EQUIVALENCE OF THE NONLINEAR SCHRÖDINGER EQUATION AND THE EQUATION OF A HEISENBERG FERROMAGNET

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The concept of gauge equivalence is introduced for nonlinear equations that can be integrated by the inverse scattering technique. It is shown that the nonlinear Schrödinger equation is equivalent to a continuous isotropic chain of Heisenberg spins.

The equations mentioned in the title have the form

$$i\psi_t+\psi_{xx}+2|\psi|^2\psi=0, \quad -\infty<x, \quad t<\infty, \tag{1}$$

which is the nonlinear Schrödinger equation in the case of attraction, and

$$\mathbf{S}_t=\mathbf{S}\times\mathbf{S}_{xx}, \ \mathbf{S}(x,t)\in\mathbf{R}^3, \ \mathbf{S}^2=1, \tag{2}$$

for a continuous chain of Heisenberg spins in the isotropic case, the isotropic Heisenberg ferromagnet. Equation (1) is encountered in nonlinear optics and in plasma physics, and its quantum variant describes a many-particle system with delta-function interaction. In [1], the method of the inverse scattering problem was applied to the nonlinear Schrödinger equation and in [2] complete integrability was proved; in [3], quasi-classical quantization of Eq. (1) was carried out. In [4], the inverse scattering method was applied to the Heisenberg ferromagnet, and in [5] a connection between the solutions of Eqs. (1) and (2) was established. Namely, it was shown in [5] that the energy and momentum densities for the solution of Eq. (2) are essentially the square of the modulus and the derivative of the argument of the solution of Eq. (1).

In the present paper, using the results of [1, 4], we prove gauge equivalence of Eqs. (1) and (2). In particular, the results of [5], with which we were acquainted after we had completed this paper, are simple consequences of this equivalence.

To introduce the concept of gauge equivalence of nonlinear equations that are integrable by the inverse scattering technique, we recall the main features of this method. It is applied to equations that arise as consistency conditions of a system of linear differential equations,

$$\Phi_x=U(x,t,\lambda)\Phi, \quad \Phi_t=V(x,t,\lambda)\Phi, \tag{3}$$

where $\Phi\in GL(n, \mathrm{C})$, U, $V\in M(n, \mathrm{C})$, $\lambda\in\mathrm{C}$. The consistency conditions have the form

$$U_t-V_x+[U, V]=0 \tag{4}$$

for all $\lambda \in \mathbf{C}$, and under the assumption that U and V are meromorphic functions of λ they give a system of nonlinear partial differential equations for the coefficients of the Laurent expansions of the functions U and V. These equations can be integrated by means of the inverse scattering technique using the system (3). From the geometrical point of view, the functions U and V can be interpreted as connection coefficients in the fiber bundle with base $\mathbf{R}^2$ and fiber $GL(n, \mathbf{C})$; Eq. (4) means that the curvature of this connection is zero (i.e., the connection is flat).

Two systems of nonlinear equations that are integrable by the inverse scattering method are said to be gauge equivalent if the corresponding flat connections U_j, V_j, $j=1, 2$, are defined in the same fiber bundle and are obtained from each other by a λ-independent gauge transformation, i.e., if

$$U_1=gU_2g^{-1}+g_xg^{-1}, \quad V_1=gV_2g^{-1}+g_tg^{-1},$$

where $g(x, t)\in GL(n, \mathrm{C})$. It is clear that in the corresponding systems of linear differential equations we then have $\Phi_1=g\Phi_2$.

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The plan of the present paper is as follows. In the first section, we give the necessary results from [1, 4] in a convenient form. In the second section, we prove the gauge equivalence of Eqs. (1) and (2), and in the third we find expressions for the first conservation laws of Eq. (2) in terms of the scattering data.

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1. Method of the Inverse Scattering Problem for the Nonlinear Schrödinger Equation and the Heisenberg Ferromagnet

It is well known (see [1], and also [6]) that Eq. (1) is the consistency condition for the system

$$\Phi_{1x}=U_1(x,t,\lambda)\Phi_1,\quad \Phi_{1t}=V_1(x,t,\lambda)\Phi_1, \tag{5}$$

where

$$U_1=A_0+\lambda A_1,\quad V_1=B_0+\lambda B_1+\lambda^2 B_2,\quad A_0=\begin{pmatrix}0 & \bar\psi\\ -\psi & 0\end{pmatrix},\quad A_1=i\sigma_3,\quad B_0=\frac{1}{i}\begin{pmatrix}|\psi|^2 & \bar\psi_x\\ \psi_x & -|\psi|^2\end{pmatrix},\quad B_1=2A_0,\quad B_2=2A_1. \tag{6}$$

Here, σ_1, σ_2, σ_3 are the Pauli matrices,

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Under the condition that the function $\psi(x)$ decreases sufficiently rapidly when $|x|\to\infty$, the first equation in (5) for matrix solutions has the matrix Jost solutions $f_1(x,\lambda)$, $g_1(x,\lambda)$, which admit the integral representations

$$f_1(x,\lambda)=e^{i\lambda x\sigma_3}+\int_x^\infty K_1(x,y)e^{i\lambda y\sigma_3}dy,\quad g_1(x,\lambda)=e^{i\lambda x\sigma_3}+\int_{-\infty}^{n} N_1(x,y)e^{i\lambda y\sigma_3}dy.$$

These solutions are related by the transition matrix $T_1(\lambda)$:

$$f_1(x,\lambda)=g_1(x,\lambda)T_1(\lambda),\quad -\infty<\lambda<\infty, \tag{7}$$

and this matrix has the form

$$T_1(\lambda)=\begin{pmatrix}a_1(\lambda) & -\overline{b_1(\lambda)}\\ b_1(\lambda) & \overline{a_1(\lambda)}\end{pmatrix}.$$

The transition matrix is unimodular and the coefficient $a_1(\lambda)$ can be analytically continued into the half-plane $\operatorname{Im}\lambda\ge 0$, where it has the asymptotic behavior $a_1(\lambda)=1+O(1/|\lambda|)$ as $|\lambda|\to\infty$. Its zeros in the upper half-plane ζ_{1j}, $j=1,\ldots,n_1$, are eigenvalues of the discrete spectrum of the first equation in (5). We introduce the reflection coefficient $r_1(\lambda)=b_1(\lambda)/a_1(\lambda)$. The scattering data for our problem is the set $\{r_1(\lambda), -\infty<\lambda<\infty, \zeta_{1j}, m_{1j}, \operatorname{Im}\zeta_{1j}>0, j=1,\ldots,n_1\}$, and the corresponding Gel'fand–Levitan–Marchenko equation is

$$K_1(x,y)+F_1(x+y)+\int_x^\infty K_1(x,z)F_1(z+y)\,dz=0 \tag{8}$$

for $x\le y$, where

$$F_1(x)=i\operatorname{Im}\varphi_1(x)\cdot\sigma_1+i\operatorname{Re}\varphi_1(x)\cdot\sigma_2,\quad \varphi_1(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}r_1(\lambda)e^{i\lambda x}d\lambda+\sum_{j=1}^{n_1}m_{1j}e^{i\zeta_{1j}x}. \tag{9}$$

For complete description of the solutions of Eq. (1), it remains to point out that the matrix A_0 can be found from the relation $A_0(x)=\sigma_3K_1(x,x)\sigma_3-K_1(x,x)$, and that the dependence of the scattering data on t is determined by the second equation in (5) and given by

$$a_1(\lambda,t)=a_1(\lambda,0),\quad b_1(\lambda,t)=e^{4i\lambda^2t}b_1(\lambda,0),\quad \zeta_{1j}(t)=\zeta_{1j}(0),\quad m_{1j}(t)=e^{4i\zeta_{1j}^2t}m_{1j}(0),\ j=1,\ldots,n_1. \tag{10}$$

We now turn to Eq. (2). Using the Pauli matrices, we write it in the form

$$S_t=\frac{1}{2i}[S,S_{xx}], \tag{11}$$

where $S=(\mathbf{S},\boldsymbol{\sigma})$, $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$,

$$S^2=I,\ S=S^+,\ \operatorname{tr}S=0. \tag{12}$$

It is readily seen (see [4]) that Eq. (11)-(12) is the consistency condition for the system

$$\Phi_{2x}=U_2(x,t,\lambda)\Phi_2,\quad \Phi_{2t}=V_2(x,t,\lambda)\Phi_2, \tag{13}$$

where

$$U_2=i\lambda S,\ V_2=\lambda SS_x+2i\lambda^2 S. \tag{14}$$

We note, although we shall not require this fact in what follows, that Eq. (11) is the consistency condition for the system (13)-(14) in the case of matrices S of arbitrary (even infinite) dimension subject to the single condition $S^2 = I$.

For Eq. (2), natural boundary conditions are

$$\lim_{|x|\to\infty}\mathbf{S}(x,t)=\mathbf{S}_0,\quad \mathbf{S}_0=(0,0,1),$$

i.e.,

$$\lim_{|x|\to\infty}S(x,t)=\sigma_3. \tag{15}$$

If S(x) tends sufficiently rapidly to its limit in (15), the Jost solutions $f_2(x,\lambda)$, $g_2(x,\lambda)$ of the first equation in (13) have the integral representations

$$f_2(x,\lambda)=e^{i\lambda x\sigma_3}+\lambda\int_x^\infty K_2(x,y)e^{i\lambda y\sigma_3}dy,\quad g_2(x,\lambda)=e^{i\lambda x\sigma_3}+\lambda\int_{-\infty}^x N_2(x,y)e^{i\lambda y\sigma_3}dy,$$

where, for example, the kernel $K_2(x, y)$ is a solution of the Goursat problem

$$K_{2x}\sigma_3+S(x)K_{2y}=0 \text{ for } x\leqslant y,\quad S(x)-\sigma_3-iK_2(x,x)+iS(x)K_2(x,x)\sigma_3=0. \tag{16}$$

These solutions are related by the unimodular transition matrix $T_2(\lambda)$:

$$f_2(x,\lambda)=g_2(x,\lambda)T_2(\lambda), \tag{17}$$

the matrix having the form

$$T_2(\lambda)=\begin{pmatrix} a_2(\lambda) & -b_2(\lambda)\\ \overline{b_2(\lambda)} & \overline{a_2(\lambda)}\end{pmatrix}.$$

In addition, $T_2(0) = I$ and the coefficient $a_2(\lambda)$ admits analytic continuation into the half-plane $\operatorname{Im}\lambda \geq 0$, where it has the asymptotic behavior $a_2(\lambda)=e^{i\alpha}+O(1/|\lambda|)$ as $|\lambda|\to\infty$, and $\alpha\in\mathbf{R}^1$. The scattering data are the set $\{r_2(\lambda),\ -\infty<\lambda<\infty,\ \zeta_{2j},\ m_{2j},\ \operatorname{Im}\zeta_{2j}>0,\ j=1,\ldots,n_2\}$, where $r_2(\lambda)=b_2(\lambda)/a_2(\lambda)$ is the transmission coefficient, ζ_{2j} are the zeros of $a_2(\lambda)$, and m_{2j} are normalizing factors for the eigenfunctions of the discrete spectrum. In our case, the Gel'fand–Levitan–Marchenko equation has the form

$$K_2(x,y)+F_2(x+y)+\int_x^\infty K_2(x,z)F_2(z+y)dz=0 \tag{18}$$

for $x \leq y$, where

$$F_2(x)=i\operatorname{Im}\varphi_2(x)\cdot\sigma_1+i\operatorname{Re}\varphi_2(x)\cdot\sigma_2,\quad \varphi_2(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{r_2(\lambda)}{\lambda}e^{i\lambda x}d\lambda+\sum_{j=1}^{n_2}\frac{m_{2j}}{\zeta_{2j}}e^{i\zeta_{2j}x}. \tag{19}$$

The matrix S(x) has the form

$$S(x)=(iK_2(x,x)+\sigma_3)\sigma_3(iK_2(x,x)+\sigma_3)^{-1},$$

and the dependence of the scattering data on t is determined by the formulas

$$a_2(\lambda,t)=a_2(\lambda,0),\quad b_2(\lambda,t)=e^{4i\lambda^2t}b_2(\lambda,0),\quad \zeta_{2j}(t)=\zeta_{2j}(0),\quad m_{2j}(t)=e^{4i\zeta_{2j}^2t}m_{2j}(0),\quad j=1,\ldots,n_2.$$

We now turn to the main proposition of the paper – the proof that Eqs. (1) and (2) are equivalent.

2. Gauge Equivalence

In this section we show that, using a λ-independent gauge transformation, any flat connection of the form (6) can be reduced to the form (14). For the proof that in this manner we obtain all flat connections of the form (14), we show that any flat connection of the form (14) can be reduced by a gauge transformation to the form (6). This will show that to every solution of Eq. (1) there corresponds a solution of (2) and vice versa. The exact formulations of these propositions together with the boundary conditions are given in Propositions 1 and 2.

PROPOSITION 1. Let $\psi(x, t)$ be a solution of Eq. (1) with the boundary conditions $\lim_{|x|\to\infty} \psi(x,t)=0$, and $g(x,t)=\Phi_1(x,t,0)$, where $\Phi_1(x,t,\lambda)$ is a solution of the system (5). Then the function

$$S(x,t)=g^{-1}(x,t)\sigma_3 g(x,t)$$

is a solution of Eq. (11). If $\Phi_1(x,t,0)=f_1(x,t,0)$, then $\lim_{\to+\infty} S(x,t)=\sigma_3$. But if also $b_1(0) = 0$, then $S(x, t)$ satisfies the condition (15).

Proof. In the system (5), we make the substitution $\Phi_1=g\Phi_2$. Since $g_x = A_0 g$, the connection coefficient U_1 goes over into $U_2 = i\lambda S$, where $S=g^{-1}\sigma_3 g$. Further, since $g_t = B_0 g$, V_1 goes over under the action of the transformation g into $V_2=\lambda g^{-1}B_1 g+2i\lambda^2 S$. But since $g^{-1}B_1 g=2g^{-1}A_0 g=2g^{-1}g_x$ and

$$SS_x=-S_xS=g^{-1}g_x-g^{-1}\sigma_3 g_x g^{-1}\sigma_3 g=g^{-1}g_x-g^{-1}\sigma_3 A_0\sigma_3 g=2g^{-1}g_x,$$

because $\sigma_3 A_0\sigma_3=-A_0$, we finally have $V_2=\lambda SS_x+2i\lambda^2 S$.

The vanishing of the curvature of the connection U_2, V_2 means that S is a solution of Eq. (11). Since the matrices A_0 and B_0 are anti-Hermitian, g is unitary; thus, S is a Hermitian matrix. Finally, if $g(x,t)=f_1(x,t,0)$ and $B_1(0) = 0$, then

$$\lim_{x\to+\infty} g(x,t)=I,\quad \lim_{x\to-\infty} g(x,t)=\begin{pmatrix} a_1(0) & 0 \\ 0 & \overline{a_1(0)} \end{pmatrix},$$

from which it follows that $S(x, t)$ satisfies the conditions (15). We have proved the proposition.

PROPOSITION 2. Let $S(x, t)$ be a solution of Eq. (11) with the boundary conditions $\lim_{x\to\pm\infty} S(x,t)=S_\pm$, where the matrices $S_\pm$ satisfy the relations (12). To within multiplication from the right by a constant unitary diagonal matrix, one can uniquely construct a unitary matrix $g(x, t)$ such that $S = g\sigma_3 g^{-1}$ and the diagonal elements of $g^{-1}g_x$ are zero. We set

$$g^{-1}g_x=\begin{pmatrix} 0 & -\overline{\psi} \\ \psi & 0 \end{pmatrix}.$$

Then $\psi(x, t)$ is a solution of Eq. (1) with the boundary conditions

$$\lim_{|x|\to\infty} \psi(x,t)=0$$

and

$$g^{-1}g_t=i\begin{pmatrix} |\psi|^2 & \overline{\psi}_x \\ \psi_x & -|\psi|^2 \end{pmatrix}.$$

Moreover, if $S_\pm=\sigma_3$, then $b_1(0) = 0$.

Proof. Suppose the unitary matrix g reduces S to diagonal form, $S = g\sigma_3 g^{-1}$. The matrix g is defined to within multiplication from the right by an arbitrary diagonal unitary matrix g_0. By the choice of g_0 we can arrange that there are zeros on the diagonal of $g_0^{-1}g^{-1}(gg_0)_x$, i.e., that

$$\sigma_3 g_0^{-1}g^{-1}(gg_0)_x+g_0^{-1}g^{-1}(gg_0)_x\sigma_3=0$$

or

$$2\sigma_3 g_{0x}=-(\sigma_3 g^{-1}g_x+g^{-1}g_x\sigma_3)g_0. \tag{20}$$

Since the matrix $\sigma_3 g^{-1}g_x+g^{-1}g_x\sigma_3$ is diagonal and anti-Hermitian, Eq. (20) determines a unitary diagonal matrix g_0. In the system (13), we now make the substitution $\Phi_2=g\Phi_1$. The coefficient U_2 then goes over into

$$U_1=-g^{-1}g_x+i\lambda\sigma_3=A_0+\lambda A_1.$$

The coefficient V_2 takes the form

$$V_1=-g^{-1}g_t+\lambda g^{-1}SS_x g+2i\lambda^2\sigma_3.$$

Since $S = g\sigma_3 g^{-1}$,

$$SS_x=2g(g^{-1})_x=-2g_x g^{-1}$$

and $g^{-1}SS_x g=2A_0$. Thus, V_1 takes the form

$$V_1=\tilde{B}_0+\lambda B_1+\lambda^2 B_2,$$

where $\tilde{B}_0=-g^{-1}g_t$. We now note that the connection U_1, V_1 is flat since U_2, V_2 was. Thus, we have the equations

$$B_{1x}=[A_1, \tilde{B}_0], \tag{21}$$

$$A_{0t}-\tilde{B}_{0x}+[A_0, \tilde{B}_0]=0. \tag{22}$$

It follows from Eq. (21) that

$$i[\sigma_3, B_0]=2\begin{pmatrix} 0 & \bar{\psi}_x \\ -\psi_x & 0 \end{pmatrix},$$

from which we obtain

$$\tilde{B}_0=\frac{1}{i}\begin{pmatrix} 0 & \bar{\psi}_x \\ \psi_x & 0 \end{pmatrix}+a\sigma_3.$$

Equation (22) now gives

$$-\psi_t+i\psi_{xx}-2a\psi=0, \quad a_x+i(\psi\bar{\psi}_x+\psi_x\bar{\psi})=0,$$

i.e., $a=-i|\psi|^2$, and ψ satisfies Eq. (1). To complete the proof, it remains to note that if $S_+=S_-=\sigma_3$, then we can choose the matrix g such that at $+\infty$ it tends to the unit matrix and at $-\infty$ to a diagonal matrix. This means that $b_1(0) = 0$.

COROLLARY 1. Let $\mathbf{S}(x, t)$ be a solution of Eq. (2) and $\psi(x, t)$ be the corresponding solution of (1). Then

$$\mathbf{S}_x^2=4|\psi|^2. \tag{23}$$

Proof. Since $SS_x=2gA_0g^{-1}$,

$$\mathbf{S}_x^2=\det S_x=4\det A_0=4|\psi|^2.$$

The following corollary is as easily proved.

COROLLARY 2. Suppose that under the assumptions of Corollary 1 the solution $S(x, t)$ satisfies the condition (15). Then $b_1(0) = 0$ and

$$T_2(\lambda)=T_1^{-1}(0)T_1(\lambda), \quad n_1=n_2=n, \quad \zeta_{1j}=\zeta_{2j}, \quad m_{2j}=\overline{a_1(0)}\,m_{1j}, \quad j=1,\ldots,n.$$

COROLLARY 3. Let $\psi(x, t)$ be a solution of Eq. (1) with the boundary conditions $\lim\limits_{|x|\to\infty} \psi(x, t)=0$, and $S(x, t)$ be the corresponding solution of Eq. (2). Then

$$\mathbf{S}_x^2\cdot(\arg\psi)_x=(\mathbf{S}, \mathbf{S}_x\times\mathbf{S}_{xx}). \tag{24}$$

Proof. From Eqs. (6) we find that

$$\operatorname{tr} A_0B_0=i(\psi\bar{\psi}_x-\psi_x\bar{\psi})$$

and since $A_0=g_xg^{-1}$, $B_0=g_tg^{-1}$, $S=g^{-1}\sigma_3g$,

$$g^{-1}S_xg=[\sigma_3, A_0], \quad g^{-1}S_tg=[\sigma_3, B_0].$$

Further, since $\sigma_3A_0\sigma_3=-A_0$,

$$\operatorname{tr}(S_xS_t)=-4\operatorname{tr} A_0B_0.$$

Thus, using Eq. (11), we obtain

$$(\mathbf{S}, \mathbf{S}_x\times\mathbf{S}_{xx})=\frac{2}{i}(\psi_x\bar{\psi}-\psi\bar{\psi}_x)=4|\psi|^2(\arg\psi)_x.$$

This proves the corollary.

Note that Eqs. (23)-(24) are the content of [5].

3. Conservation Laws

The nonlinear Schrödinger equation is a Hamiltonian system with Poisson brackets

$$\{\psi(x,t), \bar{\psi}(y,t)\}=i\delta(x-y) \tag{25}$$

and Hamiltonian

$$H_1=\int_{-\infty}^{\infty}(|\psi_x|^2-|\psi|^4)\,dx. \tag{26}$$

In [2], the complete integrability of Eq. (1) was proved and canonical action–angle variables obtained. An infinite series of conservation laws for the nonlinear Schrödinger equation was obtained in [1]. We give the first of them, which are

$$N_1=\int_{-\infty}^{\infty}|\psi|^2dx, \tag{27}$$

$$P_1=\frac{1}{2i}\int_{-\infty}^{\infty}(\psi_x\bar\psi-\psi\bar\psi_x)\,dx \tag{28}$$

and they are the particle number and the momentum. Quasiclassical quantization of Eq. (1) was carried out in [3] using the conservation laws for N, P_1, and H_1 expressed in terms of the scattering data.

Equation (2) can also be expressed in Hamiltonian form. The canonical Poisson brackets on the two-dimensional sphere in $\mathbf{R}^3$,

$$\{S_j(x,t),S_k(y,t)\}=\varepsilon_{jkl}S_l(x)\,\delta(x-y), \tag{29}$$

where ε_{jkl} is the completely antisymmetric tensor, and the Hamiltonian

$$H_2=\frac{1}{2}\int_{-\infty}^{\infty}\mathbf{S}_x^2dx \tag{30}$$

generate Eq. (2) in accordance with the rules of Hamiltonian mechanics. It was shown in [4] that the Heisenberg ferromagnet has an infinite number of conservation laws. The first of them are

$$P_2=\int_{-\infty}^{\infty}\frac{S_1S_{2x}-S_{1x}S_2}{1+S_3}dx, \tag{31}$$

$$\mathbf{M}=\int_{-\infty}^{\infty}(\mathbf{S}-\mathbf{S}_0)\,dx \tag{32}$$

and they are, respectively, the momentum and the magnetization. It is easy to see that $\frac{1}{\mathbf{S}_x^2}(\mathbf{S},\mathbf{S}_x\times\mathbf{S}_{xx})$, which was introduced in Corollary 3, is related in a simple manner to the momentum density P_2.

To obtain the infinite number of conservation laws one uses the standard technique of the inverse scattering method based on the trace identities. Expanding $\ln a_2(\lambda)$ in an asymptotic series in inverse powers of λ as $|\lambda|\to\infty$, we obtain the first series of local conservation laws, which begins with P_2, $P_2 = 2\arg a_2(\infty)$, after which there comes H_2 and, in general, we have a recursion relation for the densities c_n of these conservation laws:

$$I_n=\int_{-\infty}^{\infty}c_ndx,\quad n=1,2,\ldots, \tag{33}$$

$$c_1=\frac{1}{4}\mathbf{S}_x^2,\quad c_n=\psi\left(\frac{c_{n-1}}{\psi}\right)_x+\sum_{j+k=n-1}c_jc_k. \tag{34}$$

Here, to express the coefficients c_n in terms of the function S it is necessary to use Eqs. (23) and (24). In particular, we obtain the following expressions for P_2 and H_2 in terms of the scattering data:

$$P_2=-\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\ln|a_2(\lambda)|}{\lambda}d\lambda+4\sum_{j=1}^{n}\arg\zeta_j, \tag{35}$$

$$H_2=-\frac{4}{\pi}\int_{-\infty}^{\infty}\ln|a_2(\lambda)|\,d\lambda+8\sum_{j=1}^{n}\operatorname{Im}\zeta_j. \tag{36}$$

The second series of conservation laws (which in general are not local) can be obtained by expanding $\ln a_2(\lambda)$ in an asymptotic series in positive powers of λ in the neighborhood of the origin. In this way one obtains first the conservation law for M_3, and for the following densities there are simple recursion relations, which we do not give here because the resulting conservation laws do not have a perspicuous physical interpretation. However, in this manner one cannot obtain M_1 and M_2, since they are not in involution with M_3. We give a regular method that is based essentially on Propositions 1 and 2 and makes it possible to express the magnetization vector **M** in terms of the scattering data.

Let S be a solution of Eq. (11) with the boundary conditions (15). It follows from Propositions 1 and 2 that $S=g^{-1}\sigma_3 g$, where $g(x,t)=f_1(x,t,0)$,

$$f_{1x}=(A_0+i\lambda\sigma_3)f_1, \tag{37}$$

and $f_1(x,t,\lambda)=e^{i\lambda x\sigma_3}+o(1)$ as $x\to+\infty$.

We differentiate Eq. (37) with respect to λ and set $\lambda=0$, $\dot g(x,t)=\frac{\partial f}{\partial\lambda}(x,t,\lambda)|_{\lambda=0}$. We find that $\dot g_x=A_0\dot g+i\sigma_3 g$. Representing $\dot g$ in the form $\dot g(x,t)=g(x,t)C_1(x,t)$, we obtain $C_{1x}=ig^{-1}\sigma_3 g$ and, since $C_1\sim ix\sigma_3$ as $x\to+\infty$,

$$C_1=ix\sigma_3-i\int_x^\infty (S-\sigma_3)\,dx'.$$

On the other hand, $S=T_1^{-1}(0)\tilde g^{-1}\sigma_3\tilde g T_1(0)$, where $\tilde g(x,t)=g_1(x,t,0)$. Similarly, we find that $\dot{\tilde g}=\tilde g C_2$, where $C_{2x}=iT_1(0)ST_1^{-1}(0)$ and $C_2\sim ix\sigma_3$ as $x\to-\infty$. Recalling that $b_1(0)=0$, we obtain

$$T_1^{-1}(0)C_2T_1(0)=ix\sigma_3+i\int_{-\infty}^x (S-\sigma_3)\,dx'.$$

We now differentiate Eq. (7) with respect to λ and set $\lambda=0$, and we find that $\dot g=\dot{\tilde g}T_1(0)+\tilde g\dot T_1(0)$ or $gC_1=\tilde g C_2T_1(0)+\tilde g\dot T_1(0)$, i.e., $T_1(0)C_1-C_2T_1(0)=\dot T_1(0)$. Thus, we obtain

$$\int_{-\infty}^{\infty} (S-\sigma_3)\,dx=iT_1^{-1}(0)\dot T_1(0). \tag{38}$$

From (38) we find that

$$M_3=i\dot a_1(0)/a_1(0)=i\dot a_2(0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|a_2(\lambda)|}{\lambda^2}d\lambda-2\sum_{j=1}^{n}\frac{\operatorname{Im}\zeta_j}{|\zeta_j|^2}, \tag{39}$$

$$M=M_1-iM_2=\frac{1}{i}\frac{\dot b_1(0)}{a_1(0)}=\frac{1}{i}\dot b_2(0). \tag{40}$$

In conclusion, let us consider the proof of the complete integrability of Eq. (2). It is easy to show directly, by calculating the variational derivatives of **S** with respect to the scattering data or by using the already known canonical variables for Eq. (1) given in [2], and also Propositions 1 and 2, that the Heisenberg ferromagnet is a completely integrable Hamiltonian system, and one can find the corresponding action–angle variables. Using Eqs. (35), (36), and (39), we can readily carry out a quasiclassical quantization of Eq. (2) in the same way as in [3]. We omit here these simple arguments, since they will soon appear in a paper of P. P. Kulish and S. I. Pachevaya. We merely mention that, as usual in the case of completely integrable systems, the quantum spectrum is identical with the quasiclassical spectrum. We also point out that just as solitons were found to be bound states of the particles of the basic field for the nonlinear Schrödinger equation the quanta of the basic field – the magnons – in the case of the Heisenberg ferromagnet can form bound states with an arbitrary number of magnons.

LITERATURE CITED

1. V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz., **61**, 118 (1971).
2. V. E. Zakharov and S. V. Manakov, Teor. Mat. Fiz., **19**, 3 (1974).
3. P. P. Kulish, S. V. Manakov, L. D. Faddeev, Teor. Mat. Fiz., **28**, 38 (1976).
4. L. A. Takhtajan, Phys. Lett. A, **64**, 235 (1977).
5. M. Lakshmanan, Phys. Lett. A, **61**, 53 (1977).
6. V. E. Zakharov, Funktsional' Analiz i Ego Prilozhen. (1979) (in press).