the medium is quadratic and has the form (13a), then the general solution of the equations of motion for an unbounded medium has the form

$$a_k(t) = c(\boldsymbol{k}) \exp(-i\omega_k t).$$

In this case the amplitudes of all of the waves are independent of time.

In considering the nonlinear interaction the amplitudes of the waves in general become time functions. One may, however, consider special initial conditions for which the wave field has the form

$$a_k(t) = \sum_{i=1}^{N} A_i \delta(\boldsymbol{k} - \boldsymbol{k}_i) \exp(-i\omega_i t) + O(A^2), \tag{81}$$

where the remanent term is uniformly small in time. The wave field of the form (34) may be called a quasilinear state. All of the variables characterizing a quasilinear state are N-periodic functions of the coordinates and of time, the spatial period being stipulated by the numbers $\mathbf{k}_i$ and the time periods being dependent on the amplitudes.

The simplest quasilinear state is a stationary periodic wave – in this case N = 1 and

$$a_k(t) = A_1 \exp(-i\omega_1 t)\delta(\mathbf{k} - \mathbf{k}_1), \tag{82}$$

where $\omega_1 = \omega(\mathbf{k}_1) + T|a|^2$ (viz., (36a)). For N = 2 the quasilinear state has the form

$$a_k(t) = A_1 \exp(-i\omega_1 t)\delta(\mathbf{k} - \mathbf{k}_1) + A_2 \exp(-i\omega_2 t)\delta(\mathbf{k} - \mathbf{k}_2). \tag{83}$$

Substituting (38) into (30) and excluding the terms lying off the resonance surface (31), we find

$$\begin{aligned} \omega_1 &= \omega(\mathbf{k}_1) + T_{k_1 k_1 k_1 k_1}|A_1|^2 + 2T_{k_1 k_2 k_1 k_2}|A_2|^2, \\ \omega_2 &= \omega(\mathbf{k}_2) + T_{k_2 k_2 k_2 k_2}|A_2|^2 + 2T_{k_1 k_2 k_1 k_2}|A_1|^2. \end{aligned} \tag{84}$$

The solution (83) (the biharmonic field) exists only if the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ do not lie near the surface (28) in a layer having a width of the order of $V|a|/\omega''_k k^2$; in this case, a "secular" variation of the amplitudes A_1 and A_2 with time occurs. Analogously, for the existence of a quasilinear solution for N = 3 it is necessary for the vector $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ not to lie near the surface in a layer having the thickness $|a|^2T$ (31) and likewise for a biharmonic field to be possible to construct from each pair of vectors $(\mathbf{k}_1, \mathbf{k}_2)(\mathbf{k}_1\mathbf{k}_3)$ and $(\mathbf{k}_2\mathbf{k}_3)$.

One may formulate the conditions governing the existence of the N-wave quasilinear state analogously. Such a state creates N "dangerous zones," each of which has a width of the order of $(\Delta\omega/N)^{1/2}\omega^{1/2}/\omega''_k k$ equal to the resultant frequency shift of each of the waves. For $N \sim k^2\omega'' k/\Delta\omega$ the dangerous zones cover the entire region of the phase space in which the waves are concentrated, and a further increase of N is impossible. Besides the "dangerous zones" created by three-wave resonances, "dangerous zones" created by four-wave processes of the type (31) exist. These zones have a width of the order of $\Delta\omega/k''_k N$ and are created by pairs of waves, so that their number is of the order of N^2. For $N \ll \omega/\Delta\omega$ the overall width of these zones $\delta k_2 \approx N\Delta\omega/\omega'_k$ is much smaller than the overall width of the dangerous zones created by three-wave processes: $\delta k_2 \ll \delta k_1 \sim (\Delta\omega\omega/k\omega''_k)^{1/2}N^{1/2}$. For $N \sim k^2\omega''_k/\Delta\omega \sim 1$ the widths δk_1 and δk_2 are comparable and the zones overlap. In the reasoning presented above it was assumed that all N waves have frequencies of the same order of magnitude. In this case the dangerous zones created by the higher resonances of the type $n\omega_0 = 2\omega$ lie in the high-frequency range and do not enter into the problem. Thus, in a nonlinear medium with dispersion no more than $N \sim k^2\omega''_k/\Delta\omega$ monochromatic waves may exist regardless of whether or not decay processes are allowed or forbidden. The unidimensional case is an exception. In the unidimensional case it follows that for the condition that there is only one wave mode three-wave resonances of the type (28) are impossible, while four-wave resonance conditions (31) yield $\mathbf{k}_2 = \mathbf{k}$, $\mathbf{k}_3 = \mathbf{k}_1$ or $\mathbf{k}_2 = \mathbf{k}_1$, $\mathbf{k}_3 = \mathbf{k}$. This means that if the function a_k is sufficiently smooth, then one may isolate the term

$$H_{\rm int} = \int T_{kk'}|a_k|^2|a_{k_1}|^2 dk dk_1 \tag{41}$$

in the interaction Hamiltonian and exclude the remaining part of the Hamiltonian by means of a canonical transformation. The Hamiltonian (41) yields

$$a_k(t) = \int c(k)e^{-i\tilde{\omega}(k)t}dk, \qquad \tilde{\omega}_k = \omega_k + 2\int T_{kk'}|a_{k'}|^2 dk'.$$

This means that in the unidimensional case wide wave spectra are quasilinear and allow complete exclusion of the Hamiltonian (30). All time variations of the spectrum take place due to Hamiltonians of the type Va^3a^{*3} in this case, which yields an estimate of the characteristic variation time:

$$\frac{1}{\tau\omega} \ll \left(\frac{\Delta\omega}{\omega}\right)^4.$$

Let us also write that in a two-dimensional and a three-dimensional medium in the absence of dispersion the N-quasilinear solutions are obviously unstable – it is sufficient to place a "seeding" small wave in a "dangerous zone" in order for the amplitude of this wave to begin to grow. In a unidimensional medium stable N-quasilinear solutions are possible.

8. Completely Integrable Systems and Stochastization

Dynamic systems having a large number of degrees of freedom as a rule "mix" over a long time and behave statistically. Systems having abundant sets of integrals of motion (completely integrable systems in the first place) constitute an exception. A completely integrable system with N degrees of freedom has N independent integrals of motion which are functions of the state (i.e., they do not depend explicitly on time), all of these integrals of motion being in the involution – i.e., the Poisson brackets between all I_n are equal to zero:

$$\{I_n, I_m\} = 0. \tag{85}$$

The examples of integrable systems are a set of N linear oscillators, the motion of a point in a centrally-symmetrical field or along the surfaces of a body of rotation, free motion of a rigid body, motion of a point in the field of two Coulomb centers, motion of a symmetrical top in a gravity field, and certain cases of the motion of a nonsymmetrically-heavy top. If the dynamic system is completely integrable, then the conservation laws I_n which are in the involution may be taken as generalized momenta. Under these conditions the Hamiltonian H will not depend on the corresponding generalized coordinates φ_n ($H = H(I_1, \ldots, I_n)$) whose equations have the form

$$\frac{\partial \varphi_n}{\partial t} = \frac{\partial H}{\partial I_n}, \quad \varphi_n(t) = \varphi_n(0) + \frac{\partial H}{\partial I_n} t, \tag{86}$$

where the variables I_n, φ_n are called action-angle and variables. The simplicity of their time dependence makes it desirable to solve the initial problem for the integrable system according to the following scheme:

$$p_n(0),\ q_n(0) \to I_n,\ \varphi_n(0) \to I_n, \varphi_n(t) \to p_n(t),\ q_n(t). \tag{87}$$

During the first stage of this scheme the transition is accomplished from the original variables p_n, q_n and to action-angle variables, and during the last stage the transition is made back again from the action-angle variables to the original variables.

Integrable systems having a finite number of degrees of freedom are not stochasticized; instead of that they perform quasiperiodic motion with N periods. Note that under these conditions there may be unstable equilibrium points and types of motion in the integrable system. Thus, rotation of a rigid body relative to an intermediate inertial axis or motion of a point along a minimal diameter on a surface of rotation lead to the appearance of a new periodic motion with a large amplitude.

All of the integrable systems known until recently, with the exception of a system of N-independent oscillators, had a finite number (and a small number at that) degrees of freedom. After the work of the last fewy years, however, it has become clear that there is a large number of completely integrable Hamiltonian systems with a continuous number of degrees of freedom. This applies in the first place to unidimensional systems (the distinction of unidimensional systems is already clear in the example of the study of quasilinear solutions) it turns out that almost all unidimensional "standard" Hamiltonians which develop in the physics of nonlinear waves are integrable, and important examples of integrable two-dimensional and three-dimensional Hamiltonians exist. This fact poses anew the problem of stochastization of nonlinear wave fields. In an integrable system the evolution of an N-wave solution which is not quasilinear or the development of instability of an N-quasilinear solution leads to the establishment of a certain quasiperiodic motion and "reversible turbulence" (although possibly also with a large number of periods) rather than to phase mixing. Systems which are not too close to completely integrable systems are those which are actually stochasticized.

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