F_4. The normal form of a singularity of the type F_4 is $x^2 + y^3$ (the equation of the boundary is $x = 0$). The versal deformation is $x^2 + y^3 + \lambda_3 xy + \lambda_2 x + \lambda_1 y + \lambda_0$. Let us set $\langle z, z\rangle = 24\,\lambda_3(z)$. The quantity $\langle \nabla\lambda_i, \nabla\lambda_j\rangle$ occurs at the intersection of the row λ_i and the column λ_j of Table 1 (cf. [10]).

E_6. The normal form of a singularity of the type E_6 is $x^3 + y^4 + u$ (the equation of the boundary is $u = 0$). The versal deformation is

$$x^3 + y^4 + u + \lambda_5 xy^2 + \lambda_4 xy + \lambda_3 y^2 + \lambda_2 x + \lambda_1 y + \lambda_0.$$

Let us set $\langle z, z\rangle = 24\,\lambda_5(z)$. The formulas for $\langle \nabla\lambda_i, \nabla\lambda_j\rangle$ are given in Table 2.

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BENNEY EQUATIONS AND QUASICLASSICAL APPROXIMATION IN THE METHOD OF THE INVERSE PROBLEM

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The system of Benney equations, whose study is one of the main objects of the present article, was introduced in 1973 (see [1]) to describe long waves in shallow fluid with free surface in gravitational field. It has the form

$$h_t + \operatorname{div}\int_0^h u\,dz = 0, \quad (1)$$

$$u_t + (u\nabla)\,u + wu_z + \nabla h = 0, \quad (2)$$

$$w_z + \operatorname{div} u = 0. \quad (3)$$

Here $r = (x, y)$ is a two-dimensional vector in the horizontal plane, z is the vertical coordinate such that $0 < z < h$, where $h = h(r, t)$ is the form of the fluid surface, $u = u(r, t)$ is the vector of the horizontal velocity and $w = w(r, t)$ is the vertical component of velocity. The acceleration due to gravity is set equal to one. Benney

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equations are obtained from exact equations of hydrodynamics as the zero term of the expansion in the parameter h/L, where L is the characteristic wavelength; we do not take any assumption about the potential of the fluid flow for their derivation. The elementary frequency solution of the Benney equations $u_z = 0$ and $w = 0$ corresponds to the potential case. Then $u = u(r, t)$ and $h = h(r, t)$ are subjected to the classical hydrodynamical equations of shallow water

$$h_t + \operatorname{div} hu = 0,$$
$$u_t + (u\nabla) u + \Delta h = 0. \tag{4}$$

The system of Benney equations has been studied mainly in the one-dimensional variant, when the dependence on one of the coordinates in the plane x, y, say on y, is realized. Then it has the form

$$h_t + \frac{\partial}{\partial x}\int_0^h u\, dz = 0,$$
$$u_t + uu_x - u_z\int_0^z u_x dx + h_x = 0. \tag{5}$$

The corresponding one-dimensional system (4) is

$$h_t + (hu)_x = 0, \quad u_t + uu_x + h_x = 0. \tag{6}$$

We know that system (6) has an infinite number of integrals of motion. These integrals have the form

$$I = \int \Phi(h, u)\, dx, \quad h\Phi_{hh} - \Phi_{uu} = 0. \tag{7}$$

As the basic set of integrals we can use the polynomial solutions of Eq. (7). These include

$$\Phi_1 = h, \quad \Phi_2 = hu, \quad \Phi_3 = \frac{1}{2}(hu^2 + h^2), \quad \Phi_4 = \frac{1}{3}hu^3 + h^2u, \ldots . \tag{8}$$

The Benney equations (5) also have analogous integrals. They were found even in the first article of Benney [1] and studied by Miura [2] and Kupershmit and Manin in [3], who found simple generating functions for them. All these integrals depend on moments of the longitudinal velocity

$$A_n = \int_0^h u^n\, dz. \tag{9}$$

They have the form $I_n = \int \Phi_n dx$, where Φ_n is a polynomial in the moments. In particular,

$$\Phi_1 = A_0, \quad \Phi_2 = A_1, \quad \Phi_3 = A_2 + A_0^2, \ldots . \tag{10}$$

For $u_z = 0$ these integrals transform into the polynomial integrals (7) and (8). The Benney system is written in a simple manner in the moments:

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1}\frac{\partial A_0}{\partial x} = 0. \tag{11}$$

It has been shown in [3] that system (11) can be expressed in the Hamiltonian form

$$\frac{\partial A}{\partial t} + \{A, I_3\} = 0, \tag{12}$$

where $A = (A_0, A_1, \ldots)$ is the set of moments and the symbol { } denotes a Poisson bracket, having sufficiently special structure. It has also been shown that the integrals I_n with respect to this bracket commute, which enables us to consider "higher Benney equations," obtained on replacing I_3 by I_n.

The problem of the Hamiltonian structure of the two-dimensional Benney equations has not been considered.

The existence of an infinite set of commuting integrals has led to the conjecture about the applicability of the method of the inverse scattering problem to the Benney equation.

In the present article we will show that this application is indeed possible in a certain sense. Namely, the Benney system turns out to be the quasiclassical limit of an infinite system of connected Schrödinger equations, whose integrability by the method of the inverse scattering problem was established long ago (Sec. 3). The existence of the quasiclassical limit of the systems that are integrable by the method of the inverse scattering problem is a sufficiently common fact – it is decomposed in simple examples in Sec. 1. For the study of the Benney system its equivalence with a certain infinite system of hydrodynamical equations of dimension less by one turns out to be highly important. These systems are naturally connected also with equations of Vlasov type; Sec. 2 is devoted to their presentation and this connection. The passage to the language of an infinite hydrodynamical system enables us to simplify very much the solution of the problem about the Hamiltonian structure of Benney equations – they turn out to be trivial in the new variables. The most important result of the present article is the construction of new integrals of motion of the Benney system and that do not reduce the moments and the proof of their commutativity with respect to the new Hamiltonian structure. Certain generalizations of the methods used in the article are set forth in Sec. 5.

1. Quasiclassical Limits of Systems That Are Integrable by the Method of the Inverse Scattering Problem

Let us consider the system of equations in the function Ψ:

$$\Psi_{xx} + u\Psi = \lambda\Psi, \tag{13}$$

$$\Psi_t = A\Psi. \tag{14}$$

Here $A = (\partial^n/\partial x^n) + v_1(\partial^{n-2}/\partial x^{n-2}) + \ldots$, $n = 2k + 1$, is a linear differential operator of odd order. Let $u, v_n \to 0$ as $x \to \pm\infty$. The conditions of symmetry of Eqs. (13) and (14) leads to the nonlinear equation in u

$$u_t = \frac{\partial}{\partial x} P_n[u], \tag{15}$$

which represents one of the higher analogs of the KdV equation.

Let us accomplish the lengthening of scales with respect to x and t in system (13)-(14) so that the substitution $\partial/\partial x \to \varepsilon\partial/\partial x$, $\partial/\partial t \to \varepsilon\partial/\partial t$ takes place. Moreover, the nonlinear operator P_n becomes a polynomial in powers of ε. Let us now set $\varepsilon = 0$. We call the equation

$$u_t = \lim_{\varepsilon\to 0} \frac{\partial}{\partial x} P_n(\varepsilon, [u]) \tag{16}$$

so obtained, the quasiclassical limit of Eq. (15). We show that Eq. (16) can be computed directly, bypassing the computation of the initial equation (15).

For this we make the substitution

$$\varepsilon^2\Psi_{xx} + u\Psi = \lambda\Psi$$

in the equation

$$\Psi = e^{\frac{1}{\varepsilon}\int_{-\infty}^{x} \chi\, dx} \tag{17}$$

As a result, we obtain the Riccati equation

$$\varepsilon\chi_x + \chi^2 + u = \lambda. \tag{18}$$

Equation (14) also transforms into a nonlinear equation in χ, representing a polynomial in ε. Let us set $\varepsilon = 0$ in these equations. The following system of equations arises:

$$\begin{gathered} \chi^2 + u = \lambda, \\ \frac{\partial\chi}{\partial t} = \frac{\partial}{\partial x} A(\chi), \end{gathered} \tag{19}$$

where $A(\chi) = \chi^n + v\chi^{n-1} + \ldots$ is a polynomial in powers of χ. The consistency conditions for system (19) have form

$$\frac{\partial u}{\partial t} = u_x \frac{\partial A}{\partial \chi} - 2\chi \frac{\partial A}{\partial x} \tag{20}$$

(the derivative $\partial A/\partial x$ is taken for constant χ). Equating the coefficients of like powers of χ on both the sides of Eq. (20), we get a system of equations in the coefficients v_k:

$$\begin{aligned} nu_x &= 2v_{1x}, \\ (n-2)u_x v_1 &= 2v_{2x}, \\ &\cdots\cdots\cdots \\ (n-k)\,u_x\,v_{k-1} &= 2v_{kx}, \\ &\cdots\cdots\cdots, \end{aligned} \tag{21}$$

from which v_k can be found by the recurrence method. Equating the coefficients of the zero power of χ, we get

$$\frac{\partial u}{\partial t} = u_x u_{n-2} = \frac{n}{n-1}\,\frac{1}{2^{n-2}}\,\frac{\partial}{\partial x}\,u^{n-1}. \tag{22}$$

Equation (22) is also the quasiclassical limit of a higher KdV equation. For an ordinary KdV equation we have

$$n=3 \qquad u_t = \frac{3}{4}\,\frac{\partial}{\partial x}\,u^2.$$

It follows immediately from system (19) that the quantity

$$I(\lambda) = \int_{-\infty}^{\infty} \chi\,dx = \int_{-\infty}^{\infty} \sqrt{\lambda - u}\,dx \tag{23}$$

is an integral of system (22).

Taking the limit as $\lambda \to \infty$ in (23) and considering the expansion in powers of $1/\lambda$, we arrive at the statement (however, completely obvious) that the quantities $I_n = \int_{-\infty}^{\infty} u^n\,dx$ are integrals of motion of Eq. (22).

The following Poisson bracket, defined on functionals of u, is connected with operator (13):

$$\{\alpha, \beta\} = \int_{-\infty}^{\infty} \left(\frac{\delta\alpha}{\delta u}\,\frac{\partial}{\partial x}\,\frac{\delta\beta}{\delta u} - \frac{\delta\beta}{\delta u}\,\frac{\partial}{\partial x}\,\frac{\delta\alpha}{\delta u}\right) dx.$$

We compute the Poisson bracket between the functionals $I(\lambda)$ and $I(\lambda')$:

$$\{I(\lambda), I(\lambda')\} = \frac{1}{4} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\lambda - u}}\,\frac{\partial}{\partial x}\,\frac{1}{\sqrt{\lambda' - u}} - \frac{1}{\sqrt{\lambda' - u}}\,\frac{\partial}{\partial x}\,\frac{1}{\sqrt{\lambda - u}}\right) dx\,.$$

By the same token we have proved the fact (which can be easily and directly verified) that $\{I_n, I_m\} = 0$.

2. Infinite Systems of Hydrodynamical Equations and Vlasov Equations

In Sec. 3 we show that Benney system (5) and hydrodynamical system (6) are quasiclassical limits of systems that are integrable by the method of the inverse scattering problem. As a preliminary it is necessary to represent the Benney system as an infinite system of hydrodynamical equations.

Let ξ be a parameter that runs over a certain domain Ω in the k-dimensional real space $\mathbf{R}^k$. Let $\eta(r, \xi, t)$ and $u(r, \xi, t)$ be functions of ξ, the vector coordinate r, and time t. Let us consider the system of equations

$$\begin{aligned} \eta_t + \operatorname{div}\eta u &= 0, \\ u_t + (u\nabla)\,u + \nabla h &= 0, \\ h &= \int_{\Omega} \eta(r, \xi, t)\,d\xi. \end{aligned} \tag{24}$$

We will call system (24) an infinite hydrodynamical system. (We can also give other examples of infinite hydrodynamical systems. Thus, if we set

$$h(r,t) = \frac{1}{4\pi} \int_{\Omega} d\xi \int \frac{\eta(r', \xi, t)}{|r - r'|}\,dr',$$

then we get an infinite hydrodynamical system that describes a gravitating gas.) System (24) is Hamiltonian; the method of introducing the Hamiltonian structure depends on the number l of components of vector r. In the one-dimensional case it can be introduced in the following manner:

$$\eta_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta u} = 0, \quad v_t + \frac{\partial}{\partial x}\frac{\delta H}{\delta \eta} = 0, \tag{25}$$

$$H = \frac{1}{2}\int\limits_{-\infty}^{\infty} dx \int\limits_{\Omega} d\xi\, \eta v^2 + \frac{1}{2}\int\limits_{-\infty}^{\infty} h^2\, dx. \tag{26}$$

System (25) admits the potential substitution $u = \Phi_x$. In the potential case the Hamiltonian structure has the form

$$\eta_t = \frac{\delta H}{\delta \Phi}, \quad \Phi_t = -\frac{\delta H}{\delta u} \tag{27}$$

for arbitrary l.

In the one-dimensional case structures (25) and (27) are equivalent. In the general three-dimensional case the Hamiltonian structure is introduced with the help of the Clebsch variables (see, e.g., [5])

$$u = \nabla\Phi + \frac{\lambda}{\eta}\nabla\mu.$$

Now the pairs of variables (Φ, η) and (λ, μ) are canonically conjugate. In the two-dimensional case the Hamiltonian structure can be obtained by the degeneration of the three-dimensional case.

Let us also observe that the parameter ξ is defined up to an invertible transformation $\xi \to \xi' = \tilde{f}(\xi)$, $\eta(\xi) = \eta(\xi')f'(\xi)$.

The equality $u = u(r, \xi, t)$ gives a mapping of the domain Ω into a certain set $\tilde{\Omega}$ of values of the vector u.

Let us consider function $f(r, v, t)$ that is defined by the formula

$$f(r, v, t) = \int\limits_{\Omega} \eta(r, \xi, t)\, \delta(v - u(r, \xi, t))\, d\xi. \tag{28}$$

It is easily verified by a direct check that f satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial r} - \nabla h \frac{\partial f}{\partial v} = 0, \quad h = \int f\, dv. \tag{29}$$

Thus, in this case each solution of an infinite hydrodynamical system generates a certain solution of the Vlasov equation. If r is a vector of the same dimension k as ξ and the domain $\tilde{\Omega}$ can be mapped simply onto Ω, then we can sort out all the solutions of the Vlasov equation in this manner. Let this be the solution of the Cauchy problem with the condition

$$f(r, v, t)|_{t=0} = \varphi(r, v). \tag{30}$$

Let us consider the solution of the corresponding infinite hydrodynamical system with the initial conditions

$$\eta(r, \xi, t)|_{t=0} = \varphi(r, \xi), \quad u|_{t=0} = \xi. \tag{31}$$

(We do not consider here the problem of existence and uniqueness of solutions of the corresponding initial problems.)

The solutions of problems (30) and (31) will be equivalent as long as the mapping $u \to \xi$ remains invertible.

We introduce new variables in the system of Benney equations (1)-(3); we define the velocities u and w parametrically with the help of a parameter ξ $(0 < \xi < 1)$:

$$u = u(r, \xi, t), \quad z = z(r, \xi, t). \tag{32}$$

We require that the function $z(r, \xi, t)$ must satisfy the equation

$$z_t + (u\nabla)z = w. \tag{33}$$

Moreover, we set

$$z(r,\ 0,\ t) = 0, \quad z(r,\ 1,\ t) = h(r,\ t). \tag{34}$$

Simple computations show that u and η satisfy the system (24), where

$$h(r, t) = \int_0^1 \eta(r, \xi, t)\, d\xi. \tag{35}$$

Thus, the system of Benney equations (1)-(3) is equivalent to a two-dimensional infinite hydrodynamical system, where the interval $0 < \xi < 1$ plays the role of the set Ω. In the two-dimensional case it generates certain particular solutions of the Vlasov equation. In the one-dimensional case it is equivalent to the Vlasov equation, if u is a single-valued invertible function of z for each x. In this case only, the Benney system is equivalent to momental system (11). Function $f(x, v, t)$ has a simple physical meaning. We introduce the function $z = z(x, v, t)$. It follows from (28) that $f = \partial z/\partial v$.

3. Benney System as the Quasiclassical Approximation to a System, Integrals by the Method of the Inverse Scattering Problem

Each infinite hydrodynamical system has a particular solution of the form

$$\eta(r, \xi, t) = \sum_{n=1}^{N} \eta_n \delta(\xi - \xi_n), \tag{36}$$

where ξ_n is a set of points of Ω. In our case assumption (36) means that the fluid is divided into N layers with thicknesses η_n. Inside each of these layers the velocity u_n of the fluid is constant with respect to z. Equations for the quantities η_n and u_n have the form (we will be interested only in the one-dimensional case)

$$\begin{gathered} \frac{\partial \eta_n}{\partial t} + \frac{\partial}{\partial x} u_n \eta_n = 0, \\ \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} + \frac{\partial h}{\partial x} = 0, \quad h = \sum_{n=1}^{N} \eta_n. \end{gathered} \tag{37}$$

For N = 1 we get system (4) from (37).

Let us consider the Schrödinger system of nonlinear equations

$$i\psi_{nt} + \frac{1}{2}\psi_{nxx} - h\psi_n = 0, \quad h = \sum_{n=1}^{N} |\psi_n|^2. \tag{38}$$

Making the substitution

$$\psi_n = \sqrt{\eta_n}\, e^{-i \int_{-\infty}^{x} u_n dx} \tag{39}$$

in (38), we get

$$\begin{gathered} \frac{\partial \eta_n}{\partial t} + \frac{\partial}{\partial x} \eta_n u_n = 0, \\ \frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} + \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \frac{1}{\sqrt{\eta_n}} \frac{\partial^2}{\partial x^2} \sqrt{\eta_n}. \end{gathered} \tag{40}$$

Making the quasiclassical passage $\partial/\partial t \to \varepsilon \partial/\partial t$ and $\partial/\partial x \to \varepsilon \partial/\partial x$ in (39), $\varepsilon \to 0$, we see that (40) transforms into (37). Thus, the Benney system is the quasiclassical limit of the Schrödinger system of nonlinear equations (38).

System (38) can be integrated [4] by the method of the inverse scattering problem. It represents the condition of consistency of the two systems of equations for the (N + 1)-component vector χ_0, χ_k (k = 1, ..., N):

$$\chi_{0x} = -\frac{i\lambda}{2}\chi_0 + R, \tag{41}$$

$$\chi_{kx} = \frac{i\lambda}{2}\chi_k + \psi_k\chi_0, \tag{42}$$

$$\chi_{0t} = -\frac{i\lambda^2}{4}\chi_0 + \frac{\lambda}{2}R + \frac{i}{2}S - \frac{i}{2}u\chi_0, \tag{43}$$

$$\chi_{kt} = \frac{i\lambda^2}{4}\chi_k + \left(\frac{\lambda}{2}\psi_k - \frac{i}{2}\psi_{kx}\right)\chi_0 + \frac{i}{2}\psi_k R,$$

$$R = \sum_{k=1}^{N}\bar{\psi}_k\chi_k, \quad S = \sum_{k=1}^{N}\bar{\psi}_{kx}\chi_k. \tag{44}$$

We make substitution (39) and also the substitution

$$\chi_0 = e^{\frac{i}{\varepsilon}\int_{-\infty}^{x}\chi\,dx + \frac{i}{2}\lambda x}, \quad \chi_k = \xi_k e^{\frac{i}{\varepsilon}\int_{-\infty}^{x}\chi\,dx - \frac{i}{\varepsilon}\int_{-\infty}^{x}u_n dx + \frac{i}{2}\lambda x} \tag{45}$$

in Eqs. (41)-(44). We simultaneously make the transformations

$$\frac{\partial}{\partial t} \to \varepsilon\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \to \varepsilon\frac{\partial}{\partial x}$$

and take limit as $\varepsilon \to 0$.

From Eq. (42) we get

$$\xi_n = \frac{i\sqrt{\eta_n}}{u_n - \chi},$$

and Eqs. (41) and (43) take the form

$$\chi + \sum_{n=1}^{N}\frac{\eta_n}{\chi - u_n} + \lambda = 0, \tag{46}$$

$$\frac{\partial\chi}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}\chi^2 - h\right) = 0. \tag{47}$$

It is easily verified that the condition for the consistency of Eqs. (46) and (47) coincides with system (37).

4. Conservation Laws

System (46)-(47) actually arose in the theory of Benney system and has been used for the computation of integrals of motion. However, only one infinite series of integrals has been computed. We show how to compute the other series. Let us observe that Eq. (46) determines $\chi(\lambda)$ as an algebraic function on an (N + 1)-sheeted Riemannian surface. Let $\lambda \to \infty$. On each sheet of the Riemannian surface there exists an asymptotic expansion of the function χ in powers of $1/\lambda$. It follows from (47) that the quantity

$$I(\lambda) = \int_{-\infty}^{\infty}\chi\,dx \tag{48}$$

is an integral of motion. All terms of the asymptotic expansion of function $I(\lambda)$ on each sheet of the Riemannian surface are also integrals. One of these sheets is distinguished (we call it the zero sheet). On it $\chi \to -\lambda$ as $\lambda \to \infty$, and Eq. (46) can be expressed in the form

$$\chi = -\lambda - \frac{1}{\chi}\sum_{n=0}^{\infty}\frac{A_n}{\chi^n}.$$

Expanding χ in the asymptotic series $\chi = -\lambda + \sum_{n=1}^{\infty}\frac{\Phi_n}{\lambda^n}$, we compute the conservation laws $I_n^0 = \int_{-\infty}^{\infty}\Phi_n\,dx$. All Φ_n can be expressed in terms of the moments A_n; the first of them are given by Eqs. (10). These conservation laws have also been computed in [1] by Benney.

The remaining N sheets of the Riemannian surface can be defined as follows: On the k-th sheet $\chi \to u_k - \eta_k/\lambda$ as $\lambda \to \infty$. Setting $\chi = \chi^k + u_k$, we get

$$\chi^k = -\frac{\eta_k}{\lambda} - \frac{\chi^k}{\lambda}\left(u_k + \chi_k + \sum_{n\neq k}\frac{\eta_n}{\chi_k + u_k - u_n}\right).$$

$\chi_k \ll u_k - u_n$ for $\lambda \to \infty$ and

$$\chi^k = -\frac{\eta_k}{\lambda} - \frac{\chi^k}{\lambda}\left(u_k + \chi^k + \sum_{s=0}^{\infty} J_k^s \chi_k^s\right). \tag{49}$$

Here

$$J_k^s = (-1)^s \sum_{n\neq k}\frac{\eta_n}{(u_k - u_n)^{s+1}}. \tag{50}$$

Substituting χ^k in the form $\chi^k = \sum_{s=1}^{\infty}\frac{\Phi_s^k}{\lambda^k}$ in Eq. (49) and equating the coefficients of like powers of $1/\lambda$, we find new integrals of motion

$$I_n^k = \int_{-\infty}^{\infty} \Phi_n^k \, dx, \quad n = 1, \ldots, \infty, \quad k = 1, \ldots, N. \tag{52}$$

Moreover,

$$\Phi_1^k = -\eta_k, \quad \Phi_2^k = \eta_k (u_k + J_k^0) = \eta_k\left(u_k + \sum_{n\neq k}\frac{\eta_n}{u_k - u_n}\right), \ldots \quad . \tag{53}$$

All Φ_s^k are expressed in terms of η_k, u_k, and J_k^l.

In the general continuous case, for each n there exists a family of the integrals $I_n(\xi)$ that depends on a parameter ξ, $0 < \xi < 1$. To compute them we can use Eq. (48), having substituted ξ for k and having set

$$J^s(\xi) = (-1)^s \int_0^1 \frac{\eta(\xi')}{(u(\xi) - u(\xi'))^{s+1}} d\xi'. \tag{54}$$

The principal value of the integral in (54) is taken.

We also note that Eq. (46) can be written in the form

$$\chi^n + \lambda\chi^{n-1} + \ldots = 0.$$

Therefore, the integrals of motion are connected by the relation

$$I_n^0 + \sum_{k=1}^{N} I_n^k = 0. \tag{55}$$

In the continuous case

$$I_n^0 + \int_0^1 I_n(\xi)\, d\xi = 0. \tag{56}$$

The existence of the Hamiltonian structure (25) for the Benney equation enables us to define the Poisson bracket between any two functionals α and β from $\eta_n(x, t)$ and $u_n(x, t)$:

$$\{\alpha, \beta\} = \sum_{n=1}^{N} \int_{-\infty}^{\infty} dx \left(\frac{\delta\beta}{\delta u_n}\frac{\partial}{\partial x}\frac{\delta\alpha}{\delta\eta_n} + \frac{\delta\alpha}{\delta u_n}\frac{\partial}{\partial x}\frac{\delta\beta}{\delta\eta_n}\right). \tag{57}$$

Let us compute the Poisson bracket between the functionals $I(\lambda)$ and $I(\lambda')$. We have

$$\{I(\lambda), I(\lambda')\} = \sum_{n=1}^{N} \int_{-\infty}^{\infty} dx \left(\frac{\partial \chi(\lambda')}{\partial u_n} \frac{\partial}{\partial x} \frac{\partial \chi(\lambda)}{\partial \eta_n} + \frac{\partial \chi(\lambda)}{\partial u_n} \frac{\partial}{\partial x} \frac{\partial \chi(\lambda')}{\partial \eta_n} \right). \tag{58}$$

Here functions $\chi(\lambda)$ and $\chi(\lambda')$ may be situated on different sheets of the Riemannian surface. From (46) we have

$$\frac{\partial \chi(\lambda)}{\partial \eta_n} = -\frac{1}{\Delta} \frac{1}{\chi - u_n}, \quad \frac{\partial \chi(\lambda)}{\partial u_n} = \frac{1}{\Delta} \frac{1}{(\chi - u_n)^2}, \qquad \Delta = 1 - \sum_{k=1}^{N} \frac{\eta_k}{(\chi - u_k)^2}. \tag{59}$$

Substituting (59) in (58), after certain transformations we get $\{I(\lambda), I(\lambda')\} = 0$. Thus, all the integrals of the Benney system are found in involution.

5. Multidimensional Generalizations

The above-considered examples enable us to suggest a method, not at all connected with the method of the inverse problem, to construct multidimensional dynamical systems that have an infinite number of integrals of motion. Let the function $\chi = \chi(\lambda, x)$, $x = \{x_1, \ldots, x_n\}$, satisfy identically with respect to the parameter λ the following two equations simultaneously: the algebraic equation

$$F(\chi) = \lambda, \tag{60}$$

where F is a rational function of χ with coefficients depending on x_i, and the differential equation

$$\operatorname{div} A = 0. \tag{61}$$

Here $A = \{A_1, \ldots, A_n\}$ is a rational vector-valued function of χ with coefficients depending on x_i. Differentiating (60) with respect to x_i, we get

$$\frac{\partial F}{\partial \chi} \frac{\partial \chi}{\partial x_i} + \frac{\partial F}{\partial x_i} = 0. \tag{62}$$

On the other hand, (61) can be written in the form

$$\sum_{i=1}^{n} \left(\frac{\partial A_i}{\partial \chi} \frac{\partial \chi}{\partial x_i} + \frac{\partial A_i}{\partial x_i} \right) = 0. \tag{63}$$

In Eqs. (62) and (63) the derivatives $\partial F/\partial x_i$ and $\partial A/\partial x_i$ are taken for constant χ.

From Eqs. (62) and (63) we get the condition for the consistency of Eqs. (60) and (61):

$$G = \sum_{i=1}^{n} \left(\frac{\partial A_i}{\partial \chi} \frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial \chi} \frac{\partial A_i}{\partial x_i} \right) = 0. \tag{64}$$

Condition (64) represents the requirement that the rational function G of χ be equal to zero and imposes a finite number of relations on the coefficients of the functions F and A_i. A system of partial differential equations arises that obviously has an infinite number of integrals of motion.

The Benney system and the quasiclassical limits, considered in Sec. 1, of higher KdV equations are obviously systems of the type (64). We consider an essentially multidimensional example. Let the functions F and A_i have distinct simple poles:

$$F = \sum_{k=1}^{N} \frac{f_k}{\chi - u_k}, \quad A_i = \sum_{i=1}^{N_k} \frac{a_{ik}}{\chi - \varphi_{ik}}. \tag{65}$$

The quantities f_k, u_k, a_{ik}, and φ_{ik} are unknown functions of x_i. Two such functions are connected with each pole of F and A_i. Since no two poles coincide, each simple pole of F and A_i leads to a simple and a double pole of G, i.e., generates two differential equations. Thus, the number of equations in (64) is equal to the number of unknown functions. It is not necessary to write out these equations, since their applications are not known. Nevertheless, the possibility of construction of multidimensional partial differential equations, admitting an infinite number of integrals of motion, is of fundamental importance.

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SECOND TERM OF THE SPECTRAL ASYMPTOTIC EXPANSION OF THE LAPLACE – BELTRAMI OPERATOR ON MANIFOLDS WITH BOUNDARY

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INTRODUCTION

In this article we trace the main points of the proof of an asymptotic formula of Weyl.

Let X be a d-dimensional compact Riemann C^∞-manifold with boundary $Y \in C^\infty$, let Δ be the Laplace–Beltrami operator on X, let $\mathcal{A} = -\Delta + \mathcal{A}'$, where $\mathcal{A}'$ is a formal self-adjoint differential operator of the first order, let $A_\pm$: $L_2(X) \to L_2(X)$ be a self-adjoint extension of $\mathcal{A}$ to the domain consisting of functions which satisfy the boundary condition

$$\left(\frac{\partial}{\partial n} + \gamma\right) u\,|_Y = 0, \qquad \gamma \in C^\infty(Y), \tag{+}$$

or

$$u\,|_Y = 0, \tag{−}$$

and let n be the unit interior normal to Y. Let $N_\pm(\lambda)$ be the number of eigenvalues of $A_\pm$ (counting multiplicity) which do not exceed λ^2.

We consider S*X, the fiber of cotangent unit spheres over X, where we identify the points (x, ξ) and (x, η) if $x \in Y$, $\xi - \eta = kn(x)$. We will call a curve in S*X

$$\frac{d\rho}{dt} = \frac{1}{2} H_g(\rho),$$

where $g(x, \xi)$ is a Riemann quadratic form and H_g is a Hamiltonian field generated by g; in addition, a geodesic must lie, except perhaps for its end points, in $S^*(X \setminus Y)$; t (length) is an intrinsic parameter along a geodesic. We will call a curve in S*X a geodesic billiard if it consists of segments of geodesics such that the end of the preceding segment and the initial point of the following segment belong to $S^*X|_Y$ and are equivalent; the length t is an intrinsic parameter along geodesic billiards.

It is possible to show that there exists a set $\Sigma \subset S^*X$ of first Baire category and measure zero such that it is possible to draw through each point of $S^*X \setminus \Sigma$ a geodesic billiard which is infinitely long on both sides and which lies in $S^*X \setminus \Sigma$ such that each of its intervals of finite length contains a finite number of components, and all of the geodesics entering it are transverse to the boundary. Thus a measure-preserving continuous flow $\Phi(t)$ is given on $S^*X \setminus \Sigma$. We will call a point $\rho \in S^*X \setminus \Sigma$ periodic if there exists $t \neq 0$ such that $\Phi(t)\rho = \rho$.

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