Stationary gravitational solitons with axial symmetry

V. A. Belinskii and V. E. Zakharov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the U.S.S.R.

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An application of the method of the inverse scattering problem to the integration of the gravitational equations is described. The case considered is that of stationary axially symmetric gravitational fields. The procedure for constructing soliton solutions is carried through for all metric coefficients. Axially symmetric solutions with \( n \) solitons are considered.

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1. INTRODUCTION

In a previous paper we have shown\(^1\) that in the case in which the metric tensor depends on only two variables the gravitational equations form a system which is integrable by the method of the inverse scattering problem. The case was examined in which one of the variables is the time and the other is spacelike; this corresponds to cosmological and wave solutions of the equations of gravitation. It was pointed out that there is no difficulty in applying this method also to the case in which both the variables on which the metric tensor depends are spacelike, which corresponds to stationary gravitational fields. One possible interpretation of this case is that of a stationary gravitational field with axial symmetry. This class of solutions is important in the theory of gravitation, since it has a clear physical meaning. In this connection it is interesting to consider the case of axially symmetric stationary fields separately and to deal with one important point which was left incomplete. We shall explain the essence of the question, first introducing the metric and the corresponding Einstein equations.

Having in view the application to the case of stationary axially symmetric gravitational fields, we write the metric in the form\(^2\)

\[
\begin{align*}
\text{det} g &= -\varphi, \\
axial symmetry, the Einstein equations (in vacuum) for the metric (1.1), (1.2) separate into two groups. The first determines the matrix \( g \) and is of the form

\[
\begin{align*}
\left(\varphi, s, t^2\right) &= \left(\varphi, s, t^2\right)^{-1}, \\
\text{det} g &= -\varphi,
\end{align*}
\]

(1.3)

The second group of equations determines the metric coefficient \( f \) for a given solution of Eq. (1.3) and can be written in the form

\[
\begin{align*}
\left(\varphi, s, t^2\right) &= \left(\varphi, s, t^2\right)^{-1}, \\
\text{det} g &= -\varphi,
\end{align*}
\]

(1.4)

where the two-rowed matrices \( U \) and \( V \) are defined as follows:

\[
\begin{align*}
U &= \left(\varphi, s, t^2\right)^{-1}, \\
V &= \varphi, \\
\text{det} g &= -\varphi
\end{align*}
\]

(1.5)

It is easy to see that if instead of \( p \) and \( z \) we introduce the pair of complex variables \( \xi = (z + ip) \) and \( \eta = (z - ip) \), then in the variables \( \xi \) and \( \eta \) the metric (1.1) and Eqs. (1.2)-(1.6) will be formally reduced to the same form as we studied previously.\(^3\) For this reason all of the formal side of the method for the case considered here can be obtained\(^4\) from the results of our earlier paper.\(^1\) Of these results we shall present here only the basic points which are necessary for a complete exposition, and shall not go into the details of the proofs. The details can be found in Ref. 1.

Let us now turn to the point in the research which was not brought to completion in Ref. 1. As follows from what we have said, we can apply to the integration of Eqs. (1.2)-(1.6) the method given in Ref. 1, i.e., apply the method of the inverse scattering problem to the integration of the matrix equation (1.3) and thus get the major part of the metric coefficients, \( \xi_{\mu\nu} \). There then remains, however, the problem of calculating the metric coefficient \( f \), which is given in quadratures by Eqs. (1.4) and (1.5).

In Ref. 1 it was shown by direct calculations that for the simple soliton solutions given there these quadratures can be performed completely (i.e., the integrals can be calculated explicitly), and the answer for the coefficient \( f \) can be expressed explicitly in terms of the appropriate partial or background solution of the problem and elementary functions, i.e., qualitatively in the same way as the metric components \( g_{\mu\nu} \). This suggests that the same will be true also on the general case of an \( n \)-soliton solution. It turns out that this is indeed true, and the metric coefficient \( f \) in the general \( n \)-soliton as

\[
\begin{align*}
\left(\varphi, s, t^2\right) &= \left(\varphi, s, t^2\right)^{-1}, \\
\text{det} g &= -\varphi,
\end{align*}
\]

(1.6)
case, like the coefficients \( g_{\mu\nu} \), can be calculated altogether explicitly. The analysis for this point is given in Sec. 3 of the present paper.

Finally, we point out that the question of the integrability of the equations of gravitation for the case considered has also been investigated by Maisser, who proved the existence of an L-A pair for the Einstein equations, though in a somewhat different way from that followed in Ref. 1 and here [cf. Eqs. (2.1), (2.2)]. Harrison found the L-A pair for the matrix function \( \phi(x, p, z) \) for the Ernst equation corresponding to this problem.

2. THE n-SOLITON SOLUTION FOR THE MATRIX \( g \)

Using the results of Ref. 1 (as explained in the Introduction), we can easily find the L-A pair for the matrix equation (1.3) in the variables \( p \) and \( z \):

\[
D_1 \phi = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial p}, \quad D_2 \phi = \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial p},
\]

(2.1)

where the commuting differential operators \( D_1 \) and \( D_2 \) are given by

\[
D_1 = \frac{\partial}{\partial p} - \frac{1}{x^2} \frac{\partial}{\partial z}, \quad D_2 = \frac{\partial}{\partial p} + \frac{1}{x^2} \frac{\partial}{\partial z},
\]

(2.2)

and \( \lambda \) is a complex spectral parameter independent of the coordinates \( p \) and \( z \). It is not hard to verify that the conditions of compatibility of the equations (2.1) for the matrix function \( \phi(x, p, z) \) are identical with the original equations (1.3) and (1.6), if we rewrite them, and also the conditions for their compatibility, in terms of the matrices \( U \) and \( V \), in the same way as this was done previously. The required matrix \( g \) is the value of the matrix \( \phi(x, p, z) \) for \( \lambda = 0 \):

\[
\phi(0, 0, z) = 0, \quad z.
\]

(2.3)

The procedure for integrating the equations (2.1) presupposes the knowledge of some particular solution of the problem. Let \( U_0, U_1, V_0, V_1 \) be some particular solution of the equations (1.3) and (1.6), from which, with Eq. (2.1), the corresponding solution \( \phi(x, p, z) \) has been found. We then seek the solution for \( \phi \) in the form

\[
\phi(x, p, z) = \psi(x, p, z)
\]

(2.4)

and for \( \chi(x, p, z) \) we get from Eq. (2.1) the following equations:

\[
D_1 \psi = \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial p}, \quad D_2 \psi = \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial p},
\]

(2.5)

Now (as before) it can be shown that to assure that the matrix \( g \) is real and symmetric definite supplementary conditions have to be imposed on the solutions of Eq. (2.5). For the reality of \( g \) we have the requirements

\[
\chi(\tilde{z}) = \chi(z), \quad \psi(\tilde{z}) = \psi(z)
\]

(2.6)

(a bar denotes the complex conjugate), and for \( g \) to be symmetric we require

\[
\phi(\tilde{x}) = \phi(x)
\]

(2.7)

(a tilde indicates transposition). Besides this, compatibility of Eqs. (2.7) with (2.3) requires

\[
x(z) = \lambda.
\]

(2.8)

where \( I \) is the unit matrix (here, and often from now on, we omit the arguments \( x \) and \( z \) for functions for simplicity).

The soliton solutions for the matrix \( g \) correspond, as is well known, to the presence of pole singularities of the matrix \( \chi(x, p, z) \) in the complex plane of the spectral parameter \( \lambda \). Let us consider the general case, in which the matrix \( \chi \) has \( \eta \) such poles, which we assume to be simple. The matrix \( \chi(x, p, z) \) can then be represented in the form

\[
x = \lambda_0 + \sum_{n=1}^{\eta} \frac{R_n}{\lambda - \mu_n},
\]

(2.9)

where the matrices \( R_n \) and the numerical functions \( \mu_n \) now depend only on the variables \( x \) and \( z \).

We note that in Ref. 1 an expression analogous to Eq. (2.9) was written in a form which obviously satisfies the condition (2.6) and which emphasizes the fact that complex poles (i.e., complex \( \mu_n \)) of the matrix \( \chi \) can exist only as conjugate pairs. Of course these requirements still hold here, but experience shows that writing \( \chi \) in the form (2.8) considerably facilitates the calculations, which it is convenient to do by neglecting the conditions (2.6) and supposing (until the final form of the solutions is reached) that we have do with \( n \) arbitrary complex poles \( \lambda = \mu_n (k = 1, 2, \ldots, n) \). After the final form of the solution is obtained it is easy to assure that the matrix \( g \) is real by imposing definite supplementary conditions on the arbitrary constants that appear in the solution.

This procedure is possible with an even number of complex poles in the sum (2.9), and is of course equivalent to introducing the complex poles at the very start as conjugate pairs. If, on the other hand, all of the \( \mu_n \) in the sum (2.9) are real, then all of the matrices \( R_n \) will also be real and the matrix \( \chi \) then satisfies Eq. (2.9) automatically.

Substitution of the expression (2.9) into Eq. (2.5) and the supplementary condition (2.7) completely determines the pole trajectories \( \mu_n (x, z) \) and the matrices \( R_n (x, z) \). The numerical functions \( \mu_n \) are determined from the requirement that in the left sides of Eqs. (2.5) there are no poles of second order at the points \( \lambda = \mu_n \). The result is that each function \( \mu_n (x, z) \) (with each index \( k = 1, 2, \ldots, n \)) satisfies a pair of differential equations

\[
\mu_n + 3 \mu_n (\mu_n - \mu_{n+1}) = 0, \quad \mu_{n+1} = 3 \mu_n (\mu_n - \mu_{n-1}) = 0,
\]

(2.10)

whose solutions are the roots of a quadratic algebraic equation

\[
\mu_n^2 - 2 \nu_n \mu_n - \nu_n^2 = 0,
\]

(2.11)

where \( \nu_n \) are arbitrary constants (in general complex).

Accordingly, for each index \( k \) (i.e., for each pole) we have its own arbitrary constant \( \nu_k \), which determines two possible solutions for the trajectory of the pole \( \mu_n (x, z) \):

\[
\mu_n \pm \nu_n (\nu_n - \nu_{n+1})\nu_n = \nu_n (\nu_n - \nu_{n-1})\nu_n.
\]

(2.12)

The matrices \( R_n \) are degenerate, and their compo-
nents can be written in the form

$$(\epsilon_{kl}) = \delta_{kl} \epsilon_{m}^{(0)} m_{k}^{(0)}$$

(2.13)

The two-component vectors $m_{i}^{(0)}$ are found directly from Eqs. (2.5) by requiring that they be satisfied at the poles $\lambda = \mu_{k}$ and the vectors $m_{i}^{(0)}$ are then determined from the condition (2.7). The vectors $m_{i}^{(0)}$ can be expressed in terms of the given partial solution for the "wave" matrix $\delta_{ij}(\mu, \nu, x)$ taken at the value $\mu_{k}$ for the argument $\lambda$. These vectors are of the following form:

$$m_{i}^{(0)} = m_{i}^{(0)}(\epsilon_{i}^{(0)}(\mu_{k}, \nu, x)),$$

(2.14)

where $\delta_{ij}$ denotes the matrix inverse to $\delta_{ij}$. (Here and from now on summation is to be understood over repeated vector and tensor indices $a, b, c, d, f$, which run through the values 0 and 1. Summation over other indices occurs only when explicitly indicated.) In Eq. (2.14) the $m_{i}^{(0)}$ are arbitrary constants.

The vectors $m_{i}^{(0)}$ can then be determined from the following $n$-th order system of algebraic equations:

$$\sum_{j=1}^{n} P_{ij} m_{j}^{(0)} = \epsilon_{i}^{(0)}, i = 1, 2, \ldots, n,$$

(2.15)

where the matrix $P_{ij}$ is symmetric and its elements are

$$P_{ij} = \delta_{ij}^{(0)}(\epsilon_{i}^{(0)}(\mu_{k}, \nu), \nu, x,)$$

(2.16)

and the matrix $\epsilon_{i}^{(0)}(\mu_{k}, \nu, x)$ is a given particular solution of the original equations (1.3). If we introduce the symmetric matrix $I_{ij}$ inverse to the matrix $P_{ij}$:

$$\sum_{j=1}^{n} P_{ij} m_{j}^{(0)} = \epsilon_{i}^{(0)}, i = 1, 2, \ldots, n,$$

(2.17)

then we get from (2.15) for the vectors $m_{i}^{(0)}$

$$m_{i}^{(0)} = \sum_{j=1}^{n} I_{ij} \epsilon_{j}^{(0)}.$$

(2.18)

According to Eqs. (2.3), (2.4), and (2.9) the required matrix $g$ is

$$g - g(0)(0)(0) = \epsilon_{ij} (\sum_{j=1}^{n} \mu_{j}(x).$$

(2.20)

Now, using Eqs. (2.13), (2.18), and (2.19) we get the metric components $\epsilon_{ij}$

$$\epsilon_{ij} = \sum_{j=1}^{n} \mu_{j}(x) \epsilon_{ij}^{(0)} m_{j}^{(0)}.$$

(2.21)

With the expression (2.21) the matrix $g$ is obviously symmetric. Let us now consider the question of its being real. If all of the functions $\mu_{j}(x, \nu)$ are real, the components $\epsilon_{ij}$ are automatically real, if we take all of the arbitrary constants appearing in the solution to be real. In fact, the particular solution $\delta_{ij}(\mu, \nu, x)$ is always taken to satisfy the second of the conditions (2.5), and consequently $\delta_{ij}(\mu)$ is real on the real axis of the $\lambda$ plane, i.e., at the points $\lambda = \mu_{k}$. It can now be seen from Eq. (2.14) that the arbitrary constants $m_{i}^{(0)}$ that occur in the vectors $m_{i}^{(0)}$ must be taken real, and then the vectors $m_{i}^{(0)}$ will also be real. It then follows that all of the other quantities from which the matrix $g$ is constructed are real. We now suppose that there are also complex values among the functions $\mu_{1}, \mu_{2}, \ldots, \mu_{n}$. The conditions (2.9) then require that all of the complex poles appear only as conjugate pairs; for each complex pole $\lambda = \mu_{k}$ its conjugate $\lambda = \bar{\mu}_{k}$ must also appear. Suppose there is such a pair of poles $\lambda = \mu_{k}$ and $\lambda = \bar{\mu}_{k}$ with $\mu_{k} = \bar{\mu}_{k}$. To these poles there correspond vectors $m_{i}^{(0)}$ and $m_{i}^{(0)}$, which according to Eq. (2.14) are given by

$$m_{i}^{(0)} = m_{i}^{(0)}(\epsilon_{i}^{(0)}(\mu_{k}, \nu, x)),$$

$$m_{i}^{(0)} = m_{i}^{(0)}(\epsilon_{i}^{(0)}(\bar{\mu}_{k}, \nu, x)).$$

(2.22)

A simple analysis shows that the matrix $g$ will be real if for each such pair of complex-conjugate poles the arbitrary constant $m_{i}^{(0)}$ and $m_{i}^{(0)}$ are taken conjugate to each other. This means that the vectors $m_{i}^{(0)}$ and $m_{i}^{(0)}$ corresponding to each pair of conjugate poles are also conjugate to each other [$m_{i}^{(0)} = m_{i}^{(0)}$, since the function $\epsilon_{ij}(\mu, \nu, x)$ satisfies the condition $\epsilon_{ij}(\lambda) = \epsilon_{ij}(\bar{\lambda})$. Accordingly, we can formulate the following rule that determines the choice of the arbitrary constants $m_{i}^{(0)}$ in Eq. (2.14): To assure that the matrix $g$ is real, it is necessary to choose the arbitrary $m_{i}^{(0)}$ in Eq. (2.14) so that the vectors $m_{i}^{(0)}$ corresponding to real poles $\lambda = \mu_{k}$ are real and the vectors $m_{i}^{(0)}$ and $m_{i}^{(0)}$ corresponding to each pair of complex-conjugate poles $\lambda = \mu_{k}$ and $\lambda = \bar{\mu}_{k}$ are complex conjugate to each other.

Satisfying the requirements that $g$ be real and symmetric is still not enough. It must not be forgotten that $g$ must also satisfy the supplementary condition (1.3). We now calculate the determinant of the matrix $g$. The form (2.21) is not convenient for this calculation, and we use a different representation of our solution. We note that the process of perturbing the background solution $\epsilon_{ij}$ and obtaining from it the $n$-soliton solution $g_{ij}$, as described above, is formally equivalent to the introduction of the $n$-solutions one at a time successively. The first step is to go from the background metric $\epsilon_{ij}$ to the metric $g_{ij}$ containing one soliton, corresponding to the presence in the matrix $g(x)$ which we at this stage call $\epsilon_{ij}$ only one pole $\lambda = \mu_{1}$.

This one-soliton solution is easily obtained from the results given above. The matrix $g_{ij}$ and its inverse $x_{ij}^{(-1)}$ can be written in the following form

$$\epsilon_{ij} = f(x) \epsilon_{ij}^{(0)} \epsilon_{ij}^{(0)} f(x),$$

(2.22)

where the matrix $P_{ij}$ has the elements

$$(\epsilon_{ij}, \epsilon_{ij}^{(0)}, m_{i}^{(0)}, m_{j}^{(0)}).$$

(2.23)

and accordingly has the following properties:

$$P_{ij} = P_{ij}, \quad \det P_{ij} = 1, \quad \det \epsilon_{ij} = 0.$$

(2.24)

The quantities $\mu_{k}$ and $m_{k}^{(0)}$ are given by Eqs. (2.12) and (2.14) with $k = 1$. We now get for the matrix $g_{ij}$

$$\epsilon_{ij} = f(x) \epsilon_{ij}^{(0)} \epsilon_{ij}^{(0)} f(x).$$

(2.25)

It is not hard to calculate the determinant of $g_{ij}$. Owing to the general relation

$$\det (g_{ij} f) = \det F \det f$$

(which holds for an arbitrary two-rowed matrix $F$) and the properties (2.24) we get

$$\det g_{ij} = \det \epsilon_{ij} = 0.$$
\[
\det ([I-(\mu \sigma + \nu \tau)]\phi_0) = -\phi_0
\]  
(2.26)

and consequently

\[
det g = -\phi_0^2 \det g_0.
\]  
(2.27)

We can now take the solution \( g_0 \) as a new particular or background solution and repeat the operation of adding a soliton to it, that corresponding to the pole \( \lambda = \mu_1 \). To do this we form the new background matrix function \( g_1 = g_0 g_0^{-1} \), take its inverse \( g_0^{-1} \) and calculate it at the point \( \lambda = \mu_1 \), and then find the corresponding vector \( M^{(1)}_{\mu_1} \):

\[
M^{(1)} = [\mu_1 - (\mu_1 - \lambda)]M^{(0)}
\]

after which we construct the matrix \( P_1 \), in analogy with Eq. (2.23):

\[
(P_1)_{\mu_1} = (\mu_1)^n M^{(0)}_{\mu_1 \mu_1} (\mu_1)^{-n} M^{(0)}_{\mu_1 \mu_1}
\]

which matrix has the same properties (2.24) as the matrix \( P_1 \).

When we now construct the matrix \( y_1(x) \) [this matrix is calculated from the same formulas (2.22), with the index 1 replaced with 2], we get the two-soliton solution \( g_2 \):

\[
g = [I-(\mu_2 + \nu_2)\phi_0^{-1} \mu_2 - (\mu_2 + \nu_2)\phi_0^{-1} \nu_2] \phi_0.
\]

Continuing this process, we get an \( n \)-soliton solution (2.21) in the form

\[
v = \left( \prod_{k=1}^{n} [I-(\mu_k + \nu_k)\phi_0^{-1}\mu_k - (\mu_k + \nu_k)\phi_0^{-1}\nu_k] \phi_0 \right) v_0
\]  
(2.28)

where all of the matrices \( P_k \) satisfy the same conditions as the matrix \( P_1 \) does:

\[
P_k = 0, \quad \det P_k = 0.
\]  
(2.29)

Naturally the explicit form of the matrices \( P_k \) rapidly becomes cumbersome as \( k \) increases, and therefore this way of calculating solutions is less convenient than the one previously described. But the representation of the solution in the form (2.28) is useful for the study of some particular questions, and especially for calculating the determinant of the matrix \( g \). The important thing for this is only that the matrices \( P_k \) have the properties (2.29), not their specific form. The contribution from each factor in Eq. (2.29) to the determinant of \( g \) can be calculated trivially, and the result is

\[
det g = (-1)^n \phi_0 \left( \prod_{k=1}^{n} \phi_0^{-1} \right) \det g_0.
\]  
(2.30)

If we take the particular solution \( g_0 \) as satisfying by definition the condition \( \det g_0 = -\phi_0^2 \), then it follows from Eq. (2.30) that the number of solitons \( n \) must always be even, since an odd number would change the sign of \( \det g_0 \) and violate the physical signature of the metric. Accordingly, in contrast with the case investigated earlier on a physical background all stationary axially symmetric solitons (even those which correspond to real poles \( \lambda = \mu_j \)) can appear only in pairs forming bound two-soliton states.

We still have to construct an \( n \)-soliton solution \( g \) which not only satisfies Eqs. (1.3) but also the supplementary condition (1.2). We shall call such a solution a physical one and denote it by \( g^{(n)} \). Constructing it is simple if we note that \( \det g \) for any solution \( g \) of Eq. (1.3) satisfies the equation

\[
p = (p(\ln \det g), v(\ln \det g)) = 0
\]

Then it is easy to verify that the matrix

\[
g^{(n)} = p^{(n)} \phi_0
\]  
(2.31)

also satisfies Eqs. (1.3), and also the condition \( \det g^{(n)} = -\phi_0^2 \). Now supposing the number \( n \) of solitons is even and \( \det g_{\mu_1} = -\phi_0^2 \), we get from Eqs. (2.20) and (2.31) the final expression for the metric tensor:

\[
g^{(n)} = -\phi_0 \left( \prod_{k=1}^{n} \phi_0 \right) \det g^{(n)} = -\phi_0
\]  
(2.32)

where the matrix \( g \) is given by Eq. (2.21).

3. CALCULATION OF THE METRIC COEFFICIENT \( f \)

It is also convenient to do the calculation of the coefficient \( f \) in two stages. First we calculate the value of \( f \) that follows from Eqs. (1.4) and (1.5) when we substitute in them the nonphysical solution \( g \) given by Eq. (2.21), which does not satisfy the condition \( \det g = -\phi_0^2 \), and then use a simple procedure to find the physical value of the coefficient, \( f^{(n)} \), which is obtained from these same Eqs. (1.4) and (1.5) when \( g^{(n)} \) is substituted in them instead of \( g \).

To calculate \( f \) we must determine from Eqs. (2.5) the matrices \( U \) and \( V \); this can be done by equating the left and right sides of these equations at the poles \( \lambda = \mu_1 \) and \( \lambda = -i \phi \) (cf. the analogous procedure in the previous paper). Then calculating the traces \( Sp(U) \) and \( Sp(V) \) and substituting them in Eqs. (1.4) and (1.5), we find \( f \) by direct integration. It is a remarkable fact that this integration can actually be carried out. The key point in calculating the coefficient \( f \) corresponding to an \( n \)-soliton solution is to determine it for a one-soliton solution (which coefficient we denote as \( f_1 \)), described by Eqs. (2.22)–(2.27). Having done the necessary calculations with the scheme indicated above (in analogy with the way this was done in Ref. 1), we get the following result for the one-soliton solution:

\[
j_1 = C_1 \lambda^{1/2} \phi_0 \phi_0^{-1/2} \sin \theta
\]  
(3.1)

where \( C_1 \) is an arbitrary constant, \( \phi_0 \) is the particular (background) solution for the coefficient \( f_1 \), which corresponds to the solution \( g_0 \), and \( \Gamma_1 \) is the single component of the matrix (2.16), which is all that exists in this case (\( k = 1 \) and \( l = 1 \)):

\[
\Gamma_1 = (\mu_1 + \nu_1) \phi_0^{-1} \mu_1 - (\mu_1 + \nu_1) \phi_0^{-1} \nu_1
\]  
(3.2)

(the vector \( \phi_0 \) follows from Eq. (2.14) for \( h = 1 \)).

The next step in the calculations is that, taking the solution \( g_0, f_1 \) as a new particular solution and repeating the operation just performed (as was explained in the foregoing section in connection with finding the matrix \( g_0 \)), we get the coefficient \( f_2 \) that corresponds to the two-soliton solution with the poles \( \lambda = \mu_1 \) and \( \lambda = \mu_2 \). At this second step we already have to deal with only calculations of an algebraic nature, since the seed for integration appears in the whole procedure only once, in the transition from the background solution \( g_0, f_1 \) to the solution \( g_1, f_2 \), which contains one soliton.
unning the details of the calculation, we give the only the result:

$$f_n = C_n \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \det \Gamma_n.$$  

(3.3)

Here $C_n$ is an arbitrary constant, $f_n$ is the same background solution as in Eq. (3.1), and $\Gamma_{n1}$, $\Gamma_{n2}$, and $\Gamma_{n3}$ are the components of the matrix (2.16). We now have three independent components of $\Gamma_{n\nu}$, since the indices $k$ and $l$ can take two values, 1 and 2.

Equations (3.1) and (3.3) suggest that in the general $n$-soliton case the coefficient $f$ is given by the expression

$$f_n = C_n \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \det \Gamma_n.$$  

(3.4)

(where $k, l = 1, 2, \ldots, n$). Since we see from Eqs. (3.1) and (3.3) that this formulat indeed holds for $n=1$ and $n=2$, we can prove that it holds in the general case by using the method of mathematical induction. This proof is given in the Appendix to the present paper, and shows that Eq. (3.4) is indeed correct in general.

Now we must determine the physical value $f^{(n)}$ of the coefficient, i.e., the value that would be obtained from Eqs. (1.4) and (1.5) if we substituted in them the physical matrix $f^{(n)}$ of Eq. (3.3) instead of $f$. From Eq. (3.3) we get the obvious relations

$$f^{(2)} = C_2 \left( \prod_{i=1}^{2} (\omega_i - \mu) \right) \left( \prod_{i=1}^{2} (\omega_i - \mu) \right) \det \Gamma_2.$$  

(3.5)

When we now substitute in Eqs. (1.4) and (1.5) the matrices $U^{(n)}$ and $V^{(n)}$ instead of $U$ and $V$, we find that the physical coefficient $f^{(n)}$ is given by the formula

$$f^{(n)} = C_n \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \left( \prod_{i=1}^{n} (\omega_i - \mu) \right) \det \Gamma_n.$$  

(3.6)

where $f_n$ is the value of this coefficient which is given by Eq. (3.4), and the function $Q$ is defined by the equations

$$Q = \prod_{i=1}^{n} (\omega_i - \mu)^{\nu_i}.$$  

(3.7)

On substituting here the expression (2.29) for $g$, we find that these equations are integrated exactly, and the answer can be written in the form

$$Q = \text{const} \prod_{i=1}^{n} (\omega_i - \mu)^{\nu_i}.$$  

(3.8)

From this and Eqs. (3.4) and (3.5) we get the final expression for the physical value of the coefficient $f$:

$$f^{(n)} = C_n \prod_{i=1}^{n} (\omega_i - \mu)^{\nu_i} \left( \prod_{i=1}^{n} (\omega_i - \mu) \right)^{\nu_i} \det \Gamma_n.$$  

(3.9)

$[C_n^{(n)}]$ is an arbitrary constant].

For clarity we point out that the product

$$\prod_{i=1}^{n} (\omega_i - \mu)^{\nu_i}$$

is equal to 1 for $n = 1$, to $(\mu_1 - \mu_2)^{\nu_1}$ for $n = 2$, to $(\mu_1 - \mu_2)^{\nu_1} (\mu_2 - \mu_3)^{\nu_2}$ for $n = 3$, and so on. In deriving Eq. (3.7) we have assumed that no two of the quantities $\mu_1, \mu_2, \ldots, \mu_n$ are equal.

 Accordingly, the final form of the $n$-soliton solution can be written in the form

$$-\omega_i = \sqrt{\omega_i + \omega_i} \left( \prod_{i=1}^{n} (\omega_i - \mu) \right)^{\nu_i} \det \Gamma_n.$$  

(3.10)

where $f^{(n)}$ is given by Eq. (3.7) and the matrix elements $C^{(n)}$ are determined by Eqs. (2.32) and (2.21).

4. TWO-SOLITON SOLUTION ON A FLAT BACKGROUND

In this and the following sections we consider the application of the results presented above to the case in which the background metric $\gamma_{\mu\nu}$ is flat and given by the interval

$$-\omega_i = -\omega_i \left( \prod_{i=1}^{n} (\omega_i - \mu) \right)^{\nu_i}. $$  

(4.1)

That is, $f_1 = 1$ and $g_{12} = \text{diag}(-1, \lambda^2)$ with the obvious property $\text{det}(g_{12}) = -\lambda^2$. The matrix $V_2$ is equal to zero, and for the matrix $E_2$ we have $E_2 = \text{diag}(0, 2)$. From Eq. (2.1) we get the corresponding solution for $\gamma_{(2)} = \gamma_{(2)}(\lambda, \mu, \nu)$:

$$\gamma_{(2)} = \begin{pmatrix} -1 & \lambda \\ \lambda & 1 \end{pmatrix},$$  

(4.2)

which satisfies the requirement $\gamma_{(2)}(0) = g_{12}$. From this and Eq. (2.14), using Eq. (2.11), we easily find the components of the vectors $C^{(1)}$ and $C^{(2)}$:

$$\mu^{(1)} = C^{(1)}, \mu^{(2)} = C^{(2)},$$  

(4.3)

where $C^{(1)}$ and $C^{(2)}$ are arbitrary constants.

Now from (2.16) we get the matrix $N_2$:

$$N_2 = -C^{(1)} C^{(1)} C^{(2)} C^{(2)} -2 \ln(-\lambda^2 - \mu^2),$$  

(4.4)

From Eq. (2.19) we get the components of the vectors $N_{1\nu}$:

$$N_{1\nu} = C^{(1)}, N_{2\nu} = C^{(2)}.$$  

(4.5)

Together with the expressions (2.12) for the functions $\mu_{\nu}$, we now have everything necessary for constructing the $n$-soliton solutions on a flat-space background.

Let us now consider the simplest case of all. As was already stated at the end of Sec. 2, solitons on a physical background (with either complex or real poles) can appear only in pairs. Consequently, the simplest case will be a two-soliton solution, corresponding to two poles, $\lambda = \mu$ and $\lambda = \mu$. It is not hard to show by direct calculation that what we have here is just the Kerr-NUT solution. In our previous paper i it was already pointed out that a double stationary soliton on a flat background, corresponding to a pair of complex-conjugate poles, gives a Kerr-NUT solution with an "anomalously large" rotational moment (i.e., a solution without horizons and with a bare singularity). In fact, here we get precisely this situation for $\mu_1 = \mu_2$. On the other hand, if both functions, $\mu_1$ and $\mu_2$, are real, the solution corresponds to the "normal" situation, with the singularity hidden from an outside observer by horizons.

These assertions can be verified by direct calculation of the metric. Let us represent the constant $\nu_1$ and $\nu_2$
that appear in the relations (2.11) and (2.12) in the form
\[ v^2 - 2a \cos \alpha \sin \theta + \alpha^2 = 0, \]
where \( \alpha \) and \( \beta \) are new arbitrary constants. We now introduce instead of \( p \) and \( q \) two new coordinates \( \tau \) and \( \theta \):
\[ \rho = (r-a \sin \theta)^{-1} \sin \theta, \quad z = (r-a \cos \theta) \cos \theta, \]
where \( m \) is an arbitrary constant whose meaning will be clear later. Then from Eq. (2.12) it is easy to express the quantities \( \mu \) and \( \mu_2 \), in terms of the new variables \( \tau \) and \( \theta \). In this calculation we can choose the signs in the formula (2.12) either the same for \( \mu_1 \) and \( \mu_2 \) or else opposite. It is not hard to show that both cases lead to the same metric (to within linear transformations of the two coordinates \( t, \phi \) in terms of each other, and a trivial conformal transformation, multiplication of the interval with a constant).

Let us consider first the case of like signs. If we choose the plus sign in Eq. (2.12) for both values \( \mu_1 \) and \( \mu_2 \), then substituting the expressions (4.6) and (4.7), we get
\[ \mu = 2(\rho - a \cos \theta) \sin \theta, \quad n = -2(\rho - a \cos \theta) \sin \theta. \]

From this (using the expression (4.7) for \( p \)) and from Eq. (4.5) we find the components of the vectors \( C_{11} \) and \( C_{21} \), and from Eq. (4.4) we find the matrix \( D_1 \) and its inverse \( D_1^{-1} \) (in this case \( b, \ell = 1, 2 \)). After this we get from Eqs. (2.33) and (2.21) the components of the metric tensor \( g^{(4)} \) and from Eq. (3.7) the metric coefficient \( f^{(4)} \). Substitution of these quantities in the interval (3.6) gives the final form of the solution, which can be reduced by simple linear transformations of the coordinates to the standard form of the Kerr-NUT solution in Boyer-Lindquist coordinates.

Omitting details, we point out that without loss of generality we can select the arbitrary constants \( C_{11} \) and \( C_{21} \) that appear in the expressions (4.3) for the vectors \( m^{\alpha \beta} \) to two conditions:
\[ C_{11} \left[ c_{11}^{\alpha} c_{11}^{\beta} + c_{12}^{\alpha} c_{12}^{\beta} \right] = m, \quad C_{21} \left[ c_{11}^{\alpha} c_{11}^{\beta} - c_{12}^{\alpha} c_{12}^{\beta} \right] = -m, \]
which are equivalent to the requirement that the variable \( r \) indeed be the Boyer-Lindquist radial coordinate. We then introduce two arbitrary constants \( a \) and \( b \) defined by
\[ C_{11} \left[ c_{11}^{\alpha} c_{11}^{\beta} + c_{12}^{\alpha} c_{12}^{\beta} \right] = a, \quad C_{11} \left[ c_{11}^{\alpha} c_{11}^{\beta} - c_{12}^{\alpha} c_{12}^{\beta} \right] = -b. \]

From Eqs. (4.9) and (4.10) it follows that
\[ d\tau = -a'r - a \sin \theta \cos \theta d\theta. \]

Now the metric (3.8) contains only the constants \( m, a, b \) and takes the form
\[ ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2 + (3 - a^2 \sin^2 \theta) d\tau^2 \]
\[ -4a^2 \cos \theta \sin \theta d\tau d\theta + (3 - a^2 \sin^2 \theta) d\theta^2 \]
\[ -a' \sin \theta \cos \theta d\theta d\tau - a' \sin \theta \cos \theta (d\tau)^2, \]
where the variable \( \tau \) is connected with \( t \) (the original coordinate \( x = \delta t \)) by the relation
\[ t = t + \Delta \tau, \]
and the quantities \( \omega \) and \( \Delta \) are
\[ \omega = a' - 3a^2 \cos \theta, \quad \Delta = -a^2 \sin \theta \cos \theta. \]

It can be seen from this that the Kerr-NUT solution with horizons corresponds to real poles \( \lambda = \mu_1 \) and \( \lambda = \mu_2 \), since in this case the constant \( \sigma \) is real \( (m^2 + b^2 > 0) \), and the constants \( w_1 \) and \( w_2 \) and the functions \( \mu_1 \) and \( \mu_2 \) are real along with \( \sigma \). If the quantity \( \sigma \) is imaginary \( (m^2 + b^2 < 0) \), then the constants \( w_1 \) and \( w_2 \) and the functions \( \mu_1 \) and \( \mu_2 \) are complex and conjugate to each other. This case corresponds to a solution without horizons. Furthermore the metric (4.12) and the constants \( m, a, b \) are of course still real, but the original constants \( C_{11} \) as Eqs. (4.9) and (4.10) show, must be taken complex and related by \( C_{11}^{*} = C_{11}^{\dagger} \), which, as we see from Eq. (4.3), means that also \( m^{\alpha \beta} = m^{\alpha \beta} \). This agrees with the rule for choosing solutions with a complex-conjugate pair of poles that were formulated earlier in Sec. 2.

Let us now look at the second possibility for choosing the solutions of Eqs. (2.11), the one that corresponds to using different signs in Eq. (2.12). Choosing the plus sign for \( \mu_1 \) and the minus sign for \( \mu_2 \), we get
\[ m = 2(\rho - a \cos \theta) \sin \theta, \quad n = -2(\rho - a \cos \theta) \sin \theta. \]

Calculations like the foregoing ones show that in this case we again arrive at a Kerr-NUT metric, the only difference being that instead of the variables \( t, \phi \) we will now have new coordinates \( t', \phi' \), connected with the original variables \( x', \theta' \) by a linear transformation different from that in Eq. (4.13). The new relations are \( t' = c_{11}^{\alpha} t + c_{12}^{\alpha} \phi, \phi' = c_{11}^{\beta} t + c_{12}^{\beta} \phi \), where the coefficients are real only if the constant \( \sigma \) is real (i.e., if \( \mu_1 \) and \( \mu_2 \) are real), and become complex when \( \sigma \) is imaginary. This means that for imaginary \( \sigma \) the matrix is complex in the original coordinates \( t, \phi \); this is quite natural, since in this case, as can be seen from Eq. (4.15), the poles \( \lambda = \mu_1 \) and \( \lambda = \mu_2 \) do not compose a complex-conjugate pair.

Besides this, the connection between the arbitrary constants \( C_{11} \) and \( C_{21} \) and the parameters \( m, a, b \) are now different:
\[ C_{11} \left[ c_{11}^{\alpha} c_{11}^{\beta} + c_{12}^{\alpha} c_{12}^{\beta} \right] = a, \quad C_{11} \left[ c_{11}^{\alpha} c_{11}^{\beta} - c_{12}^{\alpha} c_{12}^{\beta} \right] = -b, \]
but the relation (4.11) between \( \sigma \) and the constants \( m, a, b \) is still valid.

In conclusion we point out that the only actual physical solution is that of Kerr, since the presence of the NUT parameter \( b \) makes the metric no longer asymptotically Euclidean and produces a number of nonphysical properties of the solution (the relevant analysis has been given by Misner).

5. THE \( n \)-SOLITON SOLUTION ON A FLAT BACKGROUND

In this section we consider some general properties of the \( n \)-soliton solution, confining ourselves to one of its possible types. We shall assume that on the background of a flat space with the metric (4.1) an even number of solitons are introduced, corresponding to the poles \( \lambda = \mu_1, \lambda = \mu_2, \ldots, \lambda = \mu_n \). We divide all of the functions \( \mu_1, \mu_2, \ldots, \mu_n \) into pairs and introduce the Greek index \( \gamma \), which will number these pairs and takes only the
odd values from 1 to \( n - 1 \): \( \gamma = 1, 3, \ldots, n - 1 \). We thus have \( n/2 \) pairs of pole trajectories \((\mu_\alpha, \nu_\alpha)\).

To understand the physical meaning of the solution it is helpful to examine first a special case which corresponds to a diagonal matrix \( g_{\alpha\beta} \), i.e., to a static \( n \)-soliton field remaining after the rotation has been turned off. To obtain such a special case we set all of the arbitrary constants \( C_{\alpha\beta} \) in Eq. (4.3) equal to zero, and then all the \( a_{\alpha\beta}^{(n)} \) also equal to zero. It now follows from Eq. (2.15) that all the \( d_{\alpha\beta}^{(n)} = 0 \) and the matrices \( R_{\alpha\beta} \), as we can see from Eq. (2.15) take the form

\[
R_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & d_{\alpha\beta} \end{pmatrix}.
\]

This means that all the matrices \( P_{\alpha\beta} \) in the representation (2.28) of the solution take the form

\[
P_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

in accordance with the conditions (2.29). Then from Eqs. (2.28) and (2.32) we get the following solution for the diagonal case under consideration:

\[
a_{\alpha\beta}^{(n)} = -\frac{1}{n} \sum_{j=1}^{n} a_{\alpha\beta}(\phi_j) = 0 \quad \text{for all } \alpha, \beta.
\]

The metric coefficient \( f^{(n)} \) can be found from Eq. (3.7); to do so we must calculate the determinant of the matrix \( F_{\alpha\beta}(\phi_j) \). It is simpler, however, to determine \( f^{(n)} \) directly from Eqs. (1.4) and (1.5), since the solution (5.1) is simple and easy to integrate. The result is

\[
f^{(n)} = 1 - \frac{2m}{r} - \frac{2m}{r}.
\]

We now determine from Eqs. (2.11) and (2.12) the functions \( \mu_{\alpha} \) which we have arranged in the pairs \((\mu_{\alpha}, \nu_{\alpha})\). Confining our treatment to the case when the signs in Eq. (2.13) are chosen differently for the functions of each pair, we have

\[
\mu_{\alpha} = m_{\alpha} - (\mu_{\alpha} - \mu_{\alpha}^{-1}) (\nu_{\alpha} - \nu_{\alpha}^{-1})/2.
\]

Instead of each pair of arbitrary constants \( \mu_{\alpha} \) and \( \nu_{\alpha} \), we introduce new constants \( s_{\alpha} \) and \( m_{\alpha} \), setting

\[
\mu_{\alpha} = s_{\alpha} - m_{\alpha}, \quad \nu_{\alpha} = s_{\alpha} + m_{\alpha}.
\]

If we now introduce \( n/2 \) pairs of functions \( r_{\alpha}(\nu_{\alpha}, \phi) \) and \( \delta_{\alpha}(\nu_{\alpha}, \phi) \) (giving to each pair of poles its own "radial and angular coordinates") through the relations

\[
r_{\alpha} \equiv r_{\alpha}(\nu_{\alpha}, \phi), \quad \delta_{\alpha} \equiv \delta_{\alpha}(\nu_{\alpha}, \phi).
\]

we get from Eq. (5.3)

\[
\nu_{\alpha} = 2(\nu_{\alpha} - 2m_{\alpha}) \sin^2 \frac{\phi}{2},
\]

\[
\mu_{\alpha} = 2(\nu_{\alpha} - 2m_{\alpha}) \cos^2 \frac{\phi}{2}.
\]

Using these expressions for \( \rho \) and \( \mu_{\alpha} \), we get from Eq. (5.1) the component \( g_{\alpha\beta}^{(n)} \) as the following product of \( n/2 \) factors:

\[
g_{\alpha\beta}^{(n)} = (1 - 2m_{\alpha}^{(n)})(1 - 2m_{\alpha}^{(n)}) \ldots (1 - 2m_{\alpha}^{(n)}).
\]

For the case of the two-soliton solution Eq. (5.7) will have only one factor, the Schwarzschild expression for the coefficient \( g^{(2)} \). Calculating from Eq. (5.2) the coefficient \( f^{(2)} \) for this case and writing out the integrals, we indeed get the standard expression for the Schwarzschild metric with radial coordinate \( r \) and polar angle \( \theta \). This result also follows, of course, from the general form of the two-soliton Kerr-NUT solution, given in the preceding section [case (4.15), (4.16)] with \( C_{\alpha\beta} = C_{\alpha\beta}^{(2)} = 0 \) 

To interpret the \( n \)-soliton static solution with the "potential" (5.7) we must choose a suitable radial variable. Any one of the functions \( r_{\alpha}(\nu_{\alpha}, \phi) \) could now serve as a radial coordinate, but it is most natural to define the radial variable in such a way that the dipole moment relative to it vanishes in the expansion at infinity of the Newtonian potential of the system in question. As is well known, the Newtonian potential here is \( \Phi = 1 + r g^{(n)} \), and from Eq. (5.7) we have

\[
\Phi = 1 - (1 - 2m_{\alpha}^{(n)})(1 - 2m_{\alpha}^{(n)}) \ldots (1 - 2m_{\alpha}^{(n)}).
\]

Let us try to define the "true" radial coordinate \( r \) and polar angle \( \theta \) by relations of the same form as Eq. (5.5):

\[
\rho = r(-2m_{\alpha}^{(n)}) \sin \theta, \quad z = z(-m_{\alpha}^{(n)}) \cos \theta,
\]

but with new constants \( m_{\alpha} \) and \( \nu_{\alpha} \), which are subject to definition. From Eqs. (5.9) and (5.10) we can find functions \( r_{\alpha}(\nu_{\alpha}, \phi) \) and \( \delta_{\alpha}(\nu_{\alpha}, \phi) \) and obtain their asymptotic expansions for \( r \to \infty \) (in the first approximation we have for \( r \to \infty \) simply \( r_{\alpha} = r + \delta_{\alpha} = \theta \)). Substituting these expansions into Eq. (5.8), we find the expansion of the potential \( \Phi \), and from the condition that it must contain no dipole term we can determine the constants \( m_{\alpha} \) and \( \nu_{\alpha} \).

In this way we get

\[
m_{\alpha} = \sum_{\beta=1}^{n/2} m_{\beta}, \quad \nu_{\alpha} = \left( \sum_{\beta=1}^{n/2} m_{\beta} \right) / \sum_{\beta=1}^{n/2} m_{\beta}, \quad (5.10)
\]

and then the expansion for \( \Phi \) takes the form

\[
\Phi = 2m_{\alpha}^{(n)} \cos \theta \sin \theta - 1 \cos \theta - \ldots
\]

(5.11)

where \( \phi \) is the quadrupole moment of the system. For the case of a four-soliton solution, for example, (where the index \( \gamma \) takes only the two values 1 and 3) we have

\[
m_{\alpha} = m_{\alpha} - m_{\alpha}^{(n)} = m_{\alpha} - m_{\alpha}^{(n)}.
\]

These results show that the \( n \)-soliton static solution is a localized perturbation in an asymptotically flat space. For a sufficiently remote observer such a field can be regarded as an external field produced by \( n/2 \) localized axially symmetric structures, each of which has its own mass \( m_{\alpha} \) and its center of mass lying on the axis of symmetry at the point with coordinate \( z \). The equations (5.10) show that the total mass of all these \( n/2 \) objects (or pairs of solitons) is equal to the sum of their masses, and the coordinate \( z \) of their common center of gravity is given by the usual expression of the mechanics of particles. All of the multipole moments of the field can also be expressed in definite ways in terms of the constants \( m_{\alpha} \) and \( \nu_{\alpha} \).
Here and for what follows we have adopted the following conventions about indices: \( n \) and \( m \) are constant; the letters \( k \) and \( l \) are used to denote running indices which go through the values \( 1, 2, \ldots, m \); and the Greek letters \( \alpha \), \( \beta \) denote indices (appearing later) that go through the \( m+1 \) values 0, 1, 2, \ldots, \( m \).

As we have already said,

\[
D_{n,m} = \det \Gamma_{n,m} = 0. \tag{A.3}
\]

The vectors \( \mathbf{m}^{(n)} \) in Eq. (A.2) are constructed according to the rule (2.14):

\[
m^{(n)} = m^{(n)} \{ e^{-\gamma \theta' \delta} \} \tag{A.4}
\]

Let us now consider the solution \( e_\alpha, \Phi_\beta \) obtained from another solution \( e_{\alpha'}, \Phi_{\beta'} \) by adding to it one soliton, corresponding to the pole \( \lambda = \mu _\text{a} \). In this case, according to Eqs. (2.22) and (2.25) we have

\[
\begin{align*}
e_\alpha &= (\alpha' + \beta' + \beta \mu _\alpha)^{-1} (\alpha' + \beta' + \beta \mu _\alpha), \\
\Phi &= (\alpha' + \beta' + \beta \mu _\alpha)^{-1} (\alpha' + \beta' + \beta \mu _\alpha).
\end{align*} \tag{A.5}
\]

The matrix \( P_n \) is constructed from \( \Phi_{\beta} \) and \( \Phi_{\beta'} \) according to the law (2.23):

\[
P_n = \Phi_{\beta}(x_1, \ldots, x_n)(x_{n+1}, \ldots, x_{m+1}), \tag{A.6}
\]

where the vector \( \Phi_n \) is given by the expression

\[
\Phi_n = \Phi_{\beta}(x_1, \ldots, x_n). \tag{A.7}
\]

Besides the vector \( \Phi_n \) we need the vectors \( \mathbf{m}^{(n)} \) (\( k = 1, 2, \ldots, n \)), which are given by

\[
m^{(n)} = e_\alpha^{(n)} \{ e^{-\gamma \theta' \delta} \} \tag{A.8}
\]

where \( m^{(n)} \) are the same arbitrary constants as appear in Eq. (A.4).

Now from Eq. (A.4), (A.5), and (A.7) we can obtain an expression for the vectors \( \mathbf{m}^{(n)} \) in terms of the vectors \( \mathbf{m}_k^{(n)} \) and \( \mathbf{m}_{k+1}^{(n)} \):

\[
m^{(n)} = m_{-k}^{(n)} \{ e^{-\gamma \theta' \delta} \} E_{n-k,n}^{(n)} \tag{A.9}
\]

where we have introduced the matrix \( E_{n-k,n}^{(n)} \) (\( k = 1, 2, \ldots, m \)):

\[
E_{n-k,n}^{(n)} = e_{\alpha'}^{(n)} \{ e^{-\gamma \theta' \delta} \} E_{n-k,n} \tag{A.10}
\]

Then, substituting Eqs. (A.10) and (A.9) in (A.2), we find an expression for the matrix \( 
\Gamma_{n-k,n}^{(n)} \) in terms of the matrix \( E_{n-k,n}^{(n)} \):

\[

\begin{align*}
\Gamma_{n-k,n}^{(n)} &= E_{n-k,n}^{(n)}(e_\alpha^{(n)}\{e^{-\gamma \theta' \delta}\})^{-1}E_{n-k,n}^{(n)}, \\
&= E_{n-k,n}^{(n)}(e_\alpha^{(n)}\{e^{-\gamma \theta' \delta}\})^{-1}E_{n-k,n}^{(n)}.
\end{align*} \tag{A.11}
\]

From Eq. (A.12) it follows that the determinants of the matrices \( E_{n-k,n}^{(n)} \) and \( \Gamma_{n-k,n}^{(n)} \) are connected by the relation

\[
\det E_{n-k,n}^{(n)} = \det \Gamma_{n-k,n}^{(n)}. \tag{A.13}
\]

Now from Eqs. (3.1) and (3.2) we get a connection between \( \Gamma _{n,k} \) and \( \Gamma_{n-k,n}^{(n)} \):

\[
\Gamma _{n,k} = E_{n-k,n}^{(n)} \{ e^{-\gamma \theta' \delta} \} \Gamma_{n-k,n}^{(n)}. \tag{A.14}
\]

Substituting this expression in Eq. (A.1) and using Eqs. (A.3) and (A.13), we get

\[
\Gamma _{n,k} = \det E_{n-k,n}^{(n)} \{ e^{-\gamma \theta' \delta} \} \Gamma_{n-k,n}^{(n)}. \tag{A.15}
\]
Search for unusual decays of superdense nuclei using two-meter hydrogen and propane bubble chambers


Joint Institute for Nuclear Research (Submitted 20 January 1979)

Preliminary results are reported on the determination of the upper limits of the cross sections for production of superdense nuclei, by detection of their unusual decays occurring with times in the millisecond range. A special mode of bubble-chamber operation is proposed. It is shown that by use of this technique it is possible to determine comparatively simple cross sections at the level $10^{-24}$ cm$^2$ per nucleus.

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In recent years, especially with the appearance of A. B. Migdal's theory of the pion condensate, great interest has arisen in the search for superdense nuclei. It is expected that they can have a very large binding energy, and therefore it is possible in principle to observe the decays of such nuclei, which occur with a large energy release. On the other hand it is known that for decays of ordinary nuclei in the case when the decay electrons are relativistic we have the relation

$$\tau \approx \frac{1}{E_{\text{max}}}$$

where $\tau$ is the lifetime of the nucleus and $E_{\text{max}}$ is the maximum energy of the decay electrons. If we assume that this relation will be satisfied also for decays of superdense nuclei, then for a maximum decay-electron energy $E_{\text{max}} = 18$ MeV $\tau_1$ will be 1.6 times less than the lifetime of NS and will amount to 6.7 msec, and for $E_{\text{max}} = 36$ MeV $\tau_2 = 0.2$ msec, etc. Measurement of these lifetimes and decay energies can be carried out very satisfactorily by means of bubble chambers.

The advantages of the bubble-chamber technique are a 4\pi geometry, the possibility of detecting decay particles of various types ($\mu^+$, $\gamma$, heavier particles), the accurate measurement of their energies, and also the possibility of observing "explosions" of superdense nuclei which result in stars recorded in the chamber. A major advantage of this experimental arrangement is the absence of any ordinary physical process imitating the effect.

Up to the present time there has been no experimental proof of the existence of superdense nuclei. Kulikov and Pontecorvo1 presented some data obtained by an electronic technique on determination of the upper limits of the cross sections for production of superdense nuclei, as a function of their lifetime. It is evident from these data that the region of lifetimes $\tau_2 < 5$ msec has not yet been investigated.

The experiments described in the present paper were intended to search for unusual decays, which can arise from superdense nuclei, with energy more than 16.4 MeV (the maximum energy of the decays known up to this time) and occurring with lifetimes 0.5-1000 msec.