THREE-DIMENSIONAL MODEL OF RELATIVISTIC-INVARIANT FIELD THEORY, INTEGRABLE BY THE INVERSE SCATTERING TRANSFORM

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ABSTRACT. In the space-time with signature (2,2) the self-duality equations in the specific case of potentials independent of one of the coordinates are reduced to a relativistic-invariant system in the (2-1)-dimensional space-time. A general solution of this system is constructed by means of IST. A soliton solution, finite in all directions, is discussed. It is found that there is no classical scattering of both solitons and continuous spectrum waves.

1. As is well-known [1] in the space-time with signature (4,0) the Yang–Mills equations can be reduced to a self-duality equation

$$F_{ik} = \pm \tfrac{1}{2} \epsilon_{iklm} F_{lm}. \tag{1}$$

In [2] it has been shown that IST can be applied to system (1), allowing one to find all its local solutions. However, in the elliptic case there are still certain complications associated with extension of local solutions to the whole space. Despite significant progress [3], this problem is not so far solved.

For the gauge group SU(N) system (1) is equivalent to an equation for the positive-definite matrix χ, $\det \chi = 1$

$$(\chi^{-1}\chi_z)_{\bar{z}} + (\chi^{-1}\chi_y)_{\bar{y}} = 0 \quad (\partial_y = \partial_3 - i\partial_4,\, \partial_z = \partial_1 - i\partial_2). \tag{2}$$

This equation can well be considered in the (2,2) signatured space-time (substitution $\partial_y \to i\partial_y$, $\partial_{\bar{y}} \to i\partial_{\bar{y}}$). If χ is independent of one of the coordinates, (2) reduced to the system of evolution equations in the two-dimensional space

$$(\chi^{-1}\chi_t)_t = (\chi^{-1}\chi_z)_{\bar{z}}$$

or, alternatively, to the system:

$$\begin{aligned} &A_t = B_{\bar{z}}, \qquad B_t = A_z - [A, B] \\ &(A = \chi^{-1}\chi_t, \qquad B = \chi^{-1}\chi_z). \end{aligned} \tag{3}$$

Namely, the latter system is a subject of our study.

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The O(4) group acts on solutions of Equation (2). Hence, one can also define Lorentz transformations on solutions of (3), although its action is nonlinear and nonlocal.

2. System (2) is but a compatibility condition of two linear equations for the matrix ψ:

$$(\lambda\partial_{\bar{z}} - \lambda^{-1}(\partial_z + B) + A)\psi = 0, \tag{4}$$

$$(\partial_t + \lambda\partial_{\bar{z}} + A)\psi = 0. \tag{5}$$

Therefore, one can apply the IST to it. The relevant scattering matrix (for $A, B \to 0$ at $x \to \infty$) is constructed as follows. Let λ be a complex parameter belonging to the unit circle: $\lambda = e^{-i\varphi}$. In terms of variables

$$\xi = \cos\varphi x_1 + \sin\varphi x_2, \qquad \eta = -\sin\varphi x_1 + \cos\varphi x_2.$$

Equation (4) takes the form

$$\frac{\partial\psi}{\partial\eta} = u(\varphi, \xi, \eta)\psi \equiv \frac{1}{2i}(e^{i\varphi} B - A)\psi. \tag{6}$$

Its solution tending to unity in the limit of $\eta \to -\infty$, is a function of ξ when $\eta \to +\infty$:

$$S(\varphi, \xi) = \psi(\xi, \eta)|_{\eta = \infty} = P\exp\left(\int_{-\infty}^{\infty} d\eta\, U(\varphi, \xi, \eta)\right). \tag{7}$$

Here P denotes η-ordering. $S(\varphi, \xi)$ is just a scattering matrix. A mapping $A, B \to S$ can be called a non-abelian Radon transform. Its inversion is a principal task of IST and will be considered below. Here we emphasize that, in terms of S, the Cauchy problem for Equation (3) can be solved trivially. Indeed, from (5) it follows that

$$S(\varphi, \xi, t) = S(\varphi, \xi - t, 0). \tag{8}$$

3. System (3) possesses a great variety of soliton solutions having quite unusual properties. These solutions can be constructed by the standard method (see, for instance, [2, 5].) A soliton is described by a single-pole function $\psi(\lambda)$:

$$\psi(\lambda) = 1 + \frac{\lambda_0 - \bar{\lambda}_0^{-1}}{\lambda - \lambda_0} R(t, z, \bar{z}), \quad R^2 = R. \tag{9}$$

For such a function to be a solution of (4) and (5) it is necessary and sufficient that the vector φ_i which defines a one-dimensional Hermitian projector R,

$$R_{ik} = \bar{\varphi}_i\varphi_k(\sum |\varphi_l|^2)^{-1} \tag{10}$$

satisfy the system of linear equations

$$(\partial_z + \lambda_0 \partial_t)\varphi_i = 0, \qquad (\partial_t + \lambda_0^{-1} \partial_{\bar{z}})\varphi_i = 0.$$

In other words, φ_i must be entire functions of the complex variable ξ, $\varphi_i = \varphi_i(\xi)$, $\xi = \lambda_0 z + \lambda_0^{-1} \bar{z} - t$.

A solution χ of (3) is given by the expression:

$$\chi = |\lambda_0|^{-2/N}(1 + (|\lambda_0|^2 - 1)R). \tag{11}$$

Without losing generality, at $N = 2$ one can put $\varphi_1 = 1$, $\varphi_2 = \varphi(\xi)^*$. The requirement that χ should be regular on the whole plane implies that φ is a rational function:

$$\varphi = c\frac{(\xi - a_1)\dots(\xi - a_n)}{(\xi - b_1)\dots(\xi - b_m)}. \tag{12}$$

The parameters c, $a_1 \dots a_n$, $b_1 \dots b_m$ are arbitrary and correspond to certain intrinsic degrees of freedom. The complex parameter λ_0 determines the soliton velocity

$$v = \frac{2|\lambda_0|}{1 + |\lambda_0|^2}$$

whereas $-\arg \lambda_0$ is an angle between v and x_1-axis.

The solution (10)–(12) describes the two-dimensional soliton (currents A and B vanish at $x \to \infty$). It is easy to produce other interesting solutions – one-dimensional solitons or 'walls'. To do so, it is sufficient to choose φ as

$$\varphi = c\, e^{a\xi} \tag{13}$$

where again c and a are arbitrary complex parameters.

N-solitons can also be constructed by a standard procedure. They are described by $\psi(\lambda)$ of the form $\psi = \Pi\psi_i(\lambda)$, where ψ_i are functions like (9). We do not quote here an explicit formula for χ. Note that in this model, two-dimensional solitons do not interact – none of their characteristics changes after scattering. The relevant solutions for one-dimensional solitons (13) describe a crossing of moving walls and give us a realization of a classical limit of solutions of Zamolodchikov's equations [6]. An analysis of such solutions is definitely of great interest and will be presented elsewhere.

4. In this section we discuss inversion of the transformation (7). For the sake of simplicity we confine ourselves to a solitonless sector.

Introduce the Green function for the operator $\lambda\partial_{\bar{z}} - \lambda^{-1}\partial_z$ ($|\lambda| \neq 1$): $(\lambda\partial_{\bar{z}} - \lambda^{-1}\partial_z)G(z, \bar{z}) = \delta(x_1)\delta(x_2)$.

* However, now $\varphi = \varphi_2/\varphi_1$ may have poles.

It is clear that $G = 1/4\pi(\lambda z + \lambda^{-1}/\bar{z})^{-1}$ (an extra factor of $\frac{1}{2}$ in this formula is due to our convention $z = (x_1 + ix_2)/2$). Now consider solutions of Equation (4), given by the integral equation

$$\psi(z, \bar{z}) = 1 + \frac{1}{4\pi}\int \frac{(\lambda^{-1}B - A)(z', \bar{z}')}{\lambda(z - z') + \lambda^{-1}(\bar{z} - \bar{z}')} \psi(z', \bar{z}')\, dz'\, d\bar{z}'. \tag{14}$$

If, in a certain convenient sense, the currents A and B are small, then at $|\lambda| < 1$ the solution of (14) defines a function $\psi_1(\lambda)$ analytical inside the unit circle, whereas at $|\lambda| > 1$ Equation (14) defines a function $\psi_2(\lambda)$ analytical outside of the unit circle and tending to unity at the infinity:

$$\psi_2(\infty) = 1. \tag{15}$$

Now take the values of ψ_1 and ψ_2 on the unit circle $|\lambda| = 1$. Consider the asymptotics at $\eta \to \pm\infty$. Put $\lambda = (1 + \epsilon)\, e^{-i\varphi}$, $\operatorname{Im} \epsilon = 0$, $|\epsilon| \ll 1$ and note that $\lambda z + \lambda^{-1}\bar{z} = \xi + i\epsilon\eta$. Then Equation (14) takes the form

$$\psi(\xi, \eta, \varphi) = 1 + \frac{1}{4\pi}\int \frac{(e^{i\varphi} B - A)(\xi', \eta')}{(\xi - \xi') + i\epsilon(\eta - \eta')} \psi(\xi', \eta', \varphi)\, d\xi'\, d\eta'. \tag{16}$$

Let $\epsilon > 0$ (in this case we are dealing with ψ_2) and for the sake of simplicity, let the currents be on a compact support. Then at sufficiently large η, the quantity $\delta = \epsilon(\eta - \eta')$ within the integration region is positive, so in this way we obtain

$$\psi_2^{(+)}(\xi, \varphi) = 1 + \frac{1}{4\pi}\int \frac{(e^{i\varphi} B - A)(\xi', \eta')}{(\xi - \xi') + i\delta} \psi_2(\xi', \eta', \varphi)\, d\xi'\, d\eta'. \tag{17}$$

Here $\psi_2^{(+)}$ stands for the asymptotics of $\psi_2(\xi, \eta, \varphi)$ at $\eta \to \infty$. Representation (1.7) for $\psi_2^{(+)}$ implies that it can be analytically continued from the real axis onto the upper half-plane of the complex *ξ-plane.*

One may similarly prove that an asymptotics of ψ_2 at $\eta \to -\infty$ ($\psi_2^{(-)}$ is analytical in the lower half of the ξ-plane, and $\psi_1^{(-)}$ and $\psi_1^{(+)}$ are analytical in the upper and lower half-planes, respectively.

These analytical properties of asymptotics of $\psi_1^{(\pm)}(\xi, \varphi)$ and $\psi_2^{\pm}(\xi, \varphi)$ allow us to find them, one $S(\varphi, \xi)$ (7) is known. Indeed, from the definition of S one has

$$\begin{aligned} S &= \psi_2^{(+)}(\xi, \varphi)(\psi_2^{(-)}(\xi, \varphi))^{-1}, \\ S &= \psi_1^{(-)}(\xi, \varphi)(\psi_1^{(-)}(\xi, \varphi))^{-1}, \end{aligned} \tag{18}$$

i.e., S can be represented as a product of two functions one of which is analytical at $\operatorname{Im} \xi > 0$ and the other is analytical at $\operatorname{Im} \xi < 0$.

Hence, (18) is nothing else but a usual Riemann conjugation problem. Since $\det \psi_{1,2}^{(\pm)} = 1$, this is a non-degenerate problem and can be solved for at least $S - 1$ sufficiently small. The normalization condition $\psi_{1,2}^{(\pm)}(\infty, \varphi) = 1$ separates the solutions we are interested in.

The next step in the solution of the inverse problem is the following. Using the asymptotics ψ_1^- and ψ_2^- we construct the expression

$$(\psi_1^{(-)}(\xi, \varphi))^{-1}\psi_2^{(-)}(\xi, \varphi) = R(\xi, \varphi). \tag{19}$$

As is clear from (4), the ratio $\psi_1^{-1}(\xi, \eta, \varphi)\psi_2(\xi, \eta, \varphi)$ does not depend on η. Thus, for any x_1, x_2 and λ belonging to the unit circle $|\lambda| = 1$ we have

$$\psi_1^{-1}(x_1, x_2, \lambda)\psi_2(x_1, x_2, \lambda) = R(\xi(x, \varphi), \varphi). \tag{20}$$

Recall that $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are analytical inside and outside of the unit circle. Therefore, (20) is again the Riemann problem subject to normalization $\psi_2(\infty) = 1$ (see Equation (15)). Its solution enables us to construct solutions of (4) for any x_1, x_2 and, thus, find the currents A and B.

Therefore, an inversion of transform (7) consists of a successive solution of the Riemann problems (18) and (20).

To conclude this section we comment on reductions which guarantee hermiticity and unimodularity of matrices χ which generate, according to (3), currents A and B. First, unimodularity of χ implies that A and B are traceless, which implies, in its turn, a unimodularity of the S matrix (7)

$$\det S(\varphi, \xi) = 1.$$

Secondly, it can be easily proven that hermiticity of χ implies that a certain involution is defined on solutions of (4). Namely, if $\psi(\lambda)$ is a solution of (4), then also $\chi^{-1}\psi^{+-1}(1/\bar{\lambda})$ is a solution of (4). Taking $\chi(\infty) = 1$, we find, first, that

$$\psi_1(\lambda) = \chi^{-1}\psi_2^{+-1}\left(\frac{1}{\bar{\lambda}}\right),$$

(so that the R-matrix is positively defined: $R = \psi_2^+\chi\psi_2$).

Secondly, the above-mentioned involution when applied to the solution defining the S-matrix, can yield

$$S^{-1}(\xi, \varphi) = S^+(\xi, \varphi)$$

i.e., S-matrices, corresponding to Hermitian χ, are unitary. Besides, (21) implies that asymptotics of two solutions ψ_1 and ψ_2 are related as

$$\psi_1^{(\pm)}(\xi, \varphi) = (\psi_2^{(\pm)}(\xi, \varphi))^{-1^+}.$$

(The corollary is that instead of two Riemann problems (18), it is sufficient to solve one of them.)

Finally, recalling the normalization of ψ_2, we obtain from (21) an expression for χ in terms of the solution of the problem (20):

$$\chi = \psi_1^{-1}(o).$$

5. In this section we describe large-time asymptotical behaviour of our solutions. In the solitonless sector χ has a purely linear asymptotics at $t \to \pm\infty$: $\chi = 1 + \alpha$, $\|\alpha\| \ll 1$ where α satisfies the wave equation. In the cylindrical coordinates at $t \to \pm\infty$ one has $\alpha^{\pm} = |t|^{-1/2} f^{\pm}(r - |t|, \varphi)$ and our task is to find a relationship between f^+ and f^-, i.e., a classical scattering matrix.

The characteristic scales of the basis of solutions is of the same order of magnitude as the basis for the initial condition. The $t \to \infty$ situation is shown in Figure 1.

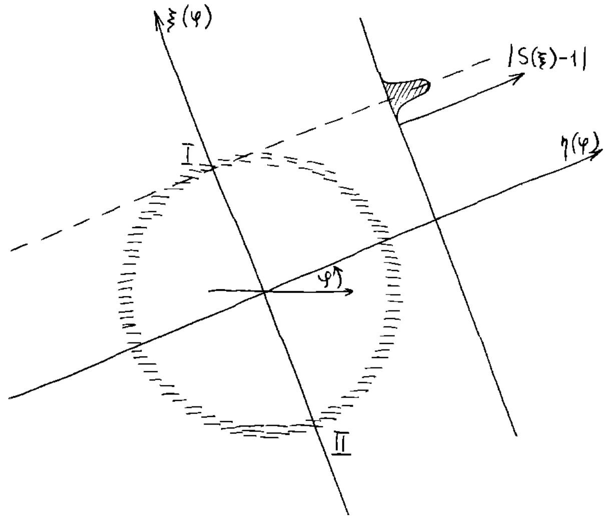

Fig. 1.

The fields do not vanish only inside the shaded circle of the radius t; inside the circle the field amplitude $\sim t^{-1/2}$ (the same is true for the currents A and B). Therefore, it is clear that all contributions to S come only along the lines ξ = const which are 'tangent' to the circle. This takes place only in the region I and II. The relevant combination (6) of currents in the region I is $-it^{-1/2}\,\partial f^+/\partial r$ whereas in the region II the leading terms in the expansion of U cancel each other (this region is 'transparent' and produces no 'shadow' $|S(\xi, \varphi)| - 1$). For this reason at $t \to \infty$ formula (7) takes the form

$$S(\varphi, \xi - t) = P \exp\left\{-i \int_{-\infty}^{\infty} f_r^+ \left(\xi - t + \frac{\zeta^2}{2}, \varphi\right) \mathrm{d}\zeta\right\}.$$

(This is an adequate illustration to the simple dependence of S on t, given above in Equation (6).)

At $t \to -\infty$ a similar calculation gives

$$S(\varphi, \xi - t) = P \exp\left\{-i \int_{-\infty}^{\infty} f_r^- \left(\xi - t + \frac{\zeta^2}{2}, \varphi + \pi\right) \mathrm{d}\zeta\right\}.$$

A comparison of the above formulas shows that

$$f^+(\varphi) = f^-(\varphi + \pi).$$

This implies an absence of any classical scattering in the considered model (both occupation numbers and phases of waves at $t \to -\infty$ and $t \to \infty$ coincide). The same holds for the scattering of continuous spectrum waves on solitons.

6. Finally, a few concluding remarks should be made. If χ is a positive-definite 2×2 matrix, then Equation (3) possesses a local real Lagrangian [4] with the positively defined corresponding Hamiltonian. We did not use the Hamiltonian approach, since the relevant action is not Lorentz invariant. For this reason we can ascribe neither mass nor angular momentum to soliton solutions found here. However, one has to bear in mind that completely integrable systems allow a variety of Hamiltonian formulations. Hopefully, for our system (3) among these formulations exists, as yet unknown, one which is relativistically invariant.

Finally note that for the matrices χ depending only on t and $|z| = r$, Equation (3) coincides with an equation for the principal chiral field on the plane

$$(\chi^{-1}\chi_t)_t = (\chi^{-1}\chi_r)_r + \frac{1}{r}\chi^{-1}\chi_r.$$

That enables one to produce solutions of the Cauchy problem for cylinder-symmetric chiral fields.

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