2b) Assume that $x_\beta = 1$, i.e., we consider a reduced element of the form $d_j y_\beta \in K_{\alpha'} \subset W_{i,t,m+2}$. Since $d_j y_\beta \in W_{i,t,m+2}$ and each element in $W_{i,t,m+2}$ has the form $c_i x$, this means that $j = i$, i.e., we consider a reduced element $d_i y_\beta = c_i(\sum_{j\in S(i)} c_j) y_\beta$. It follows from the definition of c_i and ψ_Σ that $d_i y_\beta = c_i \psi_\Sigma (c_i y_\beta)$, i.e., $d_i y_\beta \in c_i \psi_\Sigma W_\alpha$.

We have thus proved that each reduced element $x_\beta d_j y_\beta$ in the subspace $K_{\alpha'} = W_{\alpha'} \cap J$ either belongs to the subspace $c_i K_\Sigma$, or else to $c_i \psi_\Sigma W_\alpha$. Since the reduced elements of the form $x_\beta d_j y_\beta$ generate $K_{\alpha'}$, Eq. (10) and also Proposition 12 are proved.

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INTEGRATION OF NONLINEAR EQUATIONS OF MATHEMATICAL PHYSICS BY THE METHOD OF INVERSE SCATTERING. II*

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INTRODUCTION

In connection with the development of the method of the inverse scattering problem, the task of enumerating nonlinear differential equations which are integrable by this method is of fundamental interest. The first approach to the search for such equations is contained in the work [1], where the "principle" of "L–A" pairs is formulated; alternate approaches are given in the works [2, 3]. After this work it became clear that there exist infinite series of integrable equations, although only the first several equations of each series are of interest in applications.

In our previous work [4] a method was developed for constructing a broad class of integrable equations possessing an "L–A" pair (the vesture method) together with an algorithm for obtaining exact solutions of them. Other equations having an "L–A" pair were subsequently found by Calogero and his co-workers [5, 6]. The vesture method was extended to these equations in [7]. All integrable equations possessing an "L–A" pair represent conditions for the existence of a common spectrum and common eigenfunctions of two differential operators. However, right after the work [2] it became clear that it is possible to consider integrable equations representing conditions for the existence of a common spectrum of operator pencils depending rationally on the spectral parameter. It is also convenient to consider a number of physically interesting equations admitting an "L–A pair" (the sine-Gordon equation, the Bloch–Bloembergen equation, etc.) in the language of preservation of the spectrum of rational operator pencils.

The present paper is devoted to carrying over the vesture method developed in [4] to the case of spectral problems depending rationally on the spectral parameter. We hereby obtain a description of new classes of equations integrable by the method of the inverse problem together with an algorithm for constructing exact solutions of them. In the work [4] we used to this end a factorization of integral operators; in the present work we use the dual language of the matrix Riemann problem of conjugate functions analytic inside and outside a given contour. Our approach is purely local in the coordinates and enables us to

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avoid the requirement of decay of solutions of infinity.

Among the equations we find there are equations which are of interest from the point of view of applications; in particular, there are the equations of "chiral" fields — free fields in two-dimensional space-time with values in an arbitrary Lie group.

1. Matrix Riemann Problem and the Vesture Method

We consider the overdetermined system of differential equations

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi. \tag{1.1}$$

Here U and V are complex N × N matrices depending rationally on the parameter λ. The compatibility conditions for Eqs. (1.1) have the form

$$U_t - V_x + [U, V] = 0. \tag{1.2}$$

If the number of poles of the function U counting multiplicity is equal to N_1 while those of the function V is equal to N_2, then the functions U and V have $N_1 + N_2 + 2$ independent matrix functional parameters. This system of equations is the integrable system that we consider.

For example, if the poles of U and V are simple,

$$u = u_0 + \sum_{k=1}^{N_1} \frac{u_n}{\lambda - a_n}, \quad v = v_0 + \sum_{k=1}^{N_2} \frac{v_n}{\lambda - b_n}, \tag{1.3}$$

we have

$$u_{0t} - v_{0x} + [u_0, v_0] = 0, \quad u_{nt} + [u_n, R_n] = 0, \quad v_{nx} + [v_n, T_n] = 0, \tag{1.4}$$

$$R_n = u_0 + \sum_{k=1}^{N_2} \frac{v_m}{a_n - b_m}, \quad T_n = u_0 + \sum_{k=1}^{N_1} \frac{u_m}{b_n - a_m}. \tag{1.5}$$

In (1.3)-(1.5) $a_n = a_n(x)$ are given functions of x and $b_n = b_n(t)$ are given functions of t.

The indeterminacy of system (1.2) is explained by its "gauge invariance." Let U and V be some solutions of this system, and let Ψ be the corresponding solution of system (1.1). We consider the functions

$$\tilde{U} = gUg^{-1} + g_x g^{-1}, \quad \tilde{V} = gVg^{-1} + g_t g^{-1}, \tag{1.6}$$

where g is any nondegenerate, matrix-valued function of x and t. It is easy to verify that $\tilde{U}$, $\tilde{V}$ again satisfy Eq. (1.2); the corresponding solution of the system (1.1) is $\tilde{\Psi} = g\Psi$. The transformation $U, V \to \tilde{U}, \tilde{V}$ we call a gauge transformation. In application to system (1.4), (1.5) we have

$$\tilde{u}_0 = gu_0 g^{-1} + g_x g^{-1}, \quad \tilde{v}_0 = gv_0 g^{-1} + g_t g^{-1}, \tag{1.7}$$

$$\tilde{u}_n = gu_n g^{-1}, \quad \tilde{v}_n = gv_n g^{-1}. \tag{1.8}$$

System (1.2) can be completely determined by imposing an additional condition on U and V. For example, it is possible to set $u_0 \equiv 0$ (up to gauge it may hereby be assumed that $v_0 \equiv 0$). Solutions of the equations completely defined by some other condition differ from this case only by a transformation (1.6) with some matrix g. We shall call all such equations gauge equivalent.

We observe further that Eqs. (1.2) have the natural trivial solution

$$U = A(x, \lambda), \quad V = B(t, \lambda), \quad [A(x, \lambda), B(t, \lambda)] = 0. \tag{1.9}$$

The corresponding solution of Eqs. (1.1) we denote by ω.

Having one solution of the system (1.1), (1.2), e.g., A, B, ω, we shall now show how to construct new solutions of these systems depending on a functional parameter. To this end we need some facts from the theory of the matrix Riemann problem.

Let Γ be a simple closed contour in the plane of the complex variable λ. The Riemann problem consists in factoring a function G(λ) defined on the contour Γ in the form of a product

$$\psi_1(\lambda)\psi_2(\lambda) = G(\lambda), \tag{1.10}$$

where $\psi_1(\lambda)$ is analytic outside and $\psi_2(\lambda)$ analytic inside the contour Γ. The problem is considered in the algebra of square matrices of finite order N. The factorization is called regular if $\psi_{1,2}(\lambda)$ and $\psi_{1,2}^{-1}(\lambda)$ are continuous in the closure of their domains of analyticity, including the point at infinity. The solution of the Riemann problem is constant up to the replacement $\psi_1 \to \psi_1 g$, $\psi_2 \to g^{-1}\psi_2$, where g is an arbitrary, constant, nondegenerate matrix.

Let the contour Γ and the matrix function $G_0(\lambda)$ on it be given. For all x and t we define the function $G(\lambda, x, t)$ by

$$G(\lambda, x, t) = \Psi_0(\lambda, x, t) G_0(\lambda) \Psi_0^{-1}(\lambda, x, t), \tag{1.11}$$

where $\Psi_0 U, V$ is some solution of system (1.1), (1.2). For simplicity, we henceforth assume that $\Psi = \omega$, $U = A$, $V = B$, although all arguments are also valid in the general case. We consider for all x and t the Riemann problem (1.10) with the function G of (1.11). It may hereby occur that the contour Γ passes through poles of the functions A and B. At these poles we set $G \equiv 1$. This enables us to differentiate the function G with respect to x and t. Using (1.1), we find after differentiation that

$$\psi_{1x}\psi_2 + \psi_1\psi_{2x} = A\psi_1\psi_2 - \psi_1\psi_2 A. \tag{1.12}$$

We define the matrix-valued function U by the formula

$$U = -\psi_1^{-1}(\psi_{1x} - A\psi_1) = (\psi_{2x} + \psi_2 A)\psi_2^{-1}. \tag{1.13}$$

It follows from formula (1.13) that U extends from the contour Γ to the entire complex plan and is there a rational function with poles which coincide with the poles of the function A. From (1.13) it follows that

$$\psi_{1x} = A\psi_1 - \psi_1 U, \quad \psi_{2x} = U\psi_2 - \psi_2 A. \tag{1.14}$$

Similarly, differentiating (1.10) with respect to t, we define the rational function V,

$$V = -\psi_1^{-1}(\psi_{1t} - B\psi_1) = (\psi_{2t} + \psi_2 B)\psi_2^{-1}. \tag{1.15}$$

The poles of V coincide with the poles of B. Setting $\psi_1 = \omega\varphi_1^{-1}$, $\psi_2 = \varphi_2\omega^{-1}$, we see that φ_1, φ_2 satisfy Eqs. (1.1). These equations are thus compatible, and the functions U and V satisfy the system (1.2).

We consider a new solution of the Riemann problem (1.10) $\tilde{\psi}_2 = g\psi_2$, $\tilde{\psi}_1 = \psi_1 g^{-1}$, where g is any matrix-valued function of x and t. The functions U and V then undergo a gauge transformation (1.6). In this connection there is a natural means of determining system (1.7), which consists in imposing an additional condition on the Riemann problem which makes it unique. For example, it is possible to fix the value of one of the functions ψ_1, ψ_2 at some point $\lambda = \lambda'$. Thus, in the example considered above of determining system (1.5) by the condition $u_0 = v_0 = 0$ this amounts to setting $\psi_1(\infty) = I$. (We assume that $G(\infty) = I$.) The imposing of an additional condition may be called the normalization of the Riemann problem.

The procedure presented for constructing solutions of system (1.2) may be called the vesture of the "primer" solutions A, B by means of the Riemann problem with contour Γ and function $G_0(\lambda)$. By representing A and B in terms of expansions in simple fractions of the form (1.3), we obtain from (1.12), (1.13)

$$u_0 = g_{0x}g_0^{-1} + g_0 A_0 g_0^{-1}, \quad u_n = g_n A_n g_n^{-1}, \quad v_0 = g_{0t}g_0^{-1} + g_0 B_0 g_0^{-1}, \quad v_n = \tilde{g}_n B_n \tilde{g}_n^{-1}. \tag{1.16}$$

Here $g_0 = \psi_2(\infty)$, and the matrices g_n and $\tilde{g}_n$ are the values at the points $\lambda = a_n$ and $\lambda = b_n$, respectively, of functions ψ_1^{-1} or ψ_2, depending on whether the corresponding point lies inside or outside the contour. (We recall that if a_n or b_n lies on the contour, then at this point $G = I$ and $\psi_1^{-1} = \psi_2$.) Formulas (1.16) show that under vesture the invariants of the matrices u_n, v_n are preserved.

We remark further that the entire procedure presented above carries over trivially to the case where one or both of the variables are complex.

2. Reductions and Involutions. Examples

In most cases the system of Eq. (1.2) is too general to be useful in application to physics. The question therefore arises of the possibility of imposing additional conditions compatible with the system (1.2) on U and V. We call such conditions reductions. Reductions reduce the number of equations contained in the system (1.2) and simultaneously impose certain restrictions on the contour Γ and the function $G_0(\lambda)$.

We shall consider the reduction question for the example of system (1.5). Suppose that all the a_n, b_n are real. It is then possible to require that all the matrices u_n, v_n be anti-Hermitian or belong to some other Lie algebra realized by $N \times N$ matrices. A restriction hereby arises on the gauge freedom: the matrix-valued function g must belong to the Lie group corresponding to this algebra.

In order that the anti-Hermitian property be preserved under vesture, it suffices to choose the real axis as the contour Γ and assume that the function $G(\lambda)$ satisfies the involution

$$G^{+}(\bar{\lambda}) = G^{-1}(\lambda); \tag{2.1}$$

of course, the "primer" matrices A and B must also be anti-Hermitian for real λ. In a similar manner it is possible to choose the matrices u_n, v_n real or antisymmetric. In these cases the contour is again the real axis, and for G we have the involutions

$$\bar{G}(\bar{\lambda}) = G(\lambda) \tag{2.2}$$

and

$$G^{\mathrm{tr}}(\lambda) = G^{-1}(\lambda). \tag{2.3}$$

It is also possible to consider a more general situation.

Equation (1.2) can be considered as an equation on the algebra of rational functions of λ with a fixed distribution of poles and matrix coefficients. The reduction may consist in restricting Eq. (1.2) to some subalgebra of this algebra.

Suppose that on the λ plane there exists a fractional linear transformation $z(\lambda)$, such that $z^2(\lambda) = 1$, and suppose that the composition of this transformation and complex conjugation takes the set of points a_n, b_n into itself. Suppose there is given a rational function $J(\lambda)$, not depending on x and t with poles in the set a_n, b_n, which satisfies the involution

$$J^{+}(\bar{\lambda}) = J(z(\lambda)). \tag{2.4}$$

It is then possible to perform a reduction by imposing on U and V the conditions

$$U^{+}(\bar{\lambda}) = -J^{-1}(\lambda)\, U(z(\lambda))\, J(\lambda), \quad V^{+}(\bar{\lambda}) = -J^{-1}(\lambda)\; V(z(\lambda))\, J(\lambda). \tag{2.5}$$

Reductions consisting of the conditions that U and V be real and symmetric also admit similar generalizations.

The examples presented far from exhaust all possible reductions of the system (1.2); the enumeration of these reductions is one of the current problems of the theory of integrable systems.

We shall consider in more detail reductions in the system (1.4)-(1.5) in the case where $N_1 = N_2 = 1$. Suppose first of all that $a_1 = b_1 = 0$. In this case $[u_1, v_1] = 0$. We determine the system by setting $u_1 = A(x)$, where A is a diagonal matrix (in the case of general position). Then $V_1 = B(t)$, where B is also a diagonal matrix. The system of equations (1.2) now consists of Eq. (1.4) and the additional condition

$$[u_0, B] = [v_0, A]. \tag{2.6}$$

In the anti-Hermitian case where A and B are pure imaginary, $u_0^{+} = -u_0$, the system (1.4), (2.6) is the "system of n waves," known in nonlinear optics (see [8]).

Further reduction occurs if u_0 and v_0 are assumed to be real, antisymmetric matrices. The system which hereby arises corresponds in nonlinear optics to the case of "exact resonance."

Stronger reductions are possible for special forms of A and B. Suppose A and B are such that it is possible to find a matrix C which anticommutes with them,

$$AC + CA = 0, \quad BC + CB = 0. \tag{2.7}$$

The following reduction is then admissible:

$$[u_0, C] = 0, \quad [v_0, C] = 0. \tag{2.8}$$

The systems obtained here also occur in nonlinear optics.

The system (1.4), (2.6) belongs to the class of integrable systems admitting a Lax representation and having an "L–A pair" (see, e.g., [8]). A method of solving such systems was developed in our previous work [4]. The proposed scheme contains all these systems containing differentiation with respect to two variables. They all correspond to the case in which the function U has a single simple pole, while the function V has a pole of arbitrary order at the same point.

The case in which U and V each have a simple pole but at different points is of exceptional content. We restrict ourselves to the case in which the position of these points a_1, b_1 does not depend on x and t. It may be assumed with no loss of generality that $a_1 = 1$, $b_1 = -1$. Under the additional condition $u_0 = v_0 = 0$ the system of equations has the form

$$u_{1t} + \frac{1}{2}[u_1, v_1] = 0, \quad u_{1t} = v_{1x}. \tag{2.9}$$

The system (2.9) can be rewritten in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial t} g^{-1}\right) + \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial x} g^{-1}\right) = 0, \tag{2.10}$$

where $g = \varphi|_{\lambda=0}$, and φ is a solution of system (1.1). In system (2.9) it is natural to carry out reduction consisting in fixing the Lie algebra to which u_1, v_1 belong. Here g belongs to the corresponding Lie group. Further reductions are also possible which do not reduce to those described above. Thus, it is easy to verify that Eq. (2.10) admits the reductions $g^+ = g$ and $g^2 = I$. All the systems which hereby arise represent geometric models of a classical field theory — the chiral fields on Lie groups and their homogeneous spaces [9, 10].

Suppose now that the system (1.4) is determined such that $u_1 = A_1$. The Riemann problem is hereby normalized by the condition $\psi_1(1) = I$. It is then possible to set $v_0 = -{}^1/_2 \cdot v_1$, and the system acquires the form

$$-v_{1x} + [v_1, u_0 - {}^1/_2 A] = 0, \quad u_{0t} + {}^1/_2 [v_1, A] = 0. \tag{2.11}$$

System (2.11) carries the somewhat provisional name of a "u, v" system; in the simplest special case it reduces [10] to the "sine-Gordon" equation.

3. Soliton Solutions

The class of solutions described in Sec. 1 can be considerably extended if the matrix-valued functions ψ_1, ψ_2 are permitted to degenerate at a finite number of points of their domains of analyticity. We call such points zeros of the functions ψ_1, ψ_2. It is obvious that at these points $\det \psi_{1,2} = 0$, but we assume that all elements of the matrices ψ_1, ψ_2 do not vanish. We shall say that the function ψ has a simple zero at the point $\lambda = \lambda_0$ if in a neighborhood of this point $\psi^{-1} = C_0/(\lambda - \lambda_0) + C_1 + \ldots$, $\det C_0 = 0$. The function ψ in a neighborhood $\lambda \sim \lambda_0$ is $\psi = F_0 + (\lambda - \lambda_0) F_1 + \ldots$. Since $F_0C_0 = C_0F_0 = 0$, $C_0F_1 + C_1F_0 = I$,

$$\mathrm{Ker}\ F_0 = \mathrm{Im}\ C_0, \quad \mathrm{Ker}\ C_0 = \mathrm{Im}\ F_0. \tag{3.1}$$

Suppose we are given the Riemann problem on the contour Γ, and suppose that the function ψ_1 is analytic outside the contour and has N zeros $\lambda_1, \ldots, \lambda_N$, while the function ψ_2 is analytic inside the contour and has N zeros at the points $\mu_1, \ldots, \mu_N$. We prescribe two collections of subspaces

$$L_n = \mathrm{Im}\ \psi_1|_{\lambda=\lambda_n}, \quad M_n = \mathrm{Ker}\ \psi_2|_{\lambda=\mu_n}. \tag{3.2}$$

We shall show that prescribing L_n, M_n singles out a unique (up to gauge) solution of the Riemann problem. Indeed, let $\tilde\psi_1$, $\tilde\psi_2$ be another solution. Then the function χ,

$$\chi = \tilde\psi_1^{-1}\psi_1 = \tilde\psi_2\psi_2^{-1}, \tag{3.3}$$

is defined on the entire complex plane and by (3.1) has a removable singularity at the points $\lambda = \lambda_n$, $\lambda = \mu_n$. Therefore, $\tilde\psi_1$ differs from ψ by no more than a gauge transformation.

We shall describe a procedure of reducing the Riemann problem with zeros to a regular Riemann problem. Suppose that ψ_1, ψ_2 have zeros at the points λ_i, μ_i. We consider the new solution of the Riemann problem $\tilde\psi_1, \tilde\psi_2$ with the same function G which has zeros at the points $\lambda_1, \ldots, \lambda_{N-1}, \mu_1, \ldots, \mu_{N-1}$, while at these points

$$\mathrm{Ker}\ \tilde\psi_2(\mu_k) = \mathrm{Ker}\ \psi_2(\mu_k) = M_k \quad (k = 1, \ldots, N-1),$$
$$\mathrm{Im}\ \tilde\psi_1(\lambda_k) = \mathrm{Im}\ \psi_1(\lambda_k) = L_k \quad (k = 1, \ldots, N-1),$$

and we introduce the function χ by formula (3.3). As already noted, χ is regular at the points $\lambda_1, \ldots, \lambda_{N-1}, \mu_1, \ldots, \mu_{N-1}$, and hence

$$\chi = \chi_0 + \frac{\chi_1}{\lambda - \mu_N}. \tag{3.4}$$

Similarly, we find that

$$\chi^{-1} = \chi_2 + \frac{\chi_3}{\lambda - \lambda_N}. \tag{3.5}$$

By multiplying (3.4) and (3.5), we see that χ has the form

$$\chi = \left(1 - \frac{\lambda_n - \mu_n}{\lambda - \mu_n} P_n\right) \chi_0, \tag{3.6}$$

where $P_n = \frac{1}{\lambda_n - \mu_n} \chi_0 \chi_3 = \frac{1}{\mu_n - \lambda_n} \chi_1 \chi_0^{-1}$ is a projection operator, $P_n^2 = P_n$.

The projection operator is completely characterized by giving the two subspaces Ker P and Im P. We shall show that these subspaces can easily be found. Since

$$\psi_1 = \widetilde{\psi}_1 \chi = \widetilde{\psi}_1 \left(1 - \frac{\lambda_n - \mu_n}{\lambda - \mu_n} P_n\right) \chi_0, \quad \psi_2 = \chi^{-1} \widetilde{\psi}_2 = \chi_0^{-1} \left(1 - \frac{\mu_n - \lambda_n}{\lambda - \lambda_n} P_n\right) \widetilde{\psi}_2,$$

we have $M_n = \operatorname{Ker} \psi_2 (\mu_n) = \operatorname{Ker} (1 - P_n) \widetilde{\psi}_2 (\mu_n)$. Whence

$$\operatorname{Im} P_n = \operatorname{Ker} (1 - P_n) = \widetilde{\psi}_2 (\mu_n) M_n. \tag{3.7}$$

Similarly,

$$\begin{gathered} L_n = \operatorname{Im} \widetilde{\psi}_1 (\lambda_n) \chi (\lambda_n) = \operatorname{Im} \widetilde{\psi}_1 (\lambda_n) (1 - P_n) \chi_0, \\ \widetilde{\psi}_1 (\lambda_n) \operatorname{Ker} P_n = L_n, \quad \operatorname{Ker} P_n = \psi_1^{-1} (\lambda_n) L_n. \end{gathered} \tag{3.8}$$

The formulas (3.7) and (3.8) determine the projection operator in formulas (3.6). The value of the matrix χ_0 is arbitrary and is determined by the normalization of the Riemann problem. In the simplest means of normalization $\psi_1 (\infty) = \psi_2 (\infty) = I$ $\chi_0 = I$. Repeated application of this procedure leads to the formulas

$$\begin{gathered} \psi_1 = \hat{\psi}_1 \left(1 - \frac{\lambda_1 - \mu_1}{\lambda - \mu_1} P_1\right) \chi_1 \left(1 - \frac{\lambda_2 - \mu_2}{\lambda - \mu_2} P_2\right) \chi_2, \ldots, \\ \psi_2 = \chi_n^{-1} \left(1 - \frac{\mu_N - \lambda_N}{\lambda - \lambda_N} P_N\right) \ldots \chi_1^{-1} \left(1 - \frac{\mu_1 - \lambda_1}{\lambda - \lambda_1} P_1\right) \hat{\psi}_2. \end{gathered} \tag{3.9}$$

Here $(\hat{\psi}_1, \hat{\psi}_2)$ is a regular solution of the Riemann problem, and the constant, nondegenerate matrices $\chi_1, \ldots, \chi_n$ are determined by the choice of gauge.

We shall now apply the Riemann problem with zeros to construct new solutions of the system (1.2) by the method of vesture. For degenerate ψ_1, ψ_2 the functions U, V defined by formulas (1.13), (1.15) have, in general, additional poles at the points λ_n, μ_n. It must be required that the residues at these poles be equal to zero. We define the differential operators

$$\begin{gathered} D_x^{(n)} = \partial_x - A|_{\lambda=\lambda_n}, \quad \tilde{D}_x^{(n)} = \partial_x - A|_{\lambda=\mu_n}, \\ D_t^{(n)} = \partial_t - B|_{\lambda=\lambda_n}, \quad \tilde{D}_t^{(n)} = \partial_t - B|_{\lambda=\mu_n}. \end{gathered}$$

It now follows from (1.13), (1.15) that

$$c_n D_x^{(n)} F_n = 0, \quad c_n D_t^{(n)} F_n = 0, \quad \tilde{c}_n \tilde{D}_x^{(n)} \tilde{F}_n = 0, \quad \tilde{c}_n \tilde{D}_t^{(n)} \tilde{F}_n = 0. \tag{3.10}$$

Here the c_n are the residues of ψ_1^{-1}, the F_n are the values of ψ_1 at the points $\lambda = \lambda_n$, then $\tilde{c}_n$ are residues of ψ_2^{-1}, and the F_n are the values of ψ_2 at the points $\lambda = \mu_n$.

It follows from formulas (3.10) and (3.1) that the subspaces M_n are invariant under the action of the operators $\tilde{D}_x^{(n)}$, $\tilde{D}_t^{(n)}$, while the subspaces L_n are invariant under the action of $D_x^{(n)}, D_t^{(n)}$. It is clear that the subspaces

$$M_n (x, t) = \omega (x, t, \mu_n) M_n^{(0)}, \quad L_n (x, t) = \omega (x, t, \lambda_n) L_n^{(0)} \tag{3.11}$$

possess the invariance property if and only if $M_n^0, L_n^0 \subset C^N$ do not depend on (x, t). Thus, having $A, B, \omega, L_n^0, M_n^0$, and the solutions of the regular problem, we can explicitly compute all the factors in (3.9).

Let $G(\lambda) \equiv I$. In this case the solution of the regular Riemann problem is trivial, and the vesture procedure becomes purely algebraic; ψ_1, ψ_2 hereby become rational functions of λ. The corresponding functions U and V we shall call soliton solutions of the system (1.2). Calculation of the soliton solutions plays a fundamental role in applications of integrable systems to physical problems.

As an example we compute the simplest nontrivial soliton solutions of the "n-wave" system. Suppose that the matrices A and B are constant, diagonal, and purely imaginary,

$$A = i\,\mathrm{diag}\,(a_1, \ldots, a_N), \quad B = i\,\mathrm{diag}\,(b_1, \ldots, b_N),$$

and suppose that the numbers a_i are ordered as follows: $a_1 > a_2 > \ldots > a_N$. Equation (2.6) can be solved in the form $u_0 = [A, Q]$, $v_0 = [B, Q]$. The matrix Q is anti-Hermitian and has zeros on the diagonal. The equation for Q has the form

$$[A, Q_t] - [B, Q_x] + [[A, Q], [B, Q]] = 0. \tag{3.12}$$

The Riemann problem is normalized by the condition $\psi_{1,2}(\infty) = I$. It is easy to establish directly from formula (1.14) that the asymptotic expressions for the functions $\psi_{1,2}$ as $\lambda \to \infty$ have the form

$$\psi_1 \to I - (Q/\lambda), \quad \psi_2 \to I + (Q/\lambda). \tag{3.13}$$

We consider the simplest solution of Eq. (3.12) for which the function ψ_1 has a single pole at the point $\lambda = \lambda_0 = \xi + i\eta$, $\psi_1 = I + \frac{\lambda_0 - \bar{\lambda}_0}{\lambda - \lambda_0} P$. Here P is a projection operator, $P^2 = P$. Suppose that the range of the projection P is a one-dimensional space. Then

$$P_{ij} = \frac{n_i \bar{n}_j}{\sum\limits_{i=1}^{N} |n_i|^2}. \tag{3.14}$$

Obviously, $Q_{ij} = (\lambda_0 - \bar{\lambda}_0) P_{ij}$. Suppose that vesture is performed on the background of the trivial solution $U = A$, $V = B$, $[A, B] = 0$. We choose $\omega = \exp(Ax + Bt)\lambda$. Then $n_i(x, t) = c_i \exp i\lambda_0 (a_i x + b_i t)$, where c_i is an arbitrary constant complex vector. Finally, for the matrix Q we have

$$Q_{ij} = \frac{(\lambda_0 - \bar{\lambda}_0)\, c_i \bar{c}_j \exp i\,[(a_i\lambda_0 + a_j\bar{\lambda}_0)\, x + (b_i\lambda_0 + b_j\bar{\lambda}_0)\, t]}{\sum\limits_{i=1}^{N} |c_i|^2 \exp 2\eta\,(a_i x + a_j t)}. \tag{3.15}$$

The solution (3.15) for the case N = 3 was found in [8]. Analysis of its asymptotics as $t \to \pm\infty$ shows that it describes a nontrivial interaction — the decay or "gluing together" of elementary soliton solutions of various types.

4. Higher-Dimensional Generalizations

The means of constructing integrable systems described above admits a natural higher-dimensional generalization. Let U and V be square N × N matrices depending on a vector parameter $\mathbf{r} = (x_1, \ldots, x_k)$ of arbitrary dimension, and let D_1 and D_2 be linear differential operators of first order with constant coefficients,

$$D_1 = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i}, \quad D_2 = \sum_{i=1}^{k} b_i \frac{\partial}{\partial x_i}. \tag{4.1}$$

Suppose that U, V, D_1, and D_2 (i.e., a_i and b_i) depend rationally on the parameter λ, while the poles of U and D_1 are located at the points $a_1, \ldots, a_N$, and the poles of V and D_2 at the points $b_1, \ldots, b_N$. We consider the overdetermined system of equations

$$D_1\Psi = U\Psi, \quad D_2\Psi = V\Psi, \tag{4.2}$$

which generalizes system (1.1) in an obvious manner.

The compatibility condition for system (4.2) generalizes (1.2) naturally and has the form

$$D_2 U - D_1 V = [U, V]. \tag{4.3}$$

It is easily seen that (4.3), as before, represents a system of $N_1 + N_2 + 1$ differential equations for the coefficients of the functions U and V. It is obvious that the number of independent differentials in D_1, D_2, cannot exceed $N_1 + N_2 + 2$. This is the case of generalized position. In the case of general position the distribution of the poles loses its significance, and, by reducing Eq. (4.2) to a common denominator, it is possible to go over from rational functions of λ to polynomials. The simplest nontrivial example in general position arises if both polynomials have first order. The system (4.2) then has the form

$$\lambda (\partial_1 + u_1) \psi + (\partial_2 + u_2) \psi = 0, \quad \lambda (\partial_3 + u_3) \psi + (\partial_4 + u_4)\psi = 0. \tag{4.4}$$

The compatibility condition, Eq. (4.3), is now

$$\begin{gathered}\partial_1 u_3 - \partial_3 u_1 + [u_1, u_3] = 0, \quad \partial_2 u_4 - \partial_4 u_2 + [u_2, u_4] = 0, \\ \partial_1 u_4 - \partial_4 u_1 + \partial_2 u_3 - \partial_3 u_2 + [u_1, u_4] - [u_2, u_3] = 0.\end{gathered} \tag{4.5}$$

As in Sec. 1, we may assume that the vector $\mathbf{r}$ is complex. System of equations (4.5) in the general case has no physical interpretation. However, if one of the two reductions

$$\partial_4 = - \bar{\partial}_1, \quad \partial_3 = \bar{\partial}_2, \quad u_4 = u_1^+, \quad u_3 = - u_2^+ \tag{4.6}$$

or

$$\partial_4 = \bar{\partial}_2, \quad \partial_3 = - \bar{\partial}_1, \quad u_4 = -u_2^+, \quad u_3 = u_1^+ \tag{4.7}$$

is performed, system (4.5) acquires the interpretation of the duality or antiduality equations distinguishing important special solutions of the equations describing the Yang–Mills field over four-dimensional Euclidean space (see [11]) with values in the group SU_N.

Further generalization occurs if a_i, b_i are assumed to be functions of variables z_i. The differentials D_i here cease to be commutative and become elements of the algebra of k-dimensional vector fields. The system (4.3) is now augmented by the equation

$$[D_1, D_2] = 0, \tag{4.8}$$

i.e., by the equation

$$\Big(\sum_{m=1}^{k} a_m \frac{\partial}{\partial x_m}\Big) b_i - \Big(\sum_{m=1}^{k} b_m \frac{\partial}{\partial x_m}\Big) a_i = 0\,. \tag{4.9}$$

Setting

$$a_k = \sum_{n=1}^{N_1} \frac{a_{kn}}{\lambda - \varepsilon_n}, \quad b_k = \sum_{n=1}^{N_2} \frac{b_{kn}}{\lambda - \nu_n},$$

we arrive at a nontrivial system of equations for a_{kn}, b_{kn}. It is curious that the systems (4.3) and (4.9) are completely independent. System (4.9) can be reduced to any subalgebra of the algebra of vector fields, e.g., in the case of even k to the algebra of Hamiltonian Poisson brackets. Physical applications of systems of the type (4.9) are so far unknown.

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CURVATURE OF GROUPS OF DIFFEOMORPHISMS PRESERVING THE MEASURE OF THE 2-SPHERE

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In this paper, the curvatures of the groups S Diff(S^2) (diffeomorphisms of the 2-sphere S^2 preserving the standard density) equipped with the natural right-invariant Riemannian metric (weak metric) are calculated. It was shown by Arnol'd [1, 2] that the geodesics on groups of this type express flows of an ideal incompressible fluid, and negativity of the sectional curvature along two-dimensional directions is a criterion for exponential instability of flows. Steady flow on a two-dimensional torus having the velocity field sin y ∂x (i.e., a Passat flow) was, in particular, studied in [2] in detail. In this paper, the following analog of the Passat flow is studied for S^2: viz., the vector field $g = z(-y\partial x + x\partial y)$; in many two-dimensional planes cutting the field g, the curvatures turn out to be negative. The curvature values obtained are used to estimate the interval of time during which long-term dynamic weather forecasting is not possible, and results close to those of [2] are obtained. The vector field $h = -y\partial x + x\partial y$ (the curl on S^2) is also studied, for which the sectional curvatures are nonnegative. The author sincerely thanks V. I. Arnol'd for valuable advice, and also A. L. Onishchik for helpful discussions.

1. Statement of the Results

Let S^2 be defined in $\mathbf{R}^3$ by the equation $x^2 + y^2 + z^2 = 1$. We denote by SV(S^2) the Lie algebra of the group S Diff(S^2) consisting of vector fields with zero divergence. The right-invariant metric on S Diff(S^2) is defined at the identity by

$$\langle u, v\rangle = \int_{S^2} (u(x), \bar{v}(x))\, d\mu(x) \qquad (u, v \in SV(S^2)^{\mathbf{C}}).$$

It is convenient to represent vector fields on SV(S^2) by their flow functions: $v = T(f_v) = I(\operatorname{grad} f_v)$ (where I is the operator given by clockwise rotation by 90°).

We choose in the space of flow functions a basis consisting of the spherical functions (φ, θ being the standard spherical coordinates on S^2)

$$Y_m^l = \left[\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}\right]^{1/2} \frac{1}{2^l l!}(e^{i\varphi}\sin\varphi)^m \frac{d^{l+m}(\sin^l\theta)}{d(\cos\theta)^{l+m}} \quad (l \in \mathbf{N},\ m = -l, \ldots, l).$$

We remark that $\| T(Y_m^l)\|^2 = -\lambda_\Delta \| Y_m^l\|^2 = l(l+1)$, from which an orthonormal basis in SV(S^2) is formed by the vector fields $e_m^l = T(1/\sqrt{l(l+1)}Y_m^l)$.

We agree to denote by K(u, v) the curvature taken at the identity element of S Diff(S^2) along the two-dimensional plane L{u, v}.

THEOREM 1. The sectional curvatures along two-dimensional planes containing the vector field $h = -y\partial x + x\partial y$ are given by the formulas

1) $K(h, e_m^l) = \dfrac{3}{8\pi}\dfrac{m^2}{l^2(l+1)^2}$;

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