3) Suppose that the group G has a representation $G = \langle a, b, c, d | R_i = e, i = 1\text{-}n \rangle$ with a finite set of defining relations. We call the quantity $\Sigma \partial (R_i)$ the length of the corresponding representation. Among all such representations, we choose one with minimal length. The representation of H (as a subgroup), constructed using the minimal representation of G, has lesser length than the original representation of G. Therefore, the factor-representation of G, obtained from the representation of H by applying the homomorphism described in 1), has lesser length than the original minimal representation of G. This contradiction shows that the group G cannot be defined by a finite set of defining relations. The theorem is proved.

Despite the fact that the group G was defined as a group of transformations on a space with a measure, it may be defined in purely algebraic terms. In particular, there exists a simple algorithm, allowing us to answer the following question for any word W in the generators of the group G: does W represent the unit element of G, or not? We also note that G is finitely approximable and has exponential growth. We give another example. Let ξ be the transformation of the square $[0, 1] \times [0, 1]$, consisting of the cyclic permutation of its quadrants, and let the transformation η be described as in Fig. 2 (S denotes the cyclic permutation of the quadrants of the square over which it is written). Then $\xi^4 = \eta^4 = e$ and the group generated by the transformations ξ and η is an infinite periodic group.

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VARIATIONAL PRINCIPLE FOR EQUATIONS INTEGRABLE BY THE INVERSE PROBLEM METHOD

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We consider (cf. [1]) the system of nonlinear equations representing the conditions for compatibility of two linear differential equations for a square $N \times N$ nonsingular matrix function $\Psi(\xi, \eta, \lambda)$

$$\Psi_\xi = U(\xi, \eta, \lambda)\Psi, \quad \Psi_\eta = V(\xi, \eta, \lambda)\Psi. \tag{1}$$

Here U and V are rational functions of the parameter λ with distinct simple poles:

$$U = U_0 + \sum_{n=1}^{N_1} \frac{U_n(\xi, \eta)}{\lambda - a_n}, \qquad V = V_0 + \sum_{n=1}^{N_2} \frac{V_n(\xi, \eta)}{\lambda - b_n}. \tag{2}$$

The compatibility conditions for Eqs. (1) have the form

$$U_{0\eta} - V_{0\xi} = [U_0, V_0], \tag{3}$$

$$U_{n\eta} = \left[U_n, V_0 + \sum_{k=1}^{N_2} \frac{V_k}{a_n - b_k}\right], \qquad V_{n\xi} = \left[V_n, U_0 + \sum_{k=1}^{N_1} \frac{U_k}{b_n - a_k}\right]. \tag{4}$$

It follows from (3) that there exists a nonsingular matrix $g(\xi, \eta)$ such that

$$U_0 = g_\xi g^{-1}, \quad V_0 = g_\eta g^{-1}. \tag{5}$$

We introduce the notation

$$\nabla_\eta = \frac{\partial}{\partial \eta} - V_0, \qquad \nabla_\xi = \frac{\partial}{\partial \xi} - U_0.$$

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We express the matrices U_n, V_n in the form

$$U_n = \varphi_n U_n^{(0)} \varphi_n^{-1}, \quad V_n = \psi_n V_n^{(0)} \psi_n^{-1}, \tag{6}$$

where the $U_n^{(0)}(\xi)$, $V_n^{(0)}(\eta)$ are the normal Jordan forms of U_n, V_n. They are obtained by partial integration of Eqs. (4). Substituting (6) into (3), (4), we find the equations satisfied by the matrices φ_n, ψ_n,

$$\begin{aligned} &\left(\nabla_\eta \varphi_n - \sum_{m=1}^{N_2} \frac{\psi_m V_m^{(0)} \psi_m^{-1}}{a_n - b_m} \varphi_n\right) U_n^{(0)} = U_n \left(\nabla_\eta \varphi_n - \sum_{m=1}^{N_2} \frac{\psi_m V_m^{(0)} \psi_m^{-1}}{a_n - b_m} \varphi_n\right), \\ &\left(\nabla_\xi \psi_n - \sum_{m=1}^{N_1} \frac{\varphi_m U_m^{(0)} \varphi_m}{b_n - a_m} \psi_n\right) V_n^{(0)} = V_n \left(\nabla_\xi \psi_n - \sum_{m=1}^{N_1} \frac{\varphi_m U_m^{(0)} \varphi_m^{-1}}{b_n - a_m} \psi_n\right). \end{aligned} \tag{7}$$

Equations (7) can be rewritten as

$$\nabla_\eta \varphi_n - \sum_{m=1}^{N_2} \frac{\psi_m V_m^{(0)} \psi_m^{-1}}{a_n - b_m} \varphi_n = \varphi_n A_n, \quad \nabla_\xi \psi_n - \sum_{m=1}^{N_1} \frac{\varphi_m U_m^{(0)} \varphi_m^{-1}}{b_n - a_m} \psi_n = \psi_n B_n. \tag{8}$$

Here $A_n(\xi, \eta)$, $B_n(\xi, \eta)$ are arbitrary matrix functions commuting with the matrices $U_n^{(0)}$ and $V_n^{(0)}$, respectively. Their appearance in Eqs. (8) is related to the obvious nonuniqueness in determining the matrices φ_n, ψ_n.

Equations (8) imply the easily verified relation

$$\left[\nabla_\eta, \sum_{n=1}^{N_1} U_n\right] = \left[\nabla_\xi, \sum_{n=1}^{N_2} V_n\right]. \tag{9}$$

We consider the functional

$$S = \int_{\xi_1}^{\xi_2} d\xi \int_{\eta_1}^{\eta_2} d\eta \, \mathrm{Sp}\left[\sum_{n=1}^{N_1} \varphi_n^{-1} \nabla_\eta \varphi_n U_n^{(0)} - \sum_{n=1}^{N_2} \psi_n^{-1} \nabla_\xi \psi_n V_n^{(0)} - \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \frac{\psi_m V_m^{(0)} \psi_m^{-1} \varphi_n U_n^{(0)} \varphi_n^{-1}}{a_n - b_m}\right], \tag{10}$$

where ξ_1, ξ_2, η_1, η_2 are chosen arbitrarily, and find conditions such that the variation δS vanishes. It is easy to check that variation with respect to φ_n, ψ_n leads to Eqs. (7), while variation with respect to g leads to Eqs. (9).

Thus, the functional S is an action for Eqs. (7). If S is real, then Eqs. (7) are a Hamiltonian system for which the Hamiltonian and symplectic form are evaluated in the obvious way. In general, the question of whether the Hamiltonian structure constructed is unique remains open.

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