MULTIDIMENSIONAL NONLINEAR INTEGRABLE SYSTEMS AND METHODS FOR CONSTRUCTING THEIR SOLUTIONS

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A new method for constructing multidimensional nonlinear integrable systems and their solutions by means of a nonlocal Riemann problem is presented. This is the natural generalization of the method of the local Riemann problem to the case of several space variables and includes the well-known Zakharov–Shabat method of dressing by Volterra operators.

INTRODUCTION

In 1974 one of the authors of the present paper proposed, in collaboration with Shabat [1], a scheme for constructing multidimensional nonlinear integrable equations and their exact solutions, known as the "dressing method." The integrable equations considered in [1] represent the compatibility condition of the overdetermined system of equations

$$\frac{\partial \Psi}{\partial x_1} = L_1 \Psi, \qquad \frac{\partial \Psi}{\partial x_2} = L_2 \Psi, \tag{I.1}$$

and can be written in the form

$$\frac{\partial L_2}{\partial x_1} - \frac{\partial L_1}{\partial x_2} + [L_2, L_1] = 0. \tag{I.2}$$

Here $L_{1,2}$ are linear differential operators in a new variable x, generally speaking with matrix coefficients. These coefficients, which are the unknown functions for the integrable equations, depend on the three independent variables x_1, x_2, and x in a rather asymmetric manner.

Let us explain the main result of paper [1] on the simplest example. Consider the integral equation

$$K(x,z) + F(x,z) + \int_x^\infty K(x,s)F(s,z)\,ds = 0. \tag{I.3}$$

Here the $N \times N$ matrix-valued functions F and K depend also on the variables x_1, x_2, and it is assumed that F is known, whereas K is unknown. Suppose that F satisfies the system of equations

$$\frac{\partial F}{\partial x_i} = I_i \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} I_i, \qquad i = 1,2 \tag{I.4}$$

where I_i are constant commuting matrices: $[I_1, I_2] = 0$.

Differentiating Eq. (I.3) we deduce that function K satisfies the system of equations

$$\frac{\partial K}{\partial x_i} = \left(I_i \frac{\partial}{\partial x} + [I_i, Q]\right) K + \frac{\partial K}{\partial z} I_i, \quad Q(x, x_1, x_2) = K(x, x, x_1, x_2). \tag{I.5}$$

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Representing K in the form

$$K(x,z,x_1,x_2) = \int_{-\infty}^{\infty} \Psi(\lambda,x,x_1,x_2) e^{i[\lambda z+\lambda I_i x_i]}\, d\lambda, \tag{I.6}$$

we verify that Ψ satisfies Eq. (I.1) with

$$L_i = I_i \frac{\partial}{\partial x} + [I_i, Q]. \tag{I.7}$$

Equation (I.2) takes the form

$$[I_2, \frac{\partial Q}{\partial x_1}] - [I_1, \frac{\partial Q}{\partial x_2}] + I_2[I_1, \frac{\partial Q}{\partial x}] - I_1[I_2, \frac{\partial Q}{\partial x}] + [[I_2, Q],[I_1, Q]] = 0 . \tag{I.8}$$

For a suitable reduction, for example, $I^+ = I$ $J^+ = J$ $Q^+ = Q$, Eq. (I.8) turns into the "N-wave system," which is of great interest in applications. To construct effectively a solution of Eq. (I.8) one can take, for example, $F = \sum_K F_K(x) G_K(z)$ (see [2] and also [3-4]). One may naturally say that the operators L_i are obtained by "dressing" the "primer" operators $L_{0i} = I_i \frac{\partial}{\partial x}$.

The next development of the dressing method was paper [5], which deals also with Eqs. (I.1), (I.2), but under the assumption that $L_{1,2}$ are rational functions of a complex parameter λ . This leads to equations for functions of two independent variables x_1, x_2 ; the dressing is achieved by solving the Riemann problem on an arbitrary contour Γ in the complex λ-plane, i.e., by solving on this contour a certain singular integral equation. The connection between the two variants of the dressing method [6, 7]* is, at first glance, elementary. Thus, suppose that the coefficients of operators L_1, L_2 in (I.1) do not depend on x . Then, following the substitution $\Psi \sim e^{i\lambda x}$, they become polynomials in λ . In this way system (I.1)-(I.7) and Eq. (I.8) take the form

$$\frac{\partial \Psi}{\partial x_K} = (i I_K \lambda + [I_K, Q])\Psi, \tag{I.9}$$

and, respectively,

$$[I_2, \frac{\partial Q}{\partial x_1}] - [I_1, \frac{\partial Q}{\partial x_2}] + [[I_2, Q],[I_1, Q]] = 0 . \tag{I.10}$$

In the terminology of [7], Eq. (I.8) is the "first multidimensional generalization" of Eq. (I.10).

To obtain solutions which do not depend on x , one must assume that function F depends only on the difference $x - z$. Then Eq. (I.3) becomes a Wiener–Hopf equation and solves, on taking the Fourier transform, the Riemann problem; in this case the contour Γ is the real axis.

It is clear that the method developed in [5] permits the construction of a rather rich supply of solutions for Eq. (I.10), at the expense of the arbitrariness of contour Γ . A question arising naturally is how to find analogous solutions for system (I.8). Also, it is not clear how to construct the first multidimensional generalization for systems of the type considered in [5] when L_1 and L_2 are rational functions rather than polynomials. The aim of the present paper is to solve these two problmes.

*See also V. E. Zakharov's report at the International Congress of Mathematicians, Warsaw (1983).

The second of the above problems was solved to a certain extent by a direct generalization of the technique proposed in [1] (see also [7]). Actually, instead of Eqs. (I.1) the linear system

$$N_i \frac{\partial \Psi}{\partial x_i} = L_i \Psi \tag{I.11}$$

was considered, where N_i and L_i are linear differential operators in variable x. We do not intend to explain here the full relationship between these results and the results of the present paper. Our work is essentially based on the data advanced earlier by one of the authors (S. V. Manakov), that in the multidimensional case a nonlocal Riemann problem should be used instead of the local problem (see [8]). We translate the scheme of paper [1] into the language of the nonlocal Riemann problem and show that by using the nonlocal problem one can construct multidimensional integrable equations and their solutions with the same degree of effectiveness achieved by means of the local problem in the case of two-dimensional equations and their solutions.

Among the multidimensional integrable systems, one, no less popular than (I.8), is the Kadomtsev—Petviashvili (KP) equation

$$\frac{\partial}{\partial x}\left(u_t - 6uu_x - u_{xxx}\right) = 3\alpha^2 u_{yy} \tag{I.12}$$

(see [1, 9-13]). Paper [13] develops a technique for constructing solutions for this equation, which uses instead of the nonlocal Riemann problem the solution of a local $\bar{\partial}$-problem of a special form. One can show (although this is not one of our tasks here) that by means of a certain limiting process one can derive from the nonlocal Riemann problem a more universal (compared to [13]) technique for constructing solutions of multidimensional integrable systems based on a nonlocal $\bar{\partial}$-problem. The results obtained in this direction will be published separately.

1. Nonlocal Riemann Problem (discussion of an example).

Let us represent the functions F and K in (I.3) in the form

$$F(x,z) = \frac{1}{2\pi}\int F(\lambda',\lambda)e^{i(\lambda' x - \lambda z)}\,d\lambda'\,d\lambda, \tag{1.1}$$

and

$$K(x,z) = -\frac{1}{2\pi}\int K(x,\lambda)e^{i\lambda(x-z)}\,d\lambda \tag{1.2}$$

and then introduce the function

$$T(\lambda',\lambda,x,x_i) = F(\lambda',\lambda)e^{i(\lambda'-\lambda)x} = T(\lambda',\lambda) \tag{1.3}$$

(in the formulas below we omit to write the dependence of T on the variables x and x_i).

Substituting (1.1)-(1.3) in Eq. (I.3), we obtain

$$K(\lambda) = \int_{-\infty}^{\infty} T(\lambda',\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{K(\xi)T(\lambda',\lambda)}{\lambda'-\xi+i\nu}\,d\lambda'\,d\xi. \tag{1.4}$$

Consider the functions $\chi_{1,2}(\lambda)$, analytic in the upper/lower half plane, defined by the formulas

$$\chi_{1,2}(\lambda) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K(\lambda')}{\lambda-\lambda'\pm i\nu}\,d\lambda'. \tag{1.5}$$

Note that

$$K(\lambda) = \chi_2(\lambda) - \chi_1(\lambda) \Big|_{\mathrm{Im}\,\lambda = 0} . \tag{1.6}$$

It is readily verified that Eq. (I.3) is equivalent to the relation

$$\chi_2(\lambda) = \chi_1(\lambda) + \int_{-\infty}^{\infty} \chi_1(\lambda') T(\lambda', \lambda)\, d\lambda', \tag{1.7}$$

or, in a convenient symbolic form

$$\chi_2 = \chi_1 + \chi_1 * T . \tag{1.8}$$

Relation (1.7) (or (1.8)) defines on the real axis a nonlocal Riemann conjugation problem through an integral relation between two functions $\chi_{1,2}(\lambda)$ which are analytic in the upper/lower half plane and satisfy the additional constraint

$$\chi_{1,2}(\lambda) \to 1 \quad \text{as} \quad \lambda \to \infty .$$

Henceforth we shall assume that this Riemann problem (and all other) is uniquely solvable. In particular, this means that a solution of the Riemann problem (1.7)-(1.8) with the asymptotics

$$\chi_{1,2}(\lambda) \to 0 \qquad \text{as} \quad \lambda \to \infty ,$$

vanishes identically:

$$\chi_{1,2}(\lambda) = 0 . \tag{1.9}$$

The last assertion will be used in many occasions. We can write the asymptotic expansion

$$\chi_{1,2} \to 1 + \sum_{n=1}^{\infty} \frac{\varphi_n}{\lambda^n} , \quad \lambda \to \infty , \tag{1.10}$$

for the solution of Riemann's problem with the asymptotics 1 for $\lambda \to \infty$. Here

$$\varphi_n = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda^{n-1} K(\lambda)\, d\lambda . \tag{1.11}$$

Upon substituting (1.1) in (I.4) we verify that function $T(\lambda', \lambda)$ satisfies the equations

$$\frac{\partial T}{\partial x_K} = i \{ I_K \lambda' T - T I_K \lambda \} . \tag{1.12}$$

Moreover, the following relation holds:

$$\frac{\partial T}{\partial x} = i(\lambda' - \lambda) T . \tag{1.13}$$

Consider the differential operators

$$\mathcal{D}_0 \chi = \left(\frac{\partial}{\partial x} + i\lambda\right)\chi , \quad \mathcal{D}_K \chi = \frac{\partial \chi}{\partial x_K} + i \chi I_K \lambda , \tag{1.14}$$

and note that they commute. Now one can rewrite relations (1.12) and (1.13) symbolically as

$$[\mathcal{D}_0, T] = 0 , \quad [\mathcal{D}_i, T] = 0 . \tag{1.15}$$

Applying the operators $\mathcal{D}_0$ and $\mathcal{D}_i$ to relation (1.8) and using (1.15) we get

$$\mathcal{D}_0 \chi_2 = \mathcal{D}_0 \chi_1 + \mathcal{D}_0 \chi_1 * T ; \quad \mathcal{D}_i \chi_2 = \mathcal{D}_i \chi_1 + \mathcal{D}_i \chi_1 * T . \tag{1.16}$$

Furthermore, for any differential operator M of the form

$$M = \sum_K a_K \mathcal{D}^K \tag{1.17}$$

(where K is a multiindex and $\mathcal{D}^K=(\mathcal{D}_0)^\alpha(\mathcal{D}_1)^\beta(\mathcal{D}_2)^\gamma$) one has the formula

$$M\chi_2 = M\chi_1 + M\chi_1 * T. \tag{1.18}$$

Functions $M\chi_1, M\chi_2$ constitute a solution of Riemann's problem with polynomial asymptotics λ for $\lambda\to\infty$. The operators M form a ring $\mathfrak{m}$. Consider in $\mathfrak{m}$ the subset $\widetilde{\mathfrak{m}}$ of operators $\widetilde{M}$ with the property that

$$\widetilde{M}\chi_1 \to 0\,,\quad \widetilde{M}\chi_2 \to 0 \quad \text{as} \quad \lambda\to\infty\,.$$

By assertion (1.9), this implies that

$$\widetilde{M}\chi_1 = 0\,,\quad \widetilde{M}\chi_2 = 0\,. \tag{1.19}$$

The latter is clearly true for any operator of the form $M_1\widetilde{M}$, which proves that $\widetilde{\mathfrak{m}}$ is a left ideal in the ring $\mathfrak{m}$. We seek the elements of $\widetilde{\mathfrak{m}}$ in the form

$$\widetilde{M}_K = \mathcal{D}_K - I_K\mathcal{D}_0 + \mathcal{U}_K\,. \tag{1.20}$$

Substituting asymptotics (1.10) in the expression of $\widetilde{M}_K\chi$, and requiring that the latter decay for $\lambda\to\infty$, we get

$$\mathcal{U}_K = i\,[I_K, \varphi_1]\,. \tag{1.21}$$

It is convenient to view $\chi_{1,2}$ as a single function χ defined in the entire complex plane. Then

$$\widetilde{M}_K\chi = 0\,, \qquad k=1,2\,. \tag{1.22}$$

It is readily checked that the operators $\widetilde{M}_K$ form a basis in the ideal $\widetilde{\mathfrak{m}}$, i.e., that the equality $\widetilde{M}\chi=0$ implies $\widetilde{M}=A_i\widetilde{M}_i$ $(i=1,2)$, where A_i are operators. Consider the function $\Psi=\chi e^{i\lambda(x+I_Kx_K)}$ and note that it obeys the equations

$$\widetilde{R}\Psi = 0,$$

where the operators $\widetilde{R}$ are obtained from the operators $\widetilde{M}$ on replacing the "long derivatives" (1.14) by the usual ones:

$$\mathcal{D}_0 \to \frac{\partial}{\partial x}\,,\quad \mathcal{D}_K \to \frac{\partial}{\partial x_K}\,.$$

In particular, Eq. (1.22) becomes Eq. (I.9). Moreover,

$$Q = i\,\varphi_1\,. \tag{1.23}$$

Note that (1.23) is also a straightforward consequence of the definition of function χ.

In the foregoing considerations the specific form of the contour Γ (the real axis) played no special role, and they remain valid for an arbitrary contour Γ . This permits us to construct new solutions for Eq. (I.8), unknown until now. Thus, setting $T(\lambda'\lambda)=T_1(\lambda')T_2(\lambda)$ we get from (1.12) and (1.13)

$$\begin{aligned} T_1 &= e^{i(I_Kx_K+x)\lambda'} f_1(\lambda')\,, \\ T_2 &= f_2(\lambda)e^{-i(I_Kx_K+x)\lambda}\,. \end{aligned} \tag{1.24}$$

Equation (1.4) further yields

$$Q(x,x_i)=\frac{1}{2\pi}\int_\Gamma T_1(\lambda')d\lambda'\left[1-\frac{1}{2\pi}\int_\Gamma\int_\Gamma\frac{T_2(\xi)T_1(\lambda')}{\lambda'-\xi+i0}d\xi d\lambda'\right]^{-1}\int_\Gamma T_2(\lambda)d\lambda\,. \tag{1.25}$$

In (1.24) and (1.25) f_1 and f_2 are arbitrary matrix-valued functions of the variable λ, defined on contour Γ. Solution (1.25) generalizes the solution of Eq. (I.8) found in [2], We have implicitly assumed that in the framework of our new scheme Eq. (I.8) is obtained as the compatibility condition for system (1.22), or for the equivalent system

$$R_\kappa \Psi = 0\,, \qquad \kappa = 1,2$$

(which coincides with (I.1)), i.e., as a matter of fact, by means of formula (I.2). Alternatively, Eq. (I.8) can be obtained by isolating in Eq. (1.22) the terms proportional to $\frac{1}{\lambda}$ when $\lambda \to \infty$.

From (1.10), (1.20), and (1.22) it follows that

$$[\varphi_2, I_\kappa] = \frac{\partial \psi}{\partial x_\kappa} - I_\kappa \frac{\partial Q}{\partial x} + u_\kappa Q \quad ; \tag{1.25}$$

by the commutativity of the matrices I_κ

$$[[\varphi_2, I_1]\, I_2] = [[\varphi_2, I_2], I_1]\,.$$

Using this identity we can eliminate the term containing φ_2 and thus obtain Eq. (I.8).

2. Nonlocal Riemann Problem: General Case

In the previous section we have translated the technique developed in [1] for solving Eq. (I.8) in the language of the nonlocal Riemann problem and thus enlarged considerably the class of solutions by choosing abritrarily the contour.

Next we turn to the general method of applying the nonlocal Riemann problem to multidimensional integrable systems, and among them, as particular cases, the systems described in [1].

Suppose that in the complex λ plane there is given a contour Γ on which a nonlocal Riemann problem (1.8) is defined, i.e., a function χ is given, analytic in the entire plane, and whose boundary values on Γ, χ_1 and χ_2, satisfy the integral relation

$$\chi(\lambda) = \chi_1(\lambda) + \int_\Gamma \chi_1(\lambda')\, T(\lambda', \lambda)\, d\lambda'. \tag{2.1}$$

For the sake of simplicity we shall normalize the Riemann problem by $\lambda \to \infty$ for each $\chi \to 1$ as $\lambda \to \infty$.

Then, on the two distinct sides of the contour one has the representation

$$\chi = 1 + \frac{1}{2\pi i} \int_\Gamma \frac{\kappa(\lambda')}{k - \lambda' \pm i0}\, d\lambda' \;, \tag{2.2}$$

where

$$\kappa(\lambda) = \chi_2(\lambda) - \chi_1(\lambda)\Big|_{\lambda \in \Gamma}\,.$$

Function $\kappa(\lambda)$ is subject to the singular integral equation

$$\kappa(\lambda) = \int_\Gamma T(\lambda', \lambda)\, d\lambda + \frac{1}{2\pi i} \iint_{\Gamma\Gamma} \frac{\kappa(\xi)\, T(\lambda', \lambda)}{\lambda - \xi + i0}\, d\lambda' d\xi\,. \tag{2.3}$$

We shall assume that Eq. (2.3), and with it the normalized Riemann problem (2.1), are uniquely solvable.

Now consider a collection of commuting rational matrix-valued functions $I_\kappa(\lambda)$

$$(\kappa = 1 \ldots n) \qquad [I_\kappa, I_j] = 0$$

and use them to build the collection of commuting first-order differential operators

$$\mathcal{D}_K \chi = \frac{\partial \chi}{\partial x_K} + \chi I_K(\lambda) , \qquad (2.4)$$
$$[\mathcal{D}_K , \mathcal{D}_j] = 0 .$$

Here $\frac{\partial}{\partial x_K}$ are the operators of differentiation with respect to n , generally speaking complex, independent variables.

Suppose that the conjugation function $T(\lambda',\lambda)$ satisfies the conditions

$$[\mathcal{D}_i , T] = 0 ,$$

i.e., the equations

$$\frac{\partial T}{\partial x_i} = I_i(\lambda')T - T I_i(\lambda) . \qquad (2.5)$$

Then, identity (1.18) holds for any differential operator M of the form (1.17): a polynomial in the operators $\mathcal{D}_i$ with variable coefficients which do not depend on λ .

Function $M\chi$ is not a solution of Riemann's problem because it is singular at all the poles of the functions $I_K(\lambda)$, and also because, generally speaking, it has a polynomial behavior for $\lambda \to \infty$.

Again, we single out in the ring m of operators M the subset $\tilde{m}$ of all operators $\tilde{M}$ with the property that $\tilde{M}\chi$ is singularity-free in the entire complex plane (except for the contour Γ) and, in addition, $\tilde{M}\chi \to 0$ for $\lambda \to \infty$.

From the unique solvability of Riemann's problem it follows that

$$\tilde{M}\chi = 0 , \qquad (2.6)$$

So that $\tilde{m}$ is an ideal in m . The ideal $\tilde{m}$ consists of all equations $\tilde{M}$ that share the solution χ . Note that equation $\tilde{M}$ contains the parameter λ explicitly. Now passing to the function ψ by the formula

$$\psi = \chi e^{\sum I_K(\lambda) x_K} ,$$

we obtain the compatible system of equations

$$\tilde{R}\psi = 0 \qquad (2.7)$$

for function ψ ; the operators $\tilde{R}$ are obtained from $\tilde{M}$ through the substitution $\mathcal{D}_K \to \partial_K$ and do not contain the parameter λ .

Reasoning as in Sec. 1, we should look for the conditions of compatibility of Eqs. (2.7). However, the equations from $\tilde{m}$ contain, generally speaking, many arbitrary elements, because we are allowed to multiply at left by arbitrary operators of the form M . Therefore, we must construct a basis in the ideal $\tilde{m}$.

We shall examine the simple cases where one succeeds in constructing such a basis.

1. Suppose that one of the variables x_0 is distinguished by the condition that $I_0(\lambda) = i\lambda$, whereas all other functions $I_i(\lambda)$ are polynomials. Then, a basis in the ideal $\tilde{m}$ is provided by the operators of first order in $\frac{\partial}{\partial x_i}$ $(i>0)$ of the form

$$\tilde{M}_i = \mathcal{D}_i - L_i(\mathcal{D}_0) ;$$

the corresponding operators $\tilde{R}_i$ have then the form

$$\tilde{R}_i = \frac{\partial}{\partial x_i} - L_i\left(\frac{\partial}{\partial x}\right) ,$$

with L_i some polynomials. For $i = 1,2$, the class of integrable systems is exactly the class described in [1] and given by formula (I.2).

A member of this class is the KP equation (I.12). To obtain this equation we must take

$$\mathcal{D}_1 = \alpha \frac{\partial}{\partial y} - \lambda^2, \quad \mathcal{D}_2 = \frac{\partial}{\partial t} - 4\lambda^3 . \tag{2.8}$$

Applying the nonlocal Riemann problem on an arbitrary contour one obtains new classes of exact solutions of the KP equation.

2. Suppose that $I_0(\lambda) = i\lambda$, as above, but let $I_k(\lambda) = \frac{P_k(\lambda)}{Q_k(\lambda)}$ be arbitrary rational functions. P_k and Q_k are polynomials (and Q_k has scalar coefficients). Now, a basis in the ideal $\tilde{m}$ is given by operators of the form

$$\tilde{M}_i = N_i(\mathcal{D}_0)\frac{\partial}{\partial x_i} - L_i(\mathcal{D}_0) \tag{2.9}$$

(we omit the proof of this fact). The corresponding equations for function ψ take the form (I.11). The compatibility conditions that yield the nonlinear equations of interest are written in [7].

We remark that the nonlinear equations described at points 1 and 2 arise also on setting $\mathcal{D}_0 = \frac{\partial}{\partial x_0} + i\lambda I_0$, with I_0 an arbitrary nondegenerate constant matrix.

In the general case the construction of a basis in the ideal $\tilde{m}$ is not a simple task. There is however a method for computing directly the nonlinear integrable equations, which avoids the calculation of a basis ℓ in $\tilde{m}$. We have already used this method at the end of Sec. 1, in the derivation of Eq. (I.8). Let us demonstrate it on a more general example.

Let $n=3$ and

$$\mathcal{D}_i \chi = \frac{\partial}{\partial x_i}\chi + \frac{\chi A_i}{\lambda - \lambda_i}, \quad \lambda_i \neq \lambda_j ; [A_i, A_j] = 0 . \tag{2.10}$$

We denote $\chi|_{\lambda=\lambda_i} = \chi_i$ and seek the operator $\tilde{M}$ in the form

$$\tilde{M} = \mathcal{D}_1 \mathcal{D}_2 + R^1_{12}\mathcal{D}_1 + R^2_{12}\mathcal{D}_2 .$$

Function $\tilde{M}\chi$ has singularities at $\lambda = \lambda_1, \lambda_2$. Requiring that these disappear, we get

$$\begin{aligned} R^1_{12} &= -\left(\partial_2 \chi_1 + \frac{\chi_1 A_2}{\lambda_1 - \lambda_2}\right)\chi_1^{-1} , \\ R^2_{12} &= -\left(\partial_1 \chi_2 - \frac{\chi_2 A_1}{\lambda_1 - \lambda_2}\right)\chi_2^{-1} . \end{aligned} \tag{2.11}$$

By definition, $\tilde{M}\chi = 0$. In particular, this is true for $\lambda = \lambda_3$, which leads to the nonlinear equation

$$\partial_1\partial_2\chi_3 + \frac{\partial_2\chi_3 A_1}{\lambda_3 - \lambda_1} + \frac{\partial_1\chi_3 A_2}{\lambda_3 - \lambda_2} + R^2_{12}\partial_2\chi_3 + R^1_{12}\partial_1\chi_3 + \frac{\chi_3 A_1 A_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + R^1_{12}\frac{\chi_3 A_1}{\lambda_3 - \lambda_1} + R^2_{12}\frac{\chi_3 A_2}{\lambda_3-\lambda_2} = 0. \tag{2.12}$$

Two other equations relating χ_1, χ_2, and χ_3 are obtained from (2.11) and (2.12) by circular permutations.

To the best of our knowledge, Eq. (2.12) appears here for the first time. Note that all variables χ_1, χ_2, and χ_3 appear in this equation in a rather symmetric fashion.

The example discussed above suggests the following empirical procedure for constructing equations that can be integrated by using a nonlocal Riemann problem. The first step is to construct operators from $\tilde{m}$ annihilating the function χ, by confining the investigation to polynomials of degrees as low as possible. The coefficients of these operators will be expressions in the values of function χ and of its first derivatives with respect to λ (coefficients of the Taylor expansion) at the poles of the functions $I_k(\lambda)$, and also, if necessary, of the coefficients of the asymptotic expansion of χ in the neighborhood of $\lambda = \infty$. Next, we must examine all equations $\tilde{M}\chi = 0$ resulting in this way in the neighborhood of all singularities, including $\lambda = \infty$. This leads naturally to a closed system of nonlinear equations for the coefficients of the Taylor expansion of function χ at the singularities of $I_k(\lambda)$. For the moment we have no general theorem that would permit us to prove the universality of the indicated procedure, but its effectiveness was tested on a series of examples, which are too cumbersome to be discussed here.

3. Connection with the Local Riemann Problem

Equation (2.5) admits the particular solution

$$T(\lambda', \lambda) = G(\lambda)\, \delta(\lambda - \lambda') . \tag{3.1}$$

Then

$$\frac{\partial G}{\partial x_i} = [I_i(\lambda), G] . \tag{3.2}$$

In this case Riemann's problem (2.1) becomes radically simpler and reduces to the local problem

$$\chi_2(\lambda) = \chi_1(\lambda) + \chi_1(\lambda)\, G(\lambda) . \tag{3.3}$$

The calculation of the ideal $\tilde{m}$ also becomes considerably simpler. In fact, among the operators M one finds now operators of left-multiplication by an arbitrary rational function of λ. Now one can take operators $\tilde{M}$ of the form

$$\tilde{M}_i \chi = \mathcal{D}\chi - u(\lambda)\chi = \frac{\partial \chi}{\partial x_i} + \chi I_i(\lambda) - u_i(\lambda)\chi = 0 \tag{3.4}$$

with $u(\lambda)$ a rational function with the same singularities as $I(\lambda)$. Formula (3.4) is readily recognized as the basic relation of paper [5]. The function elements are expressible through the values of χ at the singularities of I_i by means of the well-known "dressing formulas." Thus, if

$$I_k(\lambda) = \Sigma \frac{I_k}{\lambda - \lambda_n} , \quad u_k = \Sigma \frac{u_n^k}{\lambda - \lambda_n} , \quad \chi|_{\lambda = \lambda_n} = \chi_n ,$$

then

$$u_n^k = \chi_n I_n^k \chi_n^{-1} \qquad \text{(see [5])}.$$

Being operators of first order, $\tilde{M}_i$ form a basis in the ideal $\tilde{m}$. The nonlinear equations of interest can be obtained by either commuting pairs of operators $\tilde{M}_i$ or by using the procedure described in Sec. 2, i.e., by restricting Eqs. (3.4) to neighborhoods of the poles of functions $I_i(\lambda)$. In the second approach the resulting equations take the so-called "spinor form" (see [14]). It is remarkable that for the nonlinear equations written in this form one can formulate a general variational principle (see [15]). This justifies the hope that in the future it will be possible to obtain a variational principle also for systems of type (2.12), connected with the nonlocal Riemann problem.

Commuting pairs of equations of the type (3.4), we obtain nonlinear integrable systems for functions which depend on two independent variables. This is in agreement with the fact that in the case of the local Riemann problem we give arbitrarily a function $G(\lambda)$ of one variable. By similar considerations, the natural number of independent variables for systems of the form (2.12) is three. This agrees with the fact that in the case under consideration one is given an arbitrary function of two variables. An attractive possible development is using nonlocal Riemann problems for constructing particular solutions of even higher-dimensional equations specified by undefined, yet compatible systems of multidimensional conditions. Another possible perspective is using nonlocal Riemann problems to find nontrivial reductions in two-dimensional systems. The results of such a reduction may turn out to be "stationary points" of even higher-dimensional integrable systems.

LITERATURE CITED

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