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It is shown that every one-dimensional differential operator whose coefficient functions depend on an arbitrary set of parameters is associated with a series of multidimensional nonlinear partial differential equations which can be integrated by means of the inverse scattering problem method.

The inverse scattering problem method was discovered in 1967 by Kruskal, Gardner, Green, and Miura [1], who integrated the well-known Korteweg-de Vries equation by means of a transformation from the potential of the one-dimensional stationary Schrödinger operator to its scattering matrix. Subsequently, in [2] a nonlinear Schrödinger equation was integrated in an analogous manner by means of the one-dimensional Dirac operator. In the following papers [3-6] it was shown that these operators are associated with infinite classes of integrable equations which can be calculated algorithmetically. In [7], a procedure was proposed for calculating these equations, together with a description of their method of solution for arbitary matrix operators of any order.

In recent papers (see [8]), Calogero has proposed a generalization of the inverse scattering problem method and he has shown that by means of the Schrödinger and Dirac operators and their matrix analogs one can integrate new classes of nonlinear differential equations which contain functions of an arbitrary number of arguments. In the present paper, we generalize Calogero's result to the case of arbitrary matrix operators of any order, and we also give a simpler proof of this result. Our approach is based to a large extent on [7].

Suppose we are given an arbitrary integral operator F and an operator K which is a Volterra operator from the right, these acting on vector functions $\psi_{n}(x),-\infty<x<\infty, 1 \leq n \leq N$, the kernels of the operators $F(x, y)$ and $K(x, y)$ being related by

$$
\begin{equation*}
F(x, y)+K(x, y)+\int_{x}^{\infty} K(x, s) F(s, y) d s=0 \tag{1}
\end{equation*}
$$

and the operators themselves by

$$
\begin{equation*}
F+K+K \cdot F=0 . \tag{2}
\end{equation*}
$$

The functions $\psi(x)$ and the operators $K$ and $F$ also depend on the vector parameter $z=\left(z_{1}, \ldots, z_{l}\right)$
Suppose we are given an operator $M$ that is differential with respect to $x$ and $z$, defined on $\psi(x, z)$, and which commutes with the operator $F: M F-F M=0$. Then (see [7]) there exists an operator $\tilde{M}$, differential with respect to x and z , such that for K related to F by the condition (2),

$$
\widetilde{M} K-K M=0 .
$$

Here $\tilde{M}=M+Q$, where $Q$ is an operator subordinated to $M$; the coefficients of the operator $Q$ can be calculated from the conditions of vanishing of the terms outside the integral in the relation

$$
\begin{equation*}
\widetilde{M}\left(\psi(x, z)+\int_{x}^{\infty} K(x, s) \psi(s, z) d s\right)=(\widetilde{M} \psi)(x, z)+\int_{x}^{\infty} K(x, s)(M \psi)(s, z) d s \tag{3}
\end{equation*}
$$

and they can be expressed by means of recursion relations in terms of a finite number of derivatives with respect to $x$ and $z$ of the kernel $K(x, y, z)$ taken at $y=x$, the set of which we denote by $\xi(x, z)$. Equation (1) is the Gel'fand-Levitan equation, which solves the inverse scattering problem for the operator $\widetilde{M}$

Suppose further that there exist two operators $M_{1}$ and $M_{2}$ such that the equations for $F$
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$$
\begin{equation*}
\left[M_{1}, F\right]=0, \quad\left[M_{2}, F\right]=0 \tag{4}
\end{equation*}
$$

\]

have a simultaneous solution. Then the following equations are compatible:

$$
\begin{equation*}
\widetilde{M}_{1} K-K M_{1}=0, \quad \widetilde{M}_{2} K-K M_{2}=0 . \tag{5}
\end{equation*}
$$

The condition of their compatibility is a system of nonlinear differential equations for the $\xi(x, z) s$ which combine the sets $\xi_{1}(x, z)$ and $\xi_{2}(x, z)$ associated with the operators $M_{1}$ and $M_{2}$. This system is the desired integrable system. Every solution $F$ of the system (4) after $K$ has been found from Eq. (1) and $\xi(x, z)$ has been calculated generates an exact solution of this system.

The problem of enumerating the systems which can be integrated by means of Eq. (1) thus reduces to the problem of enumerating the pairs of operators $\mathrm{M}_{1}$ and $\mathrm{M}_{2}$ for which Eqs. (4) have simultaneous solutions. In [7], a study is made of the class of such operator pairs defined on functions $\psi_{n}\left(x_{n}, z_{i}, z_{2}\right)$ $\left(z=\left(z_{i}, z_{2}\right)\right)$ and having the form

$$
M_{1}=\frac{\partial}{\partial z_{1}}+L_{1}, \quad M_{2}=\frac{\partial}{\partial z_{2}}+L_{2},
$$

where $L_{1,2}=L_{1,2}(x, z, \partial / \partial x)$ are matrix operators that are differential with respect to $x$ and satisfy the

$$
\begin{equation*}
\left[M_{1}, M_{2}\right]=0 \tag{6}
\end{equation*}
$$

The corresponding operators $\tilde{\mathrm{M}}_{1}$ and $\tilde{\mathrm{M}}_{2}$ have the form

$$
\widetilde{M}_{1}=\frac{\partial}{\partial z_{1}}+\widetilde{L}_{1}, \quad \widetilde{M}_{2}=\frac{\partial}{\partial z_{2}}+\widetilde{L}_{2}
$$

where $\tilde{L}_{1}$ and $\tilde{L}_{2}$ are also operators that are differential with respect to $x$ and depend on

$$
\xi_{i}(x, z)=\left.\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^{i} K(x, y)\right|_{x=y}, \quad i=1,2, \ldots, p-1
$$

where $p$ is the higher of the orders of the operators $L_{1}$ and $L_{2}$. The operators $\widetilde{M}_{1}$ and $\tilde{M}_{2}$ also satisfy the condition (6), which can be rewritten in the form

$$
\frac{\partial \widetilde{L}_{2}}{\partial Z_{1}}-\frac{\partial \widetilde{L}_{1}}{\partial Z_{2}}+\left[\widetilde{L}_{1}, \widetilde{L}_{2}\right]=0
$$

which is a generalization of Lax's relation [3]. The majority of the previously discovered systemsthat can be integrated by the inverse scattering problem method belong to this class.

The systems described by Calogero correspond to a different choice of the pair $\mathbb{M}_{1}, \mathrm{M}_{2}$. Suppose

$$
\begin{equation*}
M_{1}=L=L\left(x, z, \frac{\partial}{\partial x}\right), \quad\left(z=\left(z_{1}, \ldots, z_{i}\right)\right) \tag{7}
\end{equation*}
$$

is an arbitrary matrix operator differential with respect to x :

$$
\begin{equation*}
M_{2}=\sum_{i} f_{i}(z, L) \frac{\partial}{\partial z_{i}}, \quad i=1, \ldots, \tag{8}
\end{equation*}
$$

where $f_{i}(z, L)$ is a polynomial in $L$ with coefficients that are vector functions of $z$. We show that the operators (7) and (8) can be chosen as the pair (4). The condition $[L, F]=0$ means that the kemel $F(X, y)$ satisfies the differential equation

$$
\begin{equation*}
L\left(x, z, \frac{\partial}{\partial x}\right) F(x, y)-L^{+}\left(y, z, \frac{\partial}{\partial y}\right) F(x, y)=0 \tag{9}
\end{equation*}
$$

where $L^{+}$is the adjoint of the operator $L$; the matrices in $L^{+}$are multiplied by from the right.
It obviously follows from the condition $[L F]=0$ that $\left[f_{i}(z, L), F\right]=0$. Therefore, the condition $\left[\mathrm{M}_{2}, \mathrm{~F}\right]=0$ reduces to the condition

$$
\sum_{i} f_{i}(z, L) \frac{\partial F}{\partial z_{i}}=0
$$

i.e., to the differential equation

$$
\begin{equation*}
\sum_{i} f_{i}\left(z, L\left(x, z, \frac{\partial}{\partial x}\right)\right) \frac{\partial F(x, y)}{\partial z_{i}}=0 \tag{10}
\end{equation*}
$$

A function $F$ satisfying the conditions (9) and (10) commutes with $M_{1}$ and $M_{2}$. To calculate a system that can be integrated by means of the operators $M_{1}$ and $M_{2}$ we must find the operators $\tilde{M}_{1}$ and $\tilde{\mathrm{M}}_{2}$. This can be done by using the relation (3). It is readily verified that $\tilde{M}_{1}=\tilde{L}=L+Q_{1}, \tilde{M}_{2}=\tilde{M}_{2}+Q_{2}$, where $Q_{1}$ and $Q_{2}$ are operators that contain differentiation only with respect to $x$. The coefficients of $Q_{1}$ and $Q_{2}$ can be calculated by means of recursion relations. After the construction of $\tilde{L}$ and $\tilde{M}_{2}$, the required integrable system can be found from the relation

$$
\begin{equation*}
\left[\widetilde{L}, \widetilde{M}_{2}\right]=0 . \tag{11}
\end{equation*}
$$

One can combine the two considered classes of system by taking $M_{1}$ in the form (7) and $M_{2}$ in the form

$$
\begin{equation*}
M_{2}=\sum_{i} f_{i}(z, L) \frac{\partial}{\partial z_{i}}+L_{2}\left(x, z, \frac{\partial}{\partial x}\right) \tag{12}
\end{equation*}
$$

where $\left[L, L_{2}\right]=0$; in particular, one can set $L_{2}=g(z, L)$. If the derivative with respect to one of the parameters $t=t(z)$ is explicitly separated out in the operator (12),

$$
M_{2}=\partial / \partial t+M
$$

then the relation (11) can be written in the form of Lax's relation [3]

$$
\partial \widetilde{L} / \partial t=[\widetilde{L}, \widetilde{M}],
$$

which however implies conservation of the spectrum of the operator $\tilde{L}$ if this spectrum at $t=0$ does not depend on z .

As an example, we take

$$
L=\left(\begin{array}{ccc}
a_{1} & & 0 \\
& \ddots & \\
0 & & a_{N}
\end{array}\right) \frac{\partial}{\partial x}, \quad M_{2}=\frac{\partial}{\partial t}+L \frac{\partial}{\partial z}+\frac{\partial}{\partial z} L .
$$

It then follows from (9) and (1) that ( $\mathrm{A}=\operatorname{diag}\left(a_{1}, \ldots, a_{N}\right)$ )

$$
\begin{equation*}
i A \frac{\partial K(x, y)}{\partial x}+i \frac{\partial K(x, y)}{\partial y} A+i[A, K(x, x)] K(x, y)=0 \tag{13}
\end{equation*}
$$

the second of the equations (7) in this example has the form

$$
\begin{equation*}
\frac{\partial K(x, y)}{\partial t}+i A \frac{\partial^{2} K(x, y)}{\partial z \partial x}-i \frac{\partial^{2} K(x, y)}{\partial z \partial y} A+i[A, K(x, x)] K_{z}(x, y)+i\left(A K_{z}(x, x)+K_{z}(x, x) A\right) K(x, y)=0 \tag{14}
\end{equation*}
$$

Equations (13) and (14) mean that the matrix $u(x, t, z)=K(x, x)$ satisfies the evolution equation

$$
\frac{\partial u_{i j}}{\partial t}+i \frac{a_{i}{ }^{2}+a_{j}^{2}}{a_{i}-a_{j}} \frac{\partial^{2} u_{i j}}{\partial x \partial z}+2 i\left(a_{i}+a_{j}\right) \frac{\partial u_{i j}}{\partial z} u_{j j}+2 i a_{i}\left(u_{i i}+u_{i j}\right)_{z} u_{i j}+i \sum_{k \neq j} 2 a_{i} \frac{a_{i}-a_{k}}{a_{i}-a_{j}} u_{i \bar{s}} \frac{\partial u_{k j}}{\partial z}+i \sum_{k \neq i, j} 2 \frac{a_{i}{ }^{2}-a_{j} a_{k}}{a_{i}-a_{j}} \frac{\partial u_{i k}}{\partial z} u_{k j}=0
$$

for $i \neq j$. For the diagonal elements of $u$, we obtain directly from (13)

$$
a_{i} \frac{\partial u_{i i}}{\partial x}=-\sum_{k}\left(a_{i}-a_{k}\right) u_{i k} u_{k i}
$$

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