

DEGENERATIVE DISPERSION LAWS, MOTION INVARIANTS AND KINETIC EQUATIONS

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1. Degenerative dispersion laws

Let us consider the following problem. Imagine a homogeneous nonlinear medium in which only one type of wave with dispersion law $\omega(k)$ may propagate. Let the nonlinearity of the medium be quadratic while the equation

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2) \quad (1)$$

defines a nonzero manifold Γ whose codimension in k_1, k_2 space is unity. Equation (1) means that dispersion law $\omega(k)$ allows decay processes. The nonlinearity being weak, waves in such a system may be described statistically by introducing the average occupation numbers n_k of the state with momentum k . The time evolution of n_k is governed by the kinetic equation

$$\begin{aligned} \frac{\partial n_k}{\partial t} = & 2\pi \int \{|V_{kk_1k_2}|^2 \delta_{k-k_1-k_2} \delta_{\omega_k-\omega_{k_1}-\omega_{k_2}} (n_{k_1}n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) \\ & + 2|V_{k_1k_2}|^2 \delta_{k_1-k-k_2} \delta_{\omega_{k_1}-\omega_k-\omega_{k_2}} (n_{k_1}n_{k_2} + n_k n_{k_1} - n_k n_{k_2})\} dk_1 dk_2, \end{aligned} \quad (2)$$

where $V_{kk_1k_2}$ is the wave interaction matrix element corresponding to the interaction (1). Equation (2) is nontrivial if $V_{kk_1k_2}$ is nonzero on the manifold Γ . Equation (2) has the obvious motion invariants

$$\mathcal{E} = \int \omega_k n_k dk, \quad \mathbf{P} = \int k n_k dk, \quad (3)$$

\mathcal{E} and \mathbf{P} may be identified with energy and momentum of the wave system, respectively. Now let us discuss the question, whether (2) may have another independent motion invariant of the form

$$I = \int f(k) n_k dk. \quad (4)$$

By calculating the time derivative $\partial I / \partial t$, it is easy to prove that it may have one if and only if the equality

$$f(k_1 + k_2) = f(k_1) + f(k_2) \quad (5)$$

holds true on the manifold Γ . In other words eq. (5) must define the same manifold as (1).

Such a situation is certainly an exception. If N is the dimension of the medium, the dimension of the manifold Γ is $2N - 1 \geq N$. The analysis of the dimensions lead us to the consequence that in the general case the dispersion law $\omega(k)$ has to be unambiguously defined by the manifold Γ to within linear function of k . In case such unambiguity does not take place we shall call the dispersion law degenerative. Let us show that degenerative dispersion laws exist. Consider a two-dimensional medium ($N = 2$) and introduce the notation $k_x = p$, $k_y = q$. Let us examine the dispersion law

$$\omega(p, q) = p^3 + \frac{3q^2}{p}. \quad (6)$$

Equation (1)

$$(p_1 + p_2)^3 + 3 \frac{(q_1 + q_2)^2}{p_1 + p_2} = p_1^3 + \frac{3q_1^2}{p_1} + p_2^3 + \frac{3q_2^2}{p_2} \quad (7)$$

may be satisfied by parametrization

$$p_1 = \xi_1 - \xi_2, \quad p_2 = \xi_2 - \xi_3, \quad q_1 = \xi_1^2 - \xi_2^2, \quad q_2 = \xi_2^2 - \xi_3^2, \quad (8)$$

which directly gives coordinates on the manifold Γ . Let now $f(k)$ be of the form

$$f(k) = f(p, q) = F\left(\frac{q}{p} + p\right) - F\left(\frac{q}{p} - p\right). \quad (9)$$

Using parametrization (8) one obtains

$$f(k_1) = F(2\xi_1) - F(2\xi_2), \quad f(k_2) = F(2\xi_2) - F(2\xi_3). \quad (10)$$

In addition

$$f(k_1 + k_2) = F(2\xi_1) - F(2\xi_3). \quad (11)$$

Thus eq. (5) is satisfied and dispersion law (6) is degenerative. At the same time any dispersion law of the form (9) is degenerative. (It is worth mentioning that it is easy to obtain (6) from (9) taking $F(\xi) = \xi^3/2$.) From the above discussion it follows that for each of these dispersion laws eq. (2) has not one, but an infinite number of additional motion invariants with $f(k)$ given by (9). The function $F(\xi)$ is arbitrary.

S. V. Manakov* informed us recently of a more general example of the degenerative dispersion law. As before, $N = 2$ and the parametrization of the manifold Γ is given by

$$\begin{aligned} p_1 &= \xi_1 - \xi_2, & p_2 &= \xi_2 - \xi_3, \\ q_1 &= a(\xi_1) - a(\xi_2), & q_2 &= a(\xi_2) - a(\xi_3). \end{aligned} \quad (12)$$

Let us define a dispersion law $\omega(p_1, q_1)$ in parametric form

$$\omega(p_1, q_1) = b(\xi_1) - b(\xi_2), \quad (13)$$

where $a(\xi)$ and $b(\xi)$ are arbitrary functions of a one variable. To calculate $\omega(p_1, q_1)$ directly one has to obtain p_1, q_1 from the first pair of eqs. (12) and substitute them into (13). This procedure obviously leads us to the degenerative dispersion law for any $b(\xi)$. If $a(\xi) = \xi^2$ we obtain the case mentioned above.

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The examples of the degenerative dispersion laws for $N > 2$ are unknown. It may be supposed that formulae (12) and (13) give the most general form of the degenerative dispersion law for $N = 2$, but no proof of this fact is known.

If the decay processes (1) are impossible in the medium or the major nonlinear terms of the motion equations are cubic then the associated kinetic equation describes a four wave interaction,

$$\omega(k_1) + \omega(k_2) = \omega(k_3) + \omega(k_4), \quad k_1 + k_2 = k_3 + k_4. \quad (14)$$

Equations (14) define the $3N - 1$ -dimensional manifold in $3N$ -dimensional space. Up to now, there are no dispersion laws known, degenerative with respect to the interaction (14). It may be suggested that such dispersion laws do not exist in spaces of any dimensions $N > 1$. Now let us consider the medium in which several types of waves with dispersion laws $\omega_i(k)$, $i = 1, 2, \dots, s$ may propagate. Then the medium must be described by a system of kinetic equations. In the simplest case $s = 3$ and there is only one possible interaction process being described by the resonant condition

$$\omega_1(k_1 + k_2) = \omega_2(k_1) + \omega_3(k_2). \quad (15)$$

The associated kinetic equations for occupation numbers $n_i(k)$ are given by

$$\begin{aligned} \frac{\partial n_{1k}}{\partial t} &= \int |V_{kk_1k_2}|^2 (n_{2k_1}n_{3k_2} - n_{2k_1}n_{1k} - n_{3k_2}n_{1k}) \delta_{k-k_1-k_2} \delta_{\omega_{1k}-\omega_{2k_1}-\omega_{3k_2}} dk_1 dk_2, \\ \frac{\partial n_{2k_1}}{\partial t} &= \int |V_{kk_1k_2}|^2 (n_{2k_1}n_{1k} + n_{3k_2}n_{1k} - n_{2k_1}n_{3k_2}) \delta_{k-k_1-k_2} \delta_{\omega_{1k}-\omega_{2k_1}-\omega_{3k_2}} dk dk_2, \\ \frac{\partial n_{3k_2}}{\partial t} &= \int |V_{kk_1k_2}|^2 (n_{2k_1}n_{1k} + n_{3k_2}n_{1k} - n_{2k_1}n_{3k_2}) \delta_{k-k_1-k_2} \delta_{\omega_{1k}-\omega_{2k_1}-\omega_{3k_2}} dk dk_1. \end{aligned} \quad (16)$$

In addition to the energy and momentum integrals

$$\mathcal{E} = \int \sum_{i=1}^3 \omega_i(k) n_i(k) dk, \quad P = \int \sum_{i=1}^3 k n_i(k) dk \quad (17)$$

system (16) has another two motion invariants (Manly-Row relationships)

$$I_1 = \int (n_1(k) + n_2(k)) dk, \quad I_2 = \int (n_1(k) + n_3(k)) dk. \quad (18)$$

Let system (16) have one more motion invariant

$$I = \int \sum_i f_i(k) n_i(k) dk. \quad (19)$$

Then it is not difficult to prove that the equality

$$f_1(k_1 + k_2) = f_2(k_1) + f_3(k_2)$$

holds true on the manifold Γ . So the manifold Γ given by (15) does not define dispersion laws $\omega_i(k)$ unambiguously. In this case the set of the dispersion laws $\omega_i(k)$ may be called degenerative with respect to the interaction process (15). The example of the degenerative set of dispersion laws in two-dimensional space may easily be constructed. Namely let us take $\omega_i(p, q)$, $i = 1, 2, 3$ in the

following parametric form:

$$\begin{aligned} p &= \xi_1 - \xi_3, & p &= \xi_1 - \xi_2, & p &= \xi_2 - \xi_3, \\ q &= f(\xi_1) - h(\xi_3), & q &= f(\xi_1) - g(\xi_2), & q &= g(\xi_2) - h(\xi_3), \\ \omega_1 &= a(\xi_1) - c(\xi_3), & \omega_2 &= a(\xi_1) - b(\xi_2), & \omega_3 &= b(\xi_2) - c(\xi_3). \end{aligned} \quad (20)$$

Here f, g, h, a, b, c are arbitrary functions of a one variable. It is obvious that $\omega_i(p, q)$ form a degenerative set of the dispersion laws. In particular these functions may be linear. Then the dispersion laws become linear functions of ξ_i . It is easy to prove that they do not coincide pairwise. And conversely, any three different linear dispersion laws form a degenerative set.

Whether each given set of dispersion laws is degenerative or not may be easily tested. Let us state without detailed calculations the following result: the dispersion laws

$$\omega(p, q) = p^3 - \frac{3q^2}{p}, \quad (21)$$

$$\omega(p, q) = p^2 - q^2 \quad (22)$$

are nondegenerative with respect to the four wave interaction (14).

Now let us have a look at the following situation. Suppose there is a nondegenerative dispersion law but we know a priori that the kinetic equation has an additional motion invariant. In this case the wave interaction matrix element must be rather special. Indeed, the manifolds defined by eqs. (1) and (5) do not coincide but may have a lower dimension manifold Γ' as an intersection. In this case matrix element $V_{kk_1k_2}$ obviously equals to zero everywhere except on the manifold Γ' – and therefore is a Dirac δ -function of this manifold. In particular, if it has no singularities of the δ -function type, then the existence of the additional motion invariant means that the wave interaction matrix element is zero on the whole manifold Γ . Other three wave interaction processes and interactions with a greater number of waves may be treated in a similar way.

2. Motion invariants for integrable equations

At present, the extensive class of nonlinear wave systems having additional motion invariants is known. First of all, this class contains equations solvable using the inverse scattering transform (i.s.t.) method. Among them, the equations allowing Lax's representation of the type [1]

$$L_t - A_y + [L, A] = 0 \quad (23)$$

have been subjected to the most detailed investigations. Here L and A are differential in x operators. Variables x and y are treated as spatial variables and therefore equations of the type (23) are two-dimensional. At present there are four equations of the type (23), which proved to be applicable in physics and have been investigated to a greater or lesser extent. They are

$$u_t + \frac{1}{4} \left(u_{xxx} + 6uu_x - 3 \int u_{yy} dx' \right) = 0, \quad (24)$$

$$u_t = \frac{1}{4} \left(u_{xxx} + 6uu_x + 3 \int u_{yy} dx' \right) = 0, \quad (25)$$

$$u_{1t} = iu_2u_3, \quad u_{2t} + v_2\nabla u_2 = iu_1^*u_3, \quad u_{3t} + v_3\nabla u_3 = iu_1^*u_2, \quad (26)$$

$$2i\psi_t - \psi_{xx} + \psi_{yy} = (9/2)\psi|\psi|^2 + \phi_x\psi, \quad \phi_{xx} + \phi_{yy} = -(|\psi|^2)_x. \quad (27)$$

Equations (24), (25) represent two versions of the Kadomtzev–Petviashvili equations [2–6]. The difference in the sign of the last term leads to the essential difference in the solution behaviors. Let us call eqs. (24) and (25) KP–1 and KP–2, respectively. The set of eqs. (26) [8] is known as “2-d Three Wave Interaction” (TWI), while the set of eqs. (27) [10] is known as Davey–Stewartson equations (DS). The equation KP–2 has been investigated in more detail than others but the theory of equations of the type (23) is not so clear as that of the Korteweg de Vries’s equation and those similar to it. At present, even the Hamiltonian properties are not proved for the equations of the type (23) in the general case. Nevertheless, it is not difficult to see that the simplest equations (24)–(27) have the Hamiltonian properties that will be demonstrated below.

In the present paper we pay attention to the fact that the equations (24)–(27) have sets of the motion invariants. We calculate them using the representation (23). Motion invariants are found to be nonlocal in x , but, when u is independent of y , they appear to be the usual local integrals for the one-dimensional equations. Then we go into the question as to which possibilities described in section 1 are realized for each of the systems (24)–(27).

2.1. Equation KP–1

Let us consider at first the equation KP–1. Operators L and A have the following form [1]:

$$L = \frac{\partial^2}{\partial x^2} + u, \quad (28)$$

$$A = \frac{\partial^3}{\partial x^3} + \frac{3}{4} \left(u \frac{\partial}{\partial x} + \frac{\partial}{\partial x} u \right) + w, \quad w_x = \frac{3}{4} i u_y. \quad (29)$$

Let us obtain the motion invariants. For this purpose in the linear problem,

$$i\psi_{0y} + L\psi_0 = 0, \quad \psi_0 = \psi_\lambda e^{i\lambda y} \quad (30)$$

consider the eigenfunction

$$\psi_0 = \exp \left\{ -kx - ik^2y + \int_{-\infty}^x \chi(x', y, t, k) dx' \right\} \quad (31)$$

substituting (31) into (30) we find

$$\chi_x + \chi^2 + i \int_{-\infty}^x \chi_y dx' = 2k\chi. \quad (32)$$

Using (23) we can show that

$$\frac{d}{dt} I(k) = \frac{d}{dt} \int \int dx dy \chi(x, y, t, k) = 0 \quad (33)$$

regarded as a function of k . Since $I(k) \rightarrow 0$ as $k \rightarrow \infty$ we can expand $I(k)$ as $k \rightarrow \infty$ in powers of $1/k$:

$$I(k) = \sum \frac{I_n}{(2k)^n}, \quad I_n = \int \int dx dy \chi_n, \quad \chi(k) = \sum \frac{\chi_n}{(2k)^n}. \quad (34)$$

$$\chi_{n+1} = \sum_k \chi_k \chi_{n-k} + \hat{M} \chi_n, \quad \hat{M} = \frac{\partial}{\partial x} + i \int_{-\infty}^x \frac{\partial}{\partial y} \cdot dx' \quad (35)$$

$\chi_1 = u$.

The first three motion invariants are:

$$I_1 = \int u \, dx, \quad I_2 = \int \left(u_x + i \int_{-\infty}^x u_y \, dx' \right) dx,$$

$$I_3 = \int \left(u^2 + u_{xx} + 2i u_y - \int_{-\infty}^{xx'} u_{yy} \, dx'' \, dx' \right) dx.$$

Later we shall require quadratic (in u and its derivatives) part of I_n . With this purpose in view let us point out that in case $u \rightarrow 0$ at $|x|, |y| \rightarrow \infty$ the term with $\hat{M} \chi_n$ does not contribute to I_n and hence I_n is quadratic in χ and therefore only the part of χ linear in u will be sufficient. From (35) $\chi_{k+1} = \hat{M}^k u$ and

$$I_{n+1} = \iint \sum_k (M^{k-1} u)(M^{n-k-1} u) \, dx \, dy. \quad (36)$$

Let us now describe canonical variables in KP-1. It is not difficult to prove that KP-1 may be presented in the form

$$u_t = \frac{\partial}{\partial x} \frac{\delta I_5}{\delta u}, \quad I_5 = \iint \chi_5 \, dx \, dy, \quad (37)$$

where χ_5 can be obtained from the complete recurrent relation (35). Now we shall perform the Fourier transformation and go to the variables a_k , denoting

$$a_k = u_k \frac{1}{\sqrt{p}}, \quad u = \int_{-\infty}^{\infty} \int_0^{\infty} (u_k e^{ikr} + u_k^* e^{-ikr}) \, dp \, dq. \quad (38)$$

In terms of a_k equation KP-1 has the Hamiltonian form

$$\dot{a}_k = i \frac{\delta H}{\delta a_k} \quad (39)$$

with Hamiltonian

$$H = \int \omega_k a_k^* \tilde{a}_k \, dk + \int (V_{kk_1 k_2}^* \tilde{a}_k a_{k_1} a_{k_2} + \text{c.c.}) \delta_{k-k_1-k_2} \, dk \, dk_1 \, dk_2, \quad (40)$$

$$\omega_k = p^3 + \frac{3q^2}{p}, \quad [\text{see (6)}]$$

$$V_{kk_1 k_2} = V_{k_1 k k_2} = V_{k k_2 k_1} (= \frac{1}{2} \sqrt{p p_1 p_2} \theta(p) \theta(p_1) \theta(p_2)), \quad (41)$$

$$\theta(p) = \begin{cases} 1, & p \geq 0 \\ 0, & p < 0 \end{cases}$$

c.c. means complex conjugated.

The motion invariants may be also rewritten in canonical variables. In particular, I_5 becomes (19). The quadratic part I_n^0 of the I_n in terms of canonical variables is given by

$$I_n^0 = \frac{i^{n-1}}{2} \int \int \left[\left(p + \frac{q}{p} \right)^{n-2} - \left(-p + \frac{q}{p} \right)^{n-2} \right] |a_k|^2 dp dq. \quad (42)$$

The kinetic equation corresponding to KP-1 may be found by the standard method [7] and coincides with formula (2). In this procedure $a_k a_{k'} \rightarrow n_k \delta_{k-k'}$. The dispersion law of the equation KP-1 coincides with (6) and is degenerative. Hence the kinetic equation (2) has an infinite set of motion invariants of the form (5), where the function f is given by (9). These invariants might be calculated a priori by an averaging procedure applied to the motion invariants I_n of the equation KP-1. The kinetic equation holds in the limit $a_k \rightarrow 0$; therefore we need only the quadratic in a_k part of the integrals I_n given by the formula (42). With this procedure $|a_k|^2$ must be replaced by n_k . Formula (42) gives invariants of the form (5), (9), with $F(\xi) = \xi^n$.

2.2. Equation KP-2

Now let us look at equation KP-2. The dispersion law of KP-2 coincides with (21) and is nondecaying. The corresponding kinetic equation has the form

$$\begin{aligned} \frac{\partial n_k}{\partial t} = & \int |T_{kk_1k_2k_3}|^2 (n_{k_1} n_{k_2} n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_3} - n_k n_{k_1} n_{k_2}) \\ & \times \delta_{k+k_1-k_2-k_3} \delta_{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}} dk_1 dk_2 dk_3, \end{aligned} \quad (43)$$

where $T_{k_1k_2k_3k_4}$ is unambiguously defined on the resonant surface (14).

As was pointed out in section 1 the dispersion law (21) is nondegenerative with respect to the four wave interaction. The interaction matrix element has the form:

$$\begin{aligned} T_{k_1k_2k_3k_4} = & -2 \left[\frac{V_{k_3+k_4,k_2k_1}^* \bar{V}_{k_1+k_2,k_3k_4}}{\omega_{k_1+k_2} - \omega_{k_1} - \omega_{k_2}} \right. \\ & + \frac{V_{k_1k_3,k_1-k_3}^* \bar{V}_{k_4k_2,k_4-k_2}}{\omega_{k_4-k_2} + \omega_{k_2} - \omega_{k_4}} + \frac{V_{k_2k_4,k_2-k_4} V_{k_3k_1,k_3-k_1}}{\omega_{k_3-k_1} + \omega_{k_1} - \omega_{k_3}} \\ & \left. + \frac{V_{k_2k_3,k_2-k_3} V_{k_4k_1,k_4-k_1}}{\omega_{k_4-k_1} + \omega_{k_1} - \omega_{k_4}} + \frac{V_{k_1k_4,k_1-k_4}^* \bar{V}_{k_3k_2,k_3-k_2}}{\omega_{k_3-k_2} + \omega_{k_2} - \omega_{k_3}} \right]. \end{aligned} \quad (44)$$

As before from the existence of Lax's representation it follows that the kinetic equation (43) has the infinite set of motion invariants of the form (4), where

$$f(p, q) = i \left[\left(\frac{iq}{p} + p \right)^n - \left(\frac{-iq}{p} + p \right)^n \right]. \quad (45)$$

Matrix element (44) does not have a singularity of the δ -function type and therefore it turns into zero on the resonant surface Γ . The very extensive calculations are needed to see this fact directly. We checked it in the limit case $p^3 \ll q^2/p$ and on some sub-manifold of the whole resonant manifold Γ .

2.3. 2-d Three wave interaction

Equations (26) describe the interaction of three wave packets $u_1(r, t)$, $u_2(r, t)$, $u_3(r, t)$, $r = (x, y)$ in the frame of reference moving along with wave packet u_3 . This system of equations allows the Lax's

representation (23) [8] with operators L and A being

$$L = iC \frac{\partial}{\partial x} + [C, Q], \quad A = iB \frac{\partial}{\partial x} + [B, Q], \quad (46)$$

$$C = \text{diag}(a_1, a_2, a_3), \quad B = \text{diag}(0, b, 0), \quad Q = \begin{pmatrix} 0 & q_1 & q_3 \\ -q_1^* & 0 & q_2 \\ -q_3^* & -q_2^* & 0 \end{pmatrix}, \quad (47)$$

$$q_1 = \frac{(a_1 - a_2)^{1/2}(a_2 - a_3)^{1/2}}{b(a_1 - a_3)} u_1, \quad q_2 = \frac{(a_2 - a_3)^{1/2}}{b(a_1 - a_3)^{1/2}} u_2, \quad q_3 = \frac{(a_1 - a_2)^{1/2}}{b(a_1 - a_3)^{1/2}} u_3. \quad (48)$$

The motion invariants for this system may be obtained in a general way [9], proceeding from the linear problem for the operator L . The motion invariants appear as coefficients in the expansion of the integral of the scattering phase of the eigenfunction in powers of $1/\lambda$, where λ is a spectral parameter. These motion invariants are:

$$\beta^2 I_j^{(n)} = -\frac{1}{a_j} \int \int dx dy \sum_{k=1}^3 (a_j - a_k) Q_{jk} \Lambda_{kj}^{(n)}, \quad j = 1, 2, 3, \quad (49)$$

$$\beta^2 = [(a_1 - a_2)(a_2 - a_3)]/[b^2(a_1 - a_3)].$$

$\Lambda_{ij}^{(n)}$ are given by recurrent relations

$$\begin{aligned} (a_j - a_i) \Lambda_{ij}^{(n+1)} = & i a_j \frac{\partial \Lambda_{ij}^{(n)}}{\partial y} + i a_i a_j \frac{\partial \Lambda_{ij}^{(n)}}{\partial x} + a_j \sum_{k \neq j} (a_i - a_k) Q_{ik} \Lambda_{kj}^{(n)} \\ & + \sum_{n_1 + n_2 = n} \Lambda_{ij}^{(n_1)} \left\{ -a_j \sum_k (a_j - a_k) Q_{jk} \Lambda_{ki}^{(n_2)} + \left(1 - \frac{a_i}{a_j}\right) \sum_k (a_j - a_k) \int_{-\infty}^x \frac{\partial}{\partial y} [Q_{jk} \Lambda_{ki}^{(n_2)}] dx' \right\} \end{aligned} \quad (50)$$

with $\Lambda_{ij}^{(1)} = -a_j Q_{ij}$.

The canonical variables of the system (5) are the Fourier transforms u_{jk} of the amplitudes u_j , $j = 1, 2, 3$. In terms of these the Hamiltonian has the following form:

$$H = \sum_j \int \omega_{jk} u_{jk}^* \tilde{u}_{jk} dk + \int (\tilde{u}_{1k} u_{2k_1} u_{3k_2} + \text{c.c.}) \delta_{k-k_1-k_2} dk dk_1 dk_2 \quad (51)$$

$\omega_{1k} = 0, \quad \omega_{2k} = -(v_2, k), \quad \omega_{3k} = -(v_3, k).$

The appropriate kinetic equation coincides with (16) provided that $V_{kk_1k_2} = 1$. Now we need the quadratic part of the motion invariants. Since the $I_j^{(n)}$ given by (49) are bilinear in Q and Λ , we need only the part of Λ linear in Q . As a result we find,

$$\beta^2 I_j^{(n+1)} = \int \int \sum_i \frac{(a_i)^n Q_{ji} (\partial/\partial y + a_i \partial/\partial x)^n Q_{ij}}{(a_j - a_i)^{n-1}} dx dy, \quad (52)$$

where elements of matrix Q are given by (47), (48). Performing the Fourier transformation from u_j to u_{jk} , $j = 1, 2, 3$ we have

$$\begin{aligned} \beta^2 I_1^{(n+1)} &= \int \int \left[\left(\frac{q + a_2 p}{a_1 - a_2} \right)^n n_{2k} + \left(\frac{q + a_3 p}{a_1 - a_3} \right)^n n_{1k} \right] a_1^n dp dq, \\ \beta^2 I_2^{(n+1)} &= \int \int \left[\left(\frac{q + a_3 p}{a_2 - a_3} \right)^n n_{3k} - \left(\frac{q + a_1 p}{a_1 - a_2} \right)^n n_{2k} \right] a_2^n dp dq, \\ -\beta^2 I_3^{(n+1)} &= \int \int \left[\left(\frac{q + a_1 p}{a_1 - a_3} \right)^n n_{1k} + \left(\frac{q + a_2 p}{a_2 - a_3} \right)^n n_{3k} \right] a_3^n dp dq. \end{aligned} \quad (53)$$

The frequency resonance condition is

$$\frac{q_1 + a_1 p_1}{a_1 - a_2} = \frac{q_2 + a_3 p_2}{a_2 - a_3}. \quad (54)$$

Formula (54) is the particular case of the formula (20) with linear dispersion laws. Now one can easily prove that the kinetic equation (16) leaves $I_j^{(n)}$, $j = 1, 2, 3$ constant. Indeed, expressions for $d[I_j^{(n)}]/dt$ contain integration with respect to $dk dk_1 dk_2$ and therefore k may be replaced by k_1 in terms with n_{2k} and by k_2 in terms with n_{3k} . That leads to $dI_j^{(n)}/dt = 0$. It is important, that this fact does not depend upon the structure of the kernel of the kinetic equation $|Vkk_1k_2|^2$, but only on its symmetry properties and on the special form of the dispersion law.

2.4. Davay–Stewartson equations

DS equations [10] describe the two-dimensional waves on the surface of a finite depth liquid. They may be transformed into the following form:

$$i \frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \psi u = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) |\psi|^2. \quad (55)$$

In cone variables ($\partial^2/\partial x^2 - \partial^2/\partial y^2 \rightarrow 2\partial^2/\partial x \partial y$) system (55) allows the Lax's representation (23) [10] with operators L and A given by

$$L = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & \psi \\ -\psi^* & 0 \end{pmatrix}, \quad A = \frac{\partial^2}{\partial x^2} + \begin{pmatrix} \Sigma & \psi \\ \psi^* & \Sigma \end{pmatrix}, \quad \frac{\partial^2}{\partial x \partial y} \Sigma = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u. \quad (56)$$

The motion invariants of this system may be found in analogy with subsection 2.4.

They are given by complex expressions

$$I^{(n)} = \int \psi \Lambda^{(n)} dx dy, \quad (57)$$

where $\Lambda^{(n)}$ may be obtained from the recurrent relations (we assume $\psi \rightarrow 0$ at $|r| \rightarrow \infty$):

$$\Lambda^{(n+1)} = \left(-i \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \Lambda^{(n)} + \sum_{n_1+n_2=n} \Lambda^{(n_1)} \left\{ \psi \Lambda^{(n_2)} + 2 \int_{-\infty}^x \frac{\partial}{\partial y} [\psi \Lambda^{(n_2)}] dx' \right\}, \quad (58)$$

$$\Lambda^{(1)} = -\psi^*. \quad (59)$$

For the DS equation canonical variables are Fourier-transforms ψ_k of the field ψ . It is not difficult to rewrite the motion invariants in terms of canonical variables. For example $I^{(1)} = \int \psi_k^* \psi_k dk$, $I^{(2)} = \int (p + iq) \psi_k^* \psi_k dk$. The Hamiltonian is $\text{Re } I^{(3)}$.

$$\text{Re } I^{(3)} = \int (p^2 - q^2) \psi_k^* \psi_k dk + \int \frac{(p_1 - p_3)^2 - (q_1 - q_3)^2}{(p_1 - p_3)^2 + (q_1 - q_3)^2} \psi_{k_1} \psi_{k_2}^* \psi_{k_3}^* \psi_{k_4}^* \delta_{k_1+k_2-k_3-k_4} \prod_{i=1}^4 dk_i.$$

In accordance with relations (58) the quadratic part of the motion invariants is given by

$$I^{(n+1)} = \int \int (p + iq)^n \psi_k^* \psi_k dp dq. \quad (59)$$

Let us notice that invariants (59) correspond to eqs. (55) in the cone variables but under the inverse transformation additional inessential multipliers $(1 + i)^n$ appear in (59) only. The DS equation gives

rise to the kinetic equation (43) with $n_k = \langle \psi_k \psi_k^* \rangle$ and

$$T_{k_1 k_2 k_3 k_4} = \frac{1}{2} \left[\frac{(p_1 - p_3)^2 - (q_1 - q_3)^2}{(p_1 - p_3)^2 + (q_1 - q_3)^2} + \frac{(p_1 - p_4)^2 - (q_1 - q_4)^2}{(p_1 - p_4)^2 + (q_1 - q_4)^2} \right]. \quad (60)$$

Dispersion law for the DS equation coincides with (22) and is nondegenerative if we consider the four-wave interaction (see section 1). Since the interaction matrix element has no singularity of δ -function type it must turn into zero on the resonant surface (14). This fact may be proved directly by the following parametrization of the resonant surface (14):

$$\begin{aligned} p_1 &= p + \kappa_1, & p_3 &= p + \kappa_2, & q_1 &= Q + \eta_1, & q_3 &= Q + \eta_2, \\ p_2 &= p - \kappa_1, & p_4 &= p - \kappa_2, & q_2 &= Q - \eta_1, & q_4 &= Q - \eta_2 \end{aligned} \quad (61)$$

on condition

$$\kappa_1^2 - \kappa_2^2 - \eta_1^2 + \eta_2^2 = 0 \quad (62)$$

substituting (61) to (60) gives

$$T_{k_1 k_2 k_3 k_4} \sim (\kappa_1^2 - \kappa_2^2 - \eta_1^2 + \eta_2^2) = 0.$$

3. Conclusion

From the results of the present work an important conclusion may be drawn out. Namely one can prove the inapplicability of the i.s.t. method to various given nonlinear equations. Really, in all versions of the i.s.t. the first consequence of its applicability is the existence of the infinite number of motion invariants of the corresponding nonlinear equation. At weak nonlinearity they become quadratic invariants. In some cases it may be easily proved that such invariants do not exist. Consider for example the nonlinear Schrödinger equation.

$$i\psi_t + \Delta\psi + |\psi|^2\psi = 0. \quad (63)$$

It is not difficult to prove that its dispersion law $\omega_k = k^2$ is nondegenerative with respect to the four wave interaction (14) if the dimension of space is greater than unity. On the other hand the interaction matrix element equals to unity identically and is not zero on the resonant surface. Hence eq. (63) does not have any additional motion invariant quadratic at small amplitudes and in this case i.s.t. is inapplicable. (This must not be understood in the meaning that eq. (63) should not have an additional motion invariant at all. Generally speaking it may have motion invariants behaving at $\psi \rightarrow 0$ as ψ^4 and not being quadratic. For example one-dimensional equations of the gas dynamics have motion invariants of this type. The investigation of such motion invariants is one of the important problems of the theory of integrable systems.)

The similar analysis has to be carried out for each nonlinear system of equations provocative in the perspective of applicability of the i.s.t.

The authors hope that such analysis will promote the decrease of the number of attempts (usually fruitless) to find L - A pairs for the various systems of equations.

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