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On the Theory of Recursion Operator

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Abstract. The general structure and properties of recursion operators for Hamiltonian systems with a finite number and with a continuum of degrees of freedom are considered. Weak and strong recursion operators are introduced. The conditions which determine weak and strong recursion operators are found.

In the theory of nonlinear waves a method for the calculation of the recursion operator, which is based on the use of expansion into a power series over the fields and the momentum representation, is proposed. Within the framework of this method a recursion operator is easily calculated via the Hamiltonian of a given equation. It is shown that only the one-dimensional nonlinear evolution equations can possess a regular recursion operator. In particular, the Kadomtsev–Petviashvili equation has no regular recursion operator.

I. Introduction

The inverse scattering transform method gives a possibility of investigating in detail a wide class of both the ordinary and partial differential equations (see e.g. [1–3]). The equations, integrable by the inverse scattering transform method, possess a number of remarkable properties: solitons, infinite sets of conservation laws, infinite symmetry groups, complete integrability, etc. In turns out also that the equations, to which the inverse scattering transform method is applicable, have the pronounced recursion structure. The so-called recursion operator plays a central role in the formulation of these recursion properties. The role of the recursion operator is two-fold. Firstly, it allows one to write out the families of equations integrable by a given spectral problem in a compact form. For example, the family of equations connected with the famous Korteweg–de Vries (KdV) equation can be represented as follows:

$$\frac{\partial u(x,t)}{\partial t} + \partial L^n u = 0\,, \tag{1.1}$$

where $\partial \equiv \frac{\partial}{\partial x}$, $n=0,1,2,\ldots$, and the recursion operator L is

$$L=\partial^2+2u+2\partial^{-1}u\partial\,. \tag{1.2}$$

The KdV equation corresponds to $n=1$. The recursion operator (1.2) for the KdV family of equations was first introduced by Lenart (see [4]).

It follows from (1.1) that the recursion operator allows us to obtain the whole family, starting from one equation (e.g. with $n=0$). Recursion operators with such a property exist for the other families of equations too: see [5] and subsequent papers on this subject.

The second important role of recursion operators is associated with the Hamiltonian treatment of integrable equations. The Hamiltonian structure of Eqs. (1.1) and of the other equations, integrable by the inverse scattering transform method has been investigated, starting from [6, 7] in a variety of papers (see [1–3]). It was demonstrated in [8, 9] that the integrable equations have a very special structure from the point view of Hamiltonian formalism, namely the whole infinite sets of Hamiltonian structures correspond to these equations. For example, each of Eqs. (1.1) is a Hamiltonian one with respect to the infinite family of Poisson brackets of the form

$$\{F,H\}_n=\int_{-\infty}^{+\infty} dx\frac{\delta F}{\delta u(x)}\partial L^n\frac{\delta H}{\delta u} \qquad n=0,\pm 1,\pm 2,\ldots, \tag{1.3}$$

where the operator L is given by (1.2). In a similar manner, recursion operators determine the families of Hamiltonian structures for the other integrable equations.

So, a recursion operator is the generating operator for the family of equations connected with a given equation and simultaneously the generating operator for the family of Hamiltonian structures. Combination of these two properties in the same operator indicates the importance of recursion operators in the theory of integrable equations.

An important step in the formulation of the theory of the recursion operator was paper [10]. In this paper it was demonstrated how to calculate the recursion operator for the equations integrable by the second-order matrix spectral problem. The Hamiltonian structure of these equations has been considered in paper [11] in which the remarkable properties of the recursion operator were employed to a considerable extent. The method of calculation of recursion operators, based on the use of the spectral problem, has been further developed in [12–16]. By this method the recursion operator has been calculated for a wide class of spectral problems [12–23] (see also the paper [24, 25]).

The other methods which do not use the spectral problems were suggested in [26–29]. In papers [26] the recursion operator appears in the Hamiltonian systems which possess the Hamiltonian pairs of operators. In papers [27–29] some analogs of the recursion operator (hereditary, strong symmetry operators) were considered. By virtue of some postulates, these operators satisfy certain equations. Some solutions of these equations and thereby some examples of the recursion operators have been found. The geometrical structures, connected with the integrable equations and recursion operators, and their properties have been discussed in [30].

In the present paper we consider the different aspects of the theory of recursion operators for Hamiltonian equations. Firstly, we introduce the notions of weak and strong recursion operators. A recursion operator in a "weak" sense (briefly, a weak recursion operator) is the operator which allows us to construct recursively the infinite family of Hamiltonian equations, starting from a given Hamiltonian equation. There exist two types of weak recursion operators. A recursion operator of the first type (H-weak recursion operator) is the operator which converts the gradients of functionals into gradients. A recursion operator of the second type (Ω-weak recursion operator) is the operator which transforms symplectic forms into symplectic forms. We find the sufficient conditions which determine the weak recursion operator (for both types).

A recursion operator in a strong sense is the operator which transforms both gradients into gradients and symplectic forms into symplectic forms. The strong recursion operator generates simultaneously the infinite family of equations, starting from a given equation, and the infinite family of Hamiltonian structures for each equation from this family. The sufficient conditions for the operator L to be a strong recursion operator are given. The so-called Nijenhuis equation for L plays an important role in the theory of weak and strong recursion operators. The Hamiltonian systems both with a finite number and with a continuum of degrees of freedom are considered. The operator (1.2) is an example of the strong recursion operator.

In our paper a method is also proposed for the calculation of recursion operators in the theory of nonlinear waves in space of arbitrary dimension. This method is based on the subsequent use of the expansion of all quantities into a power series over the fields and of the momentum representation instead of a coordinate one. As a result, the equations in variational derivatives, which determine the recursion operator, convert into the system of algebraic functional equations for the coefficients of the expansions of the recursion operator and Hamiltonian. Some equations from this system offer the possibility of calculating the recursion operator via the Hamiltonian of a nonlinear equation.

In the paper we show that any Hamiltonian system of nonlinear waves possesses a formal recursion operator. In the general case, such a recursion operator is the singular operator. In the one-dimensional space there exist Hamiltonian systems which possess the regular recursion operator, i.e. the operator which generates the family of regular Hamiltonians. For some nonlinear equations the recursion operator can be a finite-order polynomial on the fields. The Hamiltonian of such an equation should satisfy a certain system of equations. In particular, it is shown that the only one-dimensional equation with three-linear Hamiltonian, which possesses a recursion operator, linear on field, is the KdV equation.

It is shown that the nonlinear evolution equations in two- and higher dimensional spaces have no regular recursion operators. In particular, the Kadomtsev–Petviashvili equation has no regular recursion operator. Thus, the regular recursion operator is a purely one-dimensional phenomenon.

All these properties of recursion operators are closely connected with the formal canonical equivalence of nonlinear equations with the Hamiltonians, which are the "entire" functionals on fields, to linear equations. Performing the inverse canonical transformation from the linear equation to the initial nonlinear equation, we simultaneously obtain the Hamiltonian of the equation and the expression for the recursion operator. The regularity problem of the recursion operator is associated now with the regularity problem of the linearizing canonical transformation.

The paper is written with the use of elementary methods only. The presented results can be, however, formulated in the invariant form as well.

The paper is organized as follows. The notions of H-weak and Ω-weak recursion operators are introduced in the second section. In the third section the sufficient conditions for the operator L_H to be a H-weak recursion operator are found. In Sect. 4 the necessary and sufficient conditions for that the operator L_Ω be a Ω-weak recursion operator are formulated. The strong recursion operator and the conditions which determine this operator are considered in Sect. 5. Recursion operators in the theory of nonlinear waves (i.e. for systems with a continuum number of degrees of freedom) and the conditions which define these operators are discussed in Sect. 6. Section 7 is devoted to the use of the expansion into a power series over the fields and of the momentum representation for the calculation of recursion operators. The problem of existence of regular recursion operators in one- and multi-dimensional spaces is discussed in Sect. 8.

II. Weak Recursion Operators

In this and next sections we will consider Hamiltonian systems for finite degrees of freedom. We would like to recall that a system of differential equations, which are defined on the $2N$-dimensional phase space, is called the Hamiltonian system if it may be represented in certain local coordinates as follows: (see e.g. [31])

$$\Omega\dot{x}=\nabla H, \tag{2.1}$$

where $x=(x^1,\ldots,x^{2N})$, $\nabla=\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^{2N}}\right)$, $\dot{x}\equiv\frac{dx}{dt}$, H is a function and Ω is a nondegenerate skewsymmetric ($\Omega_{ik}=-\Omega_{ki}$) matrix which obeys the closeness condition

$$\frac{\partial\Omega_{ij}}{\partial x^k}+\frac{\partial\Omega_{ki}}{\partial x^j}+\frac{\partial\Omega_{jk}}{\partial x^i}=0. \tag{2.2}$$

By virtue of the nondegenerateness of Ω, Eq. (2.1) can be also represented as

$$\dot{x}=\Omega^{-1}\nabla H=\{x,H\}, \tag{2.3}$$

where $\{,\}$ denotes a Poisson bracket

$$\{F,H\}=\frac{\partial F}{\partial x^i}(\Omega^{-1})^{ik}\frac{\partial H}{\partial x^k}. \tag{2.4}$$

Here and below the summation is performed over repeated indices. It is well known also that locally by appropriate change of coordinates one can convert the form Ω into the canonical one $\Omega_{(0)}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, where 1 is an identical $N\times N$ matrix (Darboux theorem).

So let us have a certain Eq. (2.1) with concrete Hamiltonian H and symplectic form Ω. What is the way in which one can recurrently multiply this equation without leaving the class of Hamiltonian equations? It is easy to see that one can do this in two ways. The first way is to multiply the right-hand side of (2.1), i.e. the gradient ∇H. The second way is to multiply the left-hand side, i.e. the symplectic form Ω.

Definition. We will refer to the operator L_H as the H-weak recursion operator if any of its power converts the gradient of H (for $H \neq \text{const}$) into the gradients: $L_H^n \nabla H = \nabla H_n$, $n=1,2,3,\ldots$. The operator L_Ω is referred to as the Ω-weak recursion operator if any of its power converts some symplectic form Ω into the symplectic forms:

$$L_\Omega^n \Omega = \Omega_n, \qquad n=1,2,3,\ldots.$$

Making use of the weak recursion operators, one can construct the following infinite families of equations, starting from Eq. (2.1):

$$\Omega \dot{x} = \nabla H_n = L_H^n \nabla H, \qquad n=0,1,2,\ldots, \tag{2.5}$$

and

$$L_\Omega^n \Omega \dot{x} = \Omega_n \dot{x} = \nabla H, \qquad n=0,1,2,\ldots. \tag{2.6}$$

Equations (2.5) are Hamiltonian ones with respect to the same symplectic form Ω and different Hamiltonian H_n. Equations (2.6) are Hamiltonian ones with respect to the same Hamiltonian H and different symplectic forms Ω_n. It is clear that any entire function of the recursion operator is the recursion operator of the same type, too.

Combining these two ways, one can obtain the most general family of the equations

$$\varphi(L_\Omega)\Omega \dot{x} = \Omega_\varphi \dot{x} = \nabla H_f = f(L_H)\nabla H, \tag{2.7}$$

which are associated with Eq. (2.1). Here $\varphi(L_\Omega)$ and $f(L_H)$ are arbitrary entire functions. For $\det L_\Omega \neq 0$, Eqs. (2.7) can be also represented in the form

$$\dot{x} = \Omega_\varphi^{-1} \nabla H_f = \Omega^{-1} \varphi^{-1}(L_\Omega) f(L_H) \nabla H. \tag{2.8}$$

III. H-Weak Recursion Operator

Here we find the conditions which determine weak recursion operators. Firstly, we consider the H-weak recursion operator. Let us note, first of all, that in order that the operator L_H transform a given gradient into gradients ($L_H^n \nabla H = \nabla H_n$) it is necessary and sufficient that it satisfy the equation

$$\frac{\partial}{\partial x^j}\left((L_H^n)_i^k \frac{\partial H}{\partial x^k}\right) = \frac{\partial}{\partial x^i}\left((L_H^n)_j^k \frac{\partial H}{\partial x^k}\right). \tag{3.1}$$

The necessity of condition (3.1) is obvious: it is the equality of cross derivatives which follows from $(L_H^n)_i^k \frac{\partial H}{\partial x^k} = \frac{\partial H_n}{\partial x^i}$. Sufficiency follows from the well known statement that for any simply connected manifold the equation $\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i} = 0$ implies $a_i = \frac{\partial \varphi}{\partial x^i}$ (see e.g. [31]).

Proposition 3.1. *The operator L_H is a H-weak recursion operator if it satisfies the system of equations*

$$\frac{\partial L^m_{Hn}}{\partial x^k}\frac{\partial H}{\partial x^m} + L^m_{Hn}\frac{\partial^2 H}{\partial x^k \partial x^m} - \frac{\partial L^m_{Hk}}{\partial x^n}\frac{\partial H}{\partial x^m} - L^m_{Hk}\frac{\partial^2 H}{\partial x^n \partial x^m} = 0, \tag{3.2}$$

$$\left(\frac{\partial L^n_{Hi}}{\partial x^k} - \frac{\partial L^n_{Hk}}{\partial x^i}\right) L^m_{Hn} + L^n_{Hi}\frac{\partial L^m_{Hk}}{\partial x^n} - L^n_{Hk}\frac{\partial L^m_{Hi}}{\partial x^n} = 0. \tag{3.3}$$

Proof. Condition (3.2) means $L_H \nabla H = \nabla H_1$. Let us prove that by virtue of (3.2) and (3.3) we also have $L^2_H \nabla H = \nabla H_2$. Let us multiply Eq. (3.2) by L^n_{Hi} and sum over n. We obtain

$$L^n_{Hi}\frac{\partial L^m_{Hn}}{\partial x^k}\frac{\partial H}{\partial x^m} + L^n_{Hi}L^m_{Hn}\frac{\partial^2 H}{\partial x^k \partial x^m} = L^n_{Hi}\frac{\partial L^m_{Hk}}{\partial x^n}\frac{\partial H}{\partial x^m} + L^n_{Hi}L^m_{Hk}\frac{\partial^2 H}{\partial x^n \partial x^m}. \tag{3.4}$$

Then multiplying (3.3) by $\frac{\partial H}{\partial x^m}$ and summing over m, we find

$$\frac{\partial L^n_{Hi}}{\partial x^k} L^m_{Hn}\frac{\partial H}{\partial x^m} + L^n_{Hi}\frac{\partial L^m_{Hk}}{\partial x^n}\frac{\partial H}{\partial x^m} = \frac{\partial L^n_{Hk}}{\partial x^i} L^m_{Hn}\frac{\partial H}{\partial x^m} + L^n_{Hk}\frac{\partial L^m_{Hi}}{\partial x^n}\frac{\partial H}{\partial x^m}. \tag{3.5}$$

Further let us sum Eq. (3.5) with Eq. (3.4) and substract Eq. (3.4) with the substituion $i \leftrightarrow k$ from the derived equation. As a result, we have

$$\frac{\partial L^n_{Hi}}{\partial x^k} L^m_{Hn}\frac{\partial H}{\partial x^m} + L^n_{Hi}\frac{\partial L^m_{Hn}}{\partial x^k}\frac{\partial H}{\partial x^m} + L^n_{Hi}L^m_{Hn}\frac{\partial^2 H}{\partial x^k \partial x^m} = \frac{\partial L^n_{Hk}}{\partial x^i} L^m_{Hn}\frac{\partial H}{\partial x^m} + L^n_{Hk}\frac{\partial L^m_{Hn}}{\partial x^i}\frac{\partial H}{\partial x^m} + L^n_{Hk}L^m_{Hn}\frac{\partial^2 H}{\partial x^i \partial x^m}, \tag{3.6}$$

i.e.

$$\frac{\partial}{\partial x^k}\left((L^2_H)^m_i \frac{\partial H}{\partial x^m}\right) = \frac{\partial}{\partial x^i}\left((L^2_H)^m_k \frac{\partial H}{\partial x^m}\right). \tag{3.7}$$

From (3.7) it follows that

$$(L^2_H)^m_i \frac{\partial H}{\partial x^m} = \frac{\partial H_2}{\partial x^i},$$

i.e. $L^2_H \nabla H = \nabla H_2$, where H_2 is a certain function.

Further let us show that $L_H^3 \nabla H = \nabla H_3$ too. Multiplying (3.3) by $L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}}$ and summing over m, we obtain

$$\frac{\partial L_{Hi}^n}{\partial x^k} L_{Hn}^m L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}} - \frac{\partial L_{Hk}^n}{\partial x^i} L_{Hn}^m L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}}$$

$$+ L_{Hi}^n \frac{\partial L_{Hk}^m}{\partial x^n} L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}} - L_{Hk}^n \frac{\partial L_{Hi}^m}{\partial x^n} L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}} = 0 . \tag{3.8}$$

Then we multiply equation (3.6) (with the substitution $i \to \varrho$) by L_{Hi}^{ϱ} and sum over ϱ. Summing the obtained equation with equation (3.8) and using (3.2), we find

$$\frac{\partial L_{Hi}^n}{\partial x^k} (L_H^2)_n^{\ell} \frac{\partial H}{\partial x^{\ell}} + L_{Hi}^n \frac{\partial L_{Hn}^m}{\partial x^k} L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}} + (L_H^2)_i^m \frac{\partial L_{Hm}^{\ell}}{\partial x^k} \frac{\partial H}{\partial x^{\ell}}$$

$$+ (L_H^3)_i^n \frac{\partial^2 H}{\partial x^k \partial x^n} - \frac{\partial L_{Hk}^n}{\partial x^i} (L_H^2)_n^{\ell} \frac{\partial H}{\partial x^{\ell}} - L_{Hk}^n \frac{\partial L_{Hn}^m}{\partial x^i} L_{Hm}^{\ell} \frac{\partial H}{\partial x^{\ell}}$$

$$- (L_H^2)_k^m \frac{\partial L_{Hm}^{\ell}}{\partial x^i} \frac{\partial H}{\partial x^{\ell}} - (L_H^3)_k^n \frac{\partial^2 H}{\partial x^i \partial x^n} = 0 ,$$

i.e.

$$\frac{\partial}{\partial x^k} \left((L_H^3)_i^n \frac{\partial H}{\partial x^n} \right) = \frac{\partial}{\partial x^i} \left((L_H^3)_k^n \frac{\partial H}{\partial x^n} \right) . \tag{3.9}$$

Thus $L_H^3 \nabla H = \nabla H_3$, where H_3 is a certain function.

In a similar manner, one can prove by induction that the equality (3.1) is valid for $n = 4, 5, 6, \ldots$ too, i.e. $L_H^n \nabla H = \nabla H_n$, $n = 1, 2, 3, 4, \ldots$. The proposition is proved.

We emphasize the important role of the quadratic equation (3.3). This equation together with the condition $L_H \nabla H = \nabla H_1$ is equivalent to the equalities $L_H^n \nabla H = \nabla H_n$ for any $n = 1, 2, 3, \ldots$. Note, however, that Eq. (3.3) does not follow from any finite subsystem of equations $L_H^n \nabla H = \nabla H_n$ $(n = 1, \ldots, M)$.

Let us now discuss some properties of Eqs. (2.5). In the general case the flows which are generated by Hamiltonians H_n do not commute with the initial flow (2.1) since $(H_0 \equiv H)$

$$\{H_n, H_0\} = \frac{\partial H_n}{\partial x^i} (\Omega^{-1})^{ik} \frac{\partial H_0}{\partial x^k}$$

$$= \frac{\partial H}{\partial x^m} (L_H^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} \neq 0 . \tag{3.10}$$

Let the operator L_H satisfy the additional constraint $(L_H \Omega)^T = -L_H \Omega$, where A^T denotes a transposed matrix A. Using this constraint, we have

$$\frac{\partial H}{\partial x^m} (L_H^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} = \frac{\partial H}{\partial x^m} (\Omega^{-1})^{mi} (L_H^n)_i^k \frac{\partial H}{\partial x^k}$$

$$= - \frac{\partial H}{\partial x^k} (L_H^n)_i^k (\Omega^{-1})^{im} \frac{\partial H}{\partial x^m} ,$$

i.e. $\frac{\partial H}{\partial x^m}(L_H^n)_i^m(\Omega^{-1})^{ik}\frac{\partial H}{\partial x^k}=0$. Therefore $\{H_n, H_0\}=0$, $n=1,2,3,\ldots$. Analogously, one can show that $\{H_n, H_m\}=0$ $(n,\ m=0,1,2,\ldots)$, i.e. in the case $(L_H\Omega)_{ik}=-(L_H\Omega)_{ki}$ all the flows from the family (2.5) commute to each other.

Thus, in the case $(L_H\Omega)^T=-L_H\Omega$ the family of Hamiltonians H_n is the infinite family of the integrals of motion for any equation of the form (2.5). Each integral of motion H_n is connected with the one-parameter symmetry group of Eq. (2.5) and, in particular, of the initial equation (2.1). In the infinitesimal form these symmetry transformations are $(x'=x+\delta x)$

$$\delta_n x=\varepsilon_n\Omega^{-1}\nabla H_n=\varepsilon_n\Omega^{-1}L_H^n\nabla H\,,\qquad n=0,1,2,3,\ldots, \tag{3.11}$$

where ε_n is the transformation parameter. If $(L_H\Omega)^T=-L_H\Omega$, Eqs. (2.5) and the symmetry transformations (3.11) can be also represented in the form $\dot{x}=(L_H^T)^n\Omega^{-1}\nabla H$, and $\delta_n x=\varepsilon_n(L_H^T)^n\Omega^{-1}H$, $n=0,\ 1,\ 2,\ 3,\ \ldots$.

IV. Ω-Weak Recursion Operator

According to the definition, the operator L_Ω is a Ω-weak recursion operator if

$$\frac{\partial(L_\Omega^n\Omega)_{ij}}{\partial x^k}+\frac{\partial(L_\Omega^n\Omega)_{ki}}{\partial x^j}+\frac{\partial(L_\Omega^n\Omega)_{jk}}{\partial x^i}=0\,, \tag{4.1}$$

and

$$(L_\Omega^n\Omega)^T=-L_\Omega^n\Omega\,, \tag{4.2}$$

for all $n=0,1,2,\ldots$.

One can show by straightforward calculations that if Eqs. (4.1) and (4.2) are satisfied for $n=0,1,2$ then they hold for $n=3$, too. As a result, they are satisfied for any n. So we have

Proposition 4.1. *If together with the form Ω the forms $L_\Omega\Omega$ and $L_\Omega^2\Omega$ are closed and skewsymmetric, then the operator L_Ω is a Ω-weak recursion operator.*

Note that all the conditions (4.2) are satisfied if $\Omega^T=-\Omega$ and $(L_\Omega\Omega)^T=-L_\Omega\Omega$.

If additionally $\det L_\Omega\neq 0$, then the closeness and skewsymmetry of the forms Ω, $L_\Omega\Omega$, $L_\Omega^2\Omega$ lead also to the closeness and skewsymmetry of the forms $L_\Omega^{-n}\Omega$, $n=1,2,3,\ldots$. Indeed, choosing the form $\Omega_2=L_\Omega^2\Omega$ as an initial one and using the closeness and skewsymmetry of the forms $L_\Omega^{-1}\Omega_2$, and $L_\Omega^{-2}\Omega_2$, one can prove that the form $L_\Omega^{-3}\Omega_2=L_\Omega^{-1}\Omega$ is closed and skewsymmetric too. Further one can easily show that all the forms $L_\Omega^{-n}\Omega_2=L_\Omega^{2-n}\Omega(n=3,4,\ldots)$ are closed ans skewsymmetric.

In the case $\det L_\Omega\neq 0$ the closeness and skewsymmetry of one of the following two sets of the forms: $L_\Omega^{-1}\Omega$, Ω, $L_\Omega\Omega$ or $L_\Omega^{-2}\Omega$, $L_\Omega^{-1}\Omega$, Ω are the sufficient conditions in order that the operator L_Ω be a Ω-weak recursion operator too.

In Proposition 4.1 and subsequent results, one can start from any given symplectic form Ω. To simplify the calculations it is convenient, however, to choose the local coordinates in such a way that $\Omega=\Omega_{(0)}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. This does not reduce the generality of the results obtained since Eqs. (4.1) and (4.2) are of invariant character.

In the case $\Omega=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, Eqs. (4.1) and (4.2) for $n=0,1,2$ are

$$\frac{\partial L^n_{\Omega i}}{\partial x^\ell}\Omega_{nm}+\frac{\partial L^n_{\Omega\ell}}{\partial x^m}\Omega_{ni}+\frac{\partial L^n_{\Omega m}}{\partial x^i}\Omega_{n\ell}=0\,, \tag{4.3}$$

$$\frac{\partial (L^2_\Omega)^n_i}{\partial x^\ell}\Omega_{nm}+\frac{\partial (L^2_\Omega)^n_\ell}{\partial x^m}\Omega_{ni}+\frac{\partial (L^2_\Omega)^n_m}{\partial x^i}\Omega_{n\ell}=0\,, \tag{4.4}$$

and

$$\Omega_{nk}L^k_{\Omega\ell}=L^k_{\Omega n}\Omega_{k\ell}\,. \tag{4.5}$$

Multiplying (4.3) by $L^m_{\Omega k}$, summing over m, and taking into account (4.5), one obtains

$$\frac{\partial L^n_{\Omega i}}{\partial x^\ell}L^m_{\Omega n}\Omega_{mk}+L^n_{\Omega k}\frac{\partial L^m_{\Omega n}}{\partial x^i}\Omega_{m\ell}=-L^n_{\Omega k}\frac{\partial L^m_{\Omega\ell}}{\partial x^n}\Omega_{mi}\,. \tag{4.6}$$

Using (4.6), it is easy to see that Eq. (4.4) is equivalent to the following one:

$$L^n_{\Omega i}\frac{QL^m_{\Omega k}}{\partial x^n}\Omega_{m\ell}+L^n_{\Omega\ell}\frac{\partial L^m_{\Omega i}}{\partial x^n}\Omega_{mk}+L^n_{\Omega k}\frac{\partial L^m_{\Omega\ell}}{\partial x^n}\Omega_{mi}=0\,. \tag{4.7}$$

Let us transform the second and third terms in (4.7), using the equality (4.5) and $\Omega_{nt}(\Omega^{-1})^{t\varrho}=\delta^\varrho_n$. As a result, we have

$$L^n_{\Omega i}\frac{\partial L^m_{\Omega k}}{\partial x^n}\Omega_{m\ell}-\frac{\partial L^\varrho_{\Omega i}}{\partial x^n}\Omega_{k\varrho}(\Omega^{-1})^{nt}L^m_{\Omega t}\Omega_{m\ell}-L^n_{\Omega k}\frac{\partial L^m_{\Omega i}}{\partial x^n}\Omega_{m\ell}=0\,. \tag{4.8}$$

Hence

$$L^n_{\Omega i}\frac{\partial L^m_{\Omega k}}{\partial x^n}-\frac{\partial L^\varrho_{\Omega i}}{\partial x^n}\Omega_{k\varrho}(\Omega^{-1})^{nt}L^m_{\Omega t}-L^n_{\Omega k}\frac{\partial L^m_{\Omega i}}{\partial x^n}=0\,. \tag{4.9}$$

We note now that by virtue of (4.5) Eq. (4.3) is equivalent to

$$\frac{\partial L^n_{\Omega i}}{\partial x^\ell}\Omega_{nm}+\frac{\partial L^n_{\Omega\ell}}{\partial x^m}\Omega_{ni}-\frac{\partial L^n_{\Omega\ell}}{\partial x^i}\Omega_{nm}=0\,. \tag{4.10}$$

If one multiplies (4.10) by $(\Omega^{-1})^{mk}$ and sums over m, one obtains

$$\begin{aligned}\frac{\partial L^k_{\Omega i}}{\partial x^\ell}-\frac{\partial L^k_{\Omega\ell}}{\partial x^i}&=-\frac{\partial L^n_{\Omega\ell}}{\partial x^m}\Omega_{ni}(\Omega^{-1})^{mk}\\&=-\frac{\partial L^n_{\Omega i}}{\partial x^m}\Omega_{\ell n}(\Omega^{-1})^{mk}\,.\end{aligned} \tag{4.11}$$

Substitution of (4.11) into (4.9), eventually, gives

$$\begin{aligned}L^n_{\Omega i}\frac{\partial L^m_{\Omega k}}{\partial x^n}&+\left(\frac{\partial L^t_{\Omega i}}{\partial x^k}-\frac{\partial L^t_{\Omega k}}{\partial x^i}\right)L^m_{\Omega t}\\&-L^n_{\Omega k}\frac{\partial L^m_{\Omega i}}{\partial x^n}=0\,.\end{aligned} \tag{4.12}$$

So the equation, which contains only the recursion operator L_Ω, follows from Eqs. (4.3)–(4.5).

Proposition 4.2. *If operator L_Ω satisfies the system of equations*

$$\left(\frac{\partial L^m_{\Omega i}}{\partial x^k}-\frac{\partial L^m_{\Omega k}}{\partial x^i}\right)L^\ell_{\Omega m}+L^n_{\Omega i}\frac{\partial L^\ell_{\Omega k}}{\partial x^n}-L^n_{\Omega k}\frac{\partial L^\ell_{\Omega i}}{\partial x^n}=0\,, \tag{4.13}$$

$$\frac{\partial(L_\Omega\Omega)_{ik}}{\partial x^\ell}+\frac{\partial(L_\Omega\Omega)_{\ell i}}{\partial x^k}+\frac{\partial(L_\Omega\Omega)_{k\ell}}{\partial x^i}=0\,, \tag{4.14}$$

$$(L_\Omega\Omega)_{k\ell}=-(L_\Omega\Omega)_{\ell k}\,, \tag{4.15}$$

then it is a Ω-weak recursion operator.

Proof. Let $\Omega=\Omega_{(0)}$. The conditions (4.13)–(4.15) are equivalent to the conditions (4.3)–(4.6). Indeed, from (4.12) and (4.11), one gets (4.9). Multiplying (4.9) by $\Omega_{m\ell}$, we obtain (4.8). Equation (4.7) follows from (4.8) and is equivalent to (4.4) due to (4.6). The conditions (4.13)–(4.15) are, therefore, equivalent to the conditions of Proposition 4.1.

Let us consider the conditions (4.13) and (4.14) in more detail for the case $\det L_\Omega \neq 0$. At first sight, these conditions are not necessary. Indeed, if one takes instead of Ω, $L_\Omega\Omega$, $L^2_\Omega\Omega$, the other sets of three closed forms: $L_\Omega^{-2}\Omega$, $L_\Omega^{-1}\Omega$, Ω or $L_\Omega^{-1}\Omega$, Ω, $L_\Omega\Omega$, then instead of (4.13) and (4.14) we will have the analogous conditions with the substituion $L_\Omega\to L_\Omega^{-1}$.

However, it is worth noting the following. Firstly, multiplying Eq. (4.13) for L_Ω^{-1} by $L^i_{\Omega\alpha}L^k_{\Omega\beta}L^\gamma_{\Omega\ell}$, summing over i, k, ℓ, and taking into account (4.13), one gets Eq. (4.13) for L_Ω. So Eq. (4.13) for L_Ω^{-1} is equivalent to the same equation for L_Ω. In other words, if L_Ω is the solution of (4.13), then L_Ω^{-1} is the solution, too.

Secondly, the conditions of closeness and skewsymmetry of the form $L_\Omega^{-1}\Omega$ together with Eq. (4.13) are equivalent to the conditions of closeness and skewsymmetry of the form $L^2_\Omega\Omega$. Indeed, putting $\Omega=\Omega_{(0)}$, multiplying (4.14) for $L_\Omega^{-1}\Omega$ by $L^i_{\Omega\alpha}L^\ell_{\Omega\beta}L^k_{\Omega\gamma}$, summing over i, k, ℓ and using (4.15), we obtain Eq. (4.7). Equation (4.9) follows from (4.7). Using (4.12) one gets (4.11). This equation is equivalent to (4.10) and, hence, to Eq. (4.3).

Thus, at $\det L_\Omega\neq 0$ the conditions (4.13)–(4.15) for L_Ω^{-1} are equivalent to the conditions (4.13)–(4.15) for L_Ω.

So in the case $\det L_\Omega \neq 0$ we have

Theorem 4.1. *The conditions* (4.13)–(4.15) *are necessary and sufficient conditions in order that the operator L_Ω be a Ω-weak recursion operator.*

The invariant form of Eq. (3.9) (or (4.13)) is the following:

$$[L^T\xi, L^T\eta]-L^T[L^T\xi,\eta]-L^T[\xi,L^T\eta]+L^{T^2}[\xi,\eta]=0\,, \tag{4.16}$$

where L^T is the transposed matrix L, ξ and η are arbitrary vector fields, and $[\xi,\eta]$ denotes the standard commutator of vector fields (see e.g. [31]): $[\xi,\eta]_i=\sum_k\left(\frac{\partial\xi_i}{\partial x^k}\eta^k-\frac{\partial\eta_i}{\partial x^k}\xi^k\right)$. Following paper [26], where Eq. (4.16) was considered for the first time, we will refer to Eq. (4.16) as the Nijenhuis equation. Note that for the Ω-weak recursion operator one has $L_\Omega^T=\Omega^{-1}L_\Omega\Omega$ due to (4.2).

An equation of the form (4.16) has been also considered in [27–30].

V. Strong Recursion Operator

Let us consider now the situation when the operator L is a recursion one both in H and Ω senses.

Definition. Operator L is the strong recursion operator if any of its power transforms the gradient into the gradients ($L^n\nabla H=\nabla H_n$) and the symplectic form into the symplectic forms ($L^n\Omega=\Omega_n$).

Possessing simultaneously the properties of both weak recursion operators, the strong recursion operator generates both the infinite family of Hamiltonians H_n and the infinite family of symplectic forms $\Omega_n=L^n\Omega$. Equations (2.5), (2.6) or (2.8) which are generated by a strong recursion operator have all the previous properties and some new ones.

Let we have an equation $\dot{x}=\Omega^{-1}\nabla H$ from the family of Eqs. (2.8), and let $\det L\neq 0$. By virtue of the properties of the strong recursion operator, we have

$$\dot{x}=\Omega^{-1}\nabla H=(L^n\Omega)^{-1}L^n\nabla H=\Omega_n^{-1}\nabla H_n, \tag{5.1}$$

where n is any integer and Ω_n are the closed symplectic forms.

So any equation generated by the strong recursion operator is a Hamiltonian one with respect to the infinite set of Hamiltonian structures (pairs Ω_n, H_n).

We denote the Poisson brackets which corresponds to the form Ω_n as $\{,\}_n:\{F,H\}_n=\frac{\partial F}{\partial x^i}(\Omega_n^{-1})^{ik}\frac{\partial H}{\partial x^k}$. Let us calculate $\{H_{n_1},H_{n_2}\}_{n_3}$. Taking into account that $(L\Omega)^T=-L\Omega$, we have

$$\{H_{n_1},H_{n_2}\}_{n_3}=(L^{n_1})_i^m\frac{\partial H}{\partial x^m}(\Omega^{-1})^{i\ell}(L^{n_2-n_3})_\ell^k\frac{\partial H}{\partial x^k}$$
$$=\frac{\partial H}{\partial x^m}(\Omega^{-1})^{mi}(L^{n_1+n_2-n_3})_i^k\frac{\partial H}{\partial x^k}.$$

Then

$$\frac{\partial H}{\partial x^m}(\Omega^{-1})^{mi}(L^n)_i^k\frac{\partial H}{\partial x^k}=\frac{\partial H}{\partial x^m}(L^n)_i^m(\Omega^{-1})^{ik}\frac{\partial H}{\partial x^k}$$
$$=-\frac{\partial H}{\partial x^k}(\Omega^{-1})^{ki}(L^n)_i^m\frac{\partial H}{\partial x^m}.$$

Therefore, $\frac{\partial H}{\partial x^m}(\Omega^{-1})^{mi}(L^n)_i^k\frac{\partial H}{\partial x^k}=0$, and, as a consequence, $\{H_{n_1},H_{n_2}\}_{n_3}=0$ for any n_1, n_2, n_3.

Thus, all Hamiltonians H_n which are generated by the strong recursion operator are in involution with respect to any Poisson bracket $\{,\}_m$ generated by this operator.

It is clear also that if the initial equation (2.1) admits the strong recursion operator L, then any equation of the form (2.8) possesses the same strong recursion operator.

So the Hamiltonian equations, which admit a strong recursion operator, have a very special structure. Firstly, they possess the infinite set of the integrals of motion in involution and they are Hamiltonian ones with respect to the infinite family of Hamiltonian structures. Secondly, the infinite families of equations are associated with such equations: any equation of these families has the same properties as the initial one.

In other words, the strong recursion operator generates the infinite family of Hamiltonian structures (Ω_n, H_m; n, $m=0, \pm 1, \pm 2, \ldots$) from the initial Hamiltonian system. Each of these Hamiltonian structures determines the dynamical system (flow). The Hamiltonian structures (Ω_n, H_m) with the same value of $n-m$ correspond to the same dynamical system. The family of Hamiltonians H_m forms the infinite set of common integrals of motion which are in involution with respect to any symplectic structure Ω_n.

The examples of strong recursion operators are well known. In the case of a continual number of the degrees of freedom they are, for example, the operator (1.2) and Eqs. (1.1), and the recursion operators which were calculated in [10–22]. Some properties of the equations which admit the strong recursion operator have been discussed as well.

The operator $\tilde{L}=\Omega^{-1}L\Omega$ is closely connected with the recursion operator L. The operator $\tilde{L}$ transforms vector fields into vector fields: $\tilde{L}^n\Omega^{-1}\nabla H=\Omega^{-1}\nabla H_n$. Equations (2.5)–(2.6) can be also represented in the form $\dot{x}=\tilde{L}^n\Omega^{-1}\nabla H$. Both operators L and $\tilde{L}$ naturally appear in the approach which is based on the spectral problems (see e.g. [20, 21]).

Let us consider now the conditions which define a strong recursion operator. These conditions are obviously the join of the conditions which determine the H- and Ω-weak recursion operators. Propositions (3.2) and (4.2) give rise to

Theorem 5.1. *If operator L satisfies the system of equations*

$$\left(\frac{\partial L_i^m}{\partial x^k}-\frac{\partial L_k^m}{\partial x^i}\right)L_m^\ell+L_i^n\frac{\partial L_k^\ell}{\partial x^n}-L_k^n\frac{\partial L_i^\ell}{\partial x^n}=0\,, \tag{5.2}$$

$$\frac{\partial L_i^n}{\partial x^k}\frac{\partial H}{\partial x^n}+L_i^n\frac{\partial^2 H}{\partial x^k\partial x^n}-\frac{\partial L_k^n}{\partial x^i}\frac{\partial H}{\partial x^n}-L_k^n\frac{\partial^2 H}{\partial x^i\partial x^n}=0\,, \tag{5.3}$$

$$\frac{\partial(L\Omega)_{ik}}{\partial x^\ell}+\frac{\partial(L\Omega)_{\ell i}}{\partial x^k}+\frac{\partial(L\Omega)_{k\ell}}{\partial x^i}=0\,, \tag{5.4}$$

$$(L\Omega)_{k\ell}=-(L\Omega)_{\ell k}\,, \tag{5.5}$$

then it is the strong recursion operator for the Hamiltonian system $\Omega\dot{x}=\nabla H$.

In the case $\det L\neq 0$, *the conditions* (5.2), (5.4) *and* (5.5) *are also necessary ones.*

So the H-weak recursion operator becomes the strong recursion operator if it also transforms the symplectic form Ω into a symplectic form. The Ω-weak recursion operator becomes a strong one if it additionally converts the gradient ∇Z into the gradient ($L\nabla H=\nabla H_1$).

We see that in the description of both the weak and strong recursion operators the quadratic equation (5.2) plays an important role. This equation is a very special one. It is a system of $2N^2(2N-1)$ equations for $(2N)^2$ quantities L_i^k $(i, k=1, \ldots, 2N)$. Nevertheless, this highly overdetermined (for $N>1$) system has a large class of solutions. As we have seen, if L is a solution of (5.2), then L^{-1} is a solution too. It is not difficult also to show, that together with L the quantity $1-\lambda L$ is also the solution for any number λ. Hence, $(1-\lambda L)^{-1}=1+\lambda L+\lambda^2 L^2+\ldots$ is the solution of (5.2) too. The simplest solution of Eq. (5.2) is $L_i^k=\delta_i^k a_i(x^i)$, where $a_i(x_i)$ are abitrary functions. By virtue of the invariance of Eq. (5.2) under the general coordinate transformations $x^i \to x^{i'}=f^i(x)$, Eq. (5.2) has also the solutions of the form $L_i^k=\sum_{\ell=1}^{2N} a_\ell(x^\ell)\frac{\partial f^k(x)}{\partial x^\ell}\frac{\partial x^\ell}{\partial f^i(x)}$, where $f^k(x)$ and $a_\ell(x^\ell)$ are arbitrary functions.

In the conclusion of this section we compare the results of the present paper with those of papers [26–28]. The key notion in papers [26] was the notion of a Hamiltonian pair, i.e. two Hamiltonian operators such that any linear superposition of them is a Hamiltonian operator too. In this approach the recursion operator appears as the "ratio" of two Hamiltonian operators from the Hamiltonian pair.

In our approach we deal with the recursion operator from the very beginning. If a system admits the Ω-weak recursion operator L_Ω, then all the form $L^n_\Omega\Omega(n=0,1,2,\ldots)$ are closed. The form $L_\Omega\Omega+\lambda L^2_\Omega\Omega+\lambda^2 L^3_\Omega\Omega+\ldots =(1-\lambda L_\Omega)^{-1}L_\Omega\Omega$, where λ is any number, is closed too. Therefore the operator $((1-\lambda L_\Omega)^{-1}L_\Omega\Omega)^{-1}=(L_\Omega\Omega)^{-1}-\lambda\Omega^{-1}$ is a Hamiltonian one for any λ, i.e. the operators $(L_\Omega\Omega)^{-1}$ and Ω^{-1} form the Hamiltonian pair.

Proposition 5.1 (Gelfand and Dorphman). *In order that the Hamiltonian operators Ω^{-1} and $(L\Omega)^{-1}$ form the Hamiltonian pair it is necessary and sufficient that the forms Ω, $L\Omega$, $L^2\Omega$ be the closed ones.*

Sufficiency immediately follows from Proposition (4.1). Let us prove the necessity. The Jacobi identity for the Hamiltonian operator $\Omega^{-1}+\lambda(L\Omega)^{-1}$ leads to the closeness of the forms Ω, $L\Omega$, and also to the equation

$$(\Omega^{-1})^{im}\frac{\partial((L\Omega)^{-1})^{jk}}{\partial x^m}+(\Omega^{-1})^{km}\frac{\partial((L\Omega)^{-1})^{ij}}{\partial x^m}+(\Omega^{-1})^{jm}\frac{\partial((L\Omega)^{-1})^{ki}}{\partial x^m}=0\,, \quad (5.6)$$

where we choose $\Omega=\Omega_{(0)}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Multiplying (5.6) by $(L\Omega)_{\alpha i}(L\Omega)_{\beta j}(L\Omega)_{\gamma k}$, and summing over i, j, k, we obtain (4.7). By virtue of (4.3), the equality (4.7) is equivalent to (4.9), i.e. to the closeness condition for the form $L^2\Omega$. The left-hand side of (5.6) is nothing but the Schouten bracket (see e.g. [26]) for the operators Ω^{-1} and $(L\Omega)^{-1}$. So the equality to zero of the Schouten bracket $[\Omega^{-1}, (L\Omega)^{-1}]$ is equivalent to the closeness of the forms $L\Omega$ and $L^2\Omega$.

If the operator L is a Ω-weak recursion operator, then any two operators $(L^{n_1}\Omega)^{-1}$ and $(L^{n_2}\Omega)^{-1}$ form the Hamiltonian pair. Indeed, by virtue of the closeness of the form

$$L^{n_1}\Omega+\lambda L^{2n_1-n_2}\Omega+\lambda^2 L^{3n_1-2n_2}\Omega+\ldots=(1-\lambda L^{n_1-n_2})^{-1}L^{n_1}\Omega\,,$$

the operator

$$((1-\lambda L^{n_1-n_2})^{-1}L^{n_1}\Omega)^{-1}=(L^{n_1}\Omega)^{-1}-\lambda(L^{n_2}\Omega)^{-1}$$

is a Hamiltonian one for any λ and n_1, n_2.

Taking into account all these results, it is not difficult to see that from the point of view of the construction of Hamiltonian systems Theorem 5.1 and Theorem 3.4 from [26] are essentially equivalent.

The recursion operator which is considered in [26] (regular operator), is the Ω-weak recursion operator in our terminology.

The fact that the operator $\Omega\tilde{L}\Omega^{-1}$, where $\tilde{L}\Omega^{-1}\nabla H=\Omega^{-1}\nabla H_1$, transforms gradients into gradients, i.e. it is the H-weak recursion operator, and also the fact that the eigenfunctions of this operator are gradients of its eigenvalues have been noted in [30].

VI. Recursion Operators in the Theory of Nonlinear Waves

Generalization of the results of the previous sections to the theory of nonlinear waves, i.e. to the case of a continual number of the degrees of freedom, is obvious enough. We have the system of n fields $u^1(x,t),\ldots,u^n(x,t)$ in d-dimensional space ($x\equiv(x_1,\ldots,x_d)$). Let us recall (see e.g. [32, 33]) that the system of n equations is a Hamiltonian one if it can be represented in the form

$$\int dx'\Omega_{\alpha\beta}(x,x')\frac{\partial u^\beta(x',t)}{\partial t}=\frac{\delta H}{\delta u^\alpha(x,t)}, \quad (\alpha=1,\ldots,n), \tag{6.1}$$

where the Hamiltonian H is a certain functional on $u^1,\ldots,u^n$, $\frac{\delta}{\delta u}$ denotes a variational derivative and $\Omega_{\alpha\beta}(x,x')$ is a kernel of the nondegenerate linear operator (which depends, in general, on $u^1,\ldots,u^n$), which satisfies the closeness

$$\frac{\delta\Omega_{\alpha\beta}(x,x')}{\delta u^\gamma(x'')}+\frac{\delta\Omega_{\gamma\alpha}(x'',x)}{\delta u^\beta(x')}+\frac{\delta\Omega_{\beta\gamma}(x',x'')}{\delta u^\alpha(x)}=0 \tag{6.2}$$

and skewsymmetry conditions

$$\Omega_{\alpha\beta}(x,x')=-\Omega_{\beta\alpha}(x',x). \tag{6.3}$$

Similar to the case of a finite number of degrees of freedom, we define the weak and strong recursion operators:

$$\int dx'(L^n_H)^\beta_\alpha(x,x')\frac{\delta H}{\delta u^\beta(x')}=\frac{\delta H_n}{\delta u^\alpha(x)}, \tag{6.4}$$

$$\int dx'(L^n_\Omega)^\beta_\alpha(x,x')\Omega_{\beta\gamma}(x',x'')=\Omega_{n\alpha\gamma}(x,x''). \tag{6.5}$$

The propositions and theorems of the previous sections hold for nonlinear waves, too. The only modification is concerned with the form of the corresponding equations. In particular, the analogs of Eqs. (5.2)–(5.5) are

$$\int d\tilde{x}\left\{\left(\frac{\delta L^{\delta}_{\alpha}(x,\tilde{x})}{\delta u^{\beta}(x')}-\frac{\delta L^{\delta}_{\beta}(x',\tilde{x})}{\delta u^{\alpha}(x)}\right)L^{\gamma}_{\delta}(\tilde{x},x'')\right.$$
$$\left.+L^{\delta}_{\alpha}(x,\tilde{x})\frac{\delta L^{\gamma}_{\beta}(x',x'')}{\delta u^{\delta}(\tilde{x})}-L^{\delta}_{\beta}(x',\tilde{x})\frac{\delta L^{\gamma}_{\alpha}(x,x'')}{\delta u^{\delta}(\tilde{x})}\right\}=0, \tag{6.6}$$

$$\int d\tilde{x}\left\{\frac{\delta L^{\varrho}_{\alpha}(x,\tilde{x})}{\delta u^{\beta}(x')}\frac{\delta H}{\delta u^{\varrho}(\tilde{x})}+L^{\varrho}_{\alpha}(x,\tilde{x})\frac{\delta^2 H}{\delta u^{\beta}(x')\delta u^{\varrho}(\tilde{x})}\right.$$
$$\left.-\frac{\delta L^{\varrho}_{\beta}(x',\tilde{x})}{\delta u^{\alpha}(x)}\cdot\frac{\delta H}{\delta u^{\varrho}(\tilde{x})}-L^{\varrho}_{\beta}(x',\tilde{x})\frac{\delta^2 H}{\delta u^{\alpha}(x)\delta u^{\varrho}(\tilde{x})}\right\}=0, \tag{6.7}$$

$$\frac{\delta(L\Omega)_{\alpha\beta}(x,x')}{\delta u^{\gamma}(x'')}+\frac{\delta(L\Omega)_{\gamma\alpha}(x'',x)}{\delta u^{\beta}(x')}+\frac{\delta(L\Omega)_{\beta\gamma}(x',x'')}{\delta u^{\alpha}(x)}=0, \tag{6.8}$$

$$(L\Omega)_{\beta\alpha}(x',x)=-(L\Omega)_{\alpha\beta}(x,x'), \tag{6.9}$$

where $(L\Omega)_{\alpha\beta}(x,x')\stackrel{\text{def}}{=}\int dx'' L^{\gamma}_{\alpha}(x,x'')\Omega_{\gamma\beta}(x'',x')$.

Theorem 6.1. *If operator L satisfies Eqs.* (6.6), (6.7), *then it is a H-weak recursion operator. If operator L satisfies Eq.* (6.6), (6.8), (6.9), *it is a Ω-weak recursion operator. In the case when operator L satisfies the whole system of Eqs.* (6.6)–(6.9), *it is the strong recursion operator.*

The strong recursion operator L generates, starting from (6.1), the infinite family of nonlinear evolution equations

$$\frac{\partial u^{\alpha}(x,t)}{\partial t}=\int dx'(\Omega^{-1}\varphi(L))^{\alpha\beta}(x,x')\frac{\partial H}{\delta u^{\beta}(x',t)}, \tag{6.10}$$

where $\varphi(L)$ is any entire (meromorphic for $\det L \neq 0$) scalar function. Each of Eqs. (6.10) possesses the infinite set of the integrals of motion H_n and is a Hamiltonian one with respect to the infinite family of Poisson brackets of the form

$$\{F,H\}_f=\int dxdx'\frac{\delta F}{\delta u^{\alpha}(x)}(\Omega^{-1}f(L))^{\alpha\beta}(x,x')\frac{\delta H}{\delta u^{\beta}(x')}, \tag{6.11}$$

where $f(L)$ is an arbitrary entire (meromorphic for $\det L \neq 0$) function.

Taking into account our further constructions, we rewirte the formulae given here in the momentum representation. Performing the Fourier transform

$$\begin{aligned}
&u^{\alpha}(x,t)=(2\pi)^{-\frac{d}{2}}\int dp\, a^{\alpha}_{p}(t)\exp(ipx),\\
&\Omega_{\alpha\beta}(x,x')=(2\pi)^{-d}\int dpdp'\Omega_{\alpha\beta,pp'}\exp(ipx+ip'x'),\\
&L^{\beta}_{\alpha}(x,x')=(2\pi)^{-d}\int dpdp' L^{\beta}_{\alpha,pp'}\exp(ipx+ip'x'),\\
&(p\equiv(p_1,\dots,p_d)),\qquad px\stackrel{\text{def}}{=}p_1x_1+\dots+p_dx_d),
\end{aligned} \tag{6.12}$$

we get the closeness condition

$$\frac{\delta\Omega_{\alpha\beta,pq}}{\delta a^{\gamma}_{-k}}+\frac{\delta\Omega_{\gamma\alpha,kp}}{\delta a^{\beta}_{-q}}+\frac{\delta\Omega_{\beta\gamma,qk}}{\delta a^{\alpha}_{-p}}=0, \tag{6.13}$$

and the skewsymmetry condition

$$\Omega_{\alpha\beta,pq}=-\Omega_{\beta\alpha,qp}. \tag{6.14}$$

Equations (6.6)–(6.9) in the momentum representation are of the form

$$\int d\tilde{p}\left\{L^{\varrho}_{\alpha,p\tilde{p}}\frac{\delta L^{\gamma}_{\beta q,k}}{\delta a^{\varrho}_{\tilde{p}}}-L^{\varrho}_{\beta,q\tilde{p}}\frac{\delta L^{\gamma}_{\alpha,pk}}{\delta a^{\varrho}_{\tilde{p}}}\right. \\ \left.+\left(\frac{\delta L^{\varrho}_{\alpha,p\tilde{p}}}{\delta a^{\beta}_{-q}}-\frac{\delta L^{\varrho}_{\beta,q\tilde{p}}}{\delta a^{\alpha}_{-p}}\right)L^{\gamma}_{\varrho,-\tilde{p}k}\right\}=0\,, \tag{6.15}$$

$$\int dk\left\{\frac{\delta L^{\varrho}_{\alpha,pk}}{\delta a^{\beta}_{-q}}\frac{\delta H}{\delta a^{\varrho}_{k}}+L^{\varrho}_{\alpha,pk}\frac{\delta^2 H}{\delta a^{\beta}_{-q}\delta a^{\varrho}_{k}}\right. \\ \left.-\frac{\delta L^{\varrho}_{\beta,qk}}{\delta a^{\alpha}_{-p}}\frac{\delta H}{\delta a^{\varrho}_{k}}-L^{\varrho}_{\beta,qk}\frac{\delta^2 H}{\delta a^{\alpha}_{-p}\delta a^{\varrho}_{k}}\right\}=0\,, \tag{6.16}$$

$$\frac{\delta(L\Omega)_{\alpha\beta,pq}}{\delta a^{\gamma}_{-k}}+\frac{\delta(L\Omega)_{\gamma\alpha,kp}}{\delta a^{\beta}_{-q}}+\frac{\delta(L\Omega)_{\beta\gamma,qk}}{\delta a^{\alpha}_{-p}}=0\,, \tag{6.17}$$

$$(L\Omega)_{\beta\alpha,qp}=-(L\Omega)_{\alpha\beta,pq}\,, \tag{6.18}$$

where $(L\Omega)_{\alpha\beta,pq}=\int dk L^{\gamma}_{\alpha,pk}\Omega_{\gamma\beta,-kq}$.

In what follows we will consider nonlinear systems which are described by one real field. In this case a local and u-independent symplectic form is of the form

$$\Omega_{(0)pq}=f_p\delta(p+q)\,, \tag{6.19}$$

where f_p is an antisymmetric function ($f_{-p}=-f_p$) and $\delta(p)$ is a Dirac-delta function. For the one-dimensional space ($d=1$) without loss of generality one can choose

$$\Omega_{(0)pq}=-\frac{i}{p}\delta(p+q)\,. \tag{6.20}$$

VII. Expansion over Nonlinearity and Recursion Operator

The problem of calculation of the recursion operator, i.e. the problem of solution of Eqs. (6.6)–(6.9) or (6.15)–(6.19) in functional derivatives, is difficult enough even in the simplest case of one field. One can simplify this problem if one restricts oneself to a certain special class of dynamical systems (Ω, H) and solutions.

For this purpose we consider the translation-invariant systems which have a smooth behaviour at $a_p\to 0$, i.e. smoothly reduce to the linear system in this limit. So we assume that the Hamiltonian and the symplectic form of the translation-invariant system (6.1) are of the form

$$H=\sum_{n=2}^{\infty}\int dq_1\ldots dq_n\delta(q_1+\ldots+q_n)V_{(n)q_1\ldots q_n}a_{q_1}\ldots a_{q_n}\,, \tag{7.1}$$

$$\Omega_{pq}=\sum_{n=0}^{\infty}\int dq_1\ldots dq_n\delta(p+q-q_1-\ldots-q_n)\Omega_{(n)pq(q_1\ldots q_n)}a_{q_1}\ldots a_{q_n}\,, \tag{7.2}$$

where $V_{(n)q_1\ldots q_n}$ are some functions which are completely symmetric on their variables and $\Omega_{(n)pq(q_1\ldots q_n)}$ are functions which are symmetric on the variables $q_1,\ldots,q_n$. For simplicity, we consider the case of one real field and successively use the momentum representation.

We will search for the recursion operator L_{pq} in the form of an "entire" function on a_p too:

$$L_{pq}=\sum_{n=0}^{\infty}\int dq_1\ldots dq_n \tilde{L}_{(n)pq(q_1\ldots q_n)}a_{q_1}\ldots a_{q_n}, \tag{7.3}$$

where $\tilde{L}_{(n)pq(q_1\ldots q_n)}$ are functions which are completely symmetric over the variables $q_1, \ldots, q_n$. The translation invariance gives a certain restriction on the form of the functions $\tilde{L}_{(n)pq(q_1\ldots q_n)}$. Namely, taking into account that $\{\vec{P}, a_q\}=i\vec{q}a_q$, where $\vec{P}$ is a total momentum, we have

$$\tilde{L}_{(n)pq(q_1\ldots q_n)}=\delta(p+q-q_1-\ldots-q_n)L_{(n)pq(q_1\ldots q_n)}, \tag{7.4}$$

where $L_{(n)pq(q_1\ldots q_n)}$ are some functions.

Further, let us consider the case of a constant symplectic form Ω_{pq} and choose it as (6.19).

Let us substitute the expressions (7.1), (7.3), (7.4), and (6.19) into Eqs. (6.15)–(6.18). The left-hand side of these equations should be equal to zero in any order on a_p. Therefore, Eqs. (6.15)–(6.18) are equivalent to the following functional equations for $L_{(n)pq(q_1\ldots q_n)}$ and $V_{(n)q_1\ldots q_n}$:

$$\begin{aligned}&\delta(p+q+k-k_1-\ldots-k_n)\underset{(k_1,\ldots,k_n)}{\mathrm{Sym}}\sum_{m=0}^{n}\{n-m+1)\\&L_{(m)p,\sum_{\ell=1}^{m}k_\ell-p(k_1\ldots k_m)}L_{(n-m+1)qk\left(\sum_{\ell=1}^{m}k_\ell-p,k_{m+1},\ldots,k_n\right)}\\&+(n-m+1)L_{(m)\sum_{\ell=1}^{m}k_\ell-k,k(k_1\ldots k_m)}L_{(n-m+1)p,k-\sum_{\ell=1}^{m}k_\ell(-q,k_{m+1},\ldots,k_n)}\\&-(p\leftrightarrow q)\}=0\qquad(n=0,1,2,\ldots),\end{aligned} \tag{7.5}$$

$$\begin{aligned}&\delta(p+q-q_1-\ldots-q_n)\underset{(k_1,\ldots,k_n)}{\mathrm{Sym}}\sum_{m=0}^{n}\{(m+1)(n-m+1)\\&\cdot L_{(m+1)p,-\sum_{\ell=1}^{n-m}k_{m+\ell}(-q,k_1,\ldots,k_m)}V_{(n-m+1)-\sum_{\ell=1}^{n-m}k_{m+\ell},k_{m+1},\ldots,k_n)}\\&+(n-m+2)(n-m+1)L_{(m)p,-p+\sum_{\ell=1}^{m}k_\ell(k_1,\ldots,k_m)}V_{(n-m+2)-q,-p+\sum_{\ell=1}^{m}k_\ell,k_{m+1},\ldots,k_n}\\&-(p\leftrightarrow q)\}=0\qquad(n=0,1,2,\ldots),\end{aligned} \tag{7.6}$$

$$\begin{aligned}&\delta(p+q+k-k_2-\ldots-k_n)\{f_qL_{(n)pq(-k,k_2,\ldots,k_n)}+f_pL_{(n)kp(-q,k_2,\ldots,k_n)}\\&+f_kL_{(n)qk(-p,k_2,\ldots,k_n)}K=0\qquad(n=1,2,3,\ldots),\end{aligned} \tag{7.7}$$

$$\delta(p+q-k_1-\ldots-k_n)(f_qL_{(n)pq(k_1\ldots k_n)}+f_pL_{(n)qp(k_1\ldots k_n)})=0\qquad(n=0,1,2,\ldots), \tag{7.8}$$

where $\underset{(k_1,\ldots,k_n)}{\mathrm{Sym}}$ denotes the complete symmetrization over the variables $k_1, \ldots, k_n$.

The system of algebraic functional equations (7.5)–(7.8) is the complete system of equations for the calculations of all functions $L_{(n)pq(q_1\ldots q_n)}$, which determine the recursion operator L.

Here we present the simplest examples of equations (7.5), (7.6). Equations (7.5) with $n=0$ and $n=1$ are of the form

$$(L_{(0)p+q,-p-q}-L_{(0)q,-q})L_{(1)p,-p-q(-q)}-(L_{(0)p+q,-p-q}-L_{(0)p,-p})L_{(1)q,-p-q(-p)}=0, \tag{7.9}$$

$$\begin{aligned}&2(L_{(0)p,-p}-L_{(0)-k,k})L_{(2)q,k(-p,p+q+k)}-2(L_{(0)q,-q}-L_{(0)-k,k})L_{(2)p,k(-q,p+q+k)}\\&+L_{(1)p,q+k(p+q+k)}L_{(1)q,k(q+k)}+L_{(1)p+q,k(p+q+k)}L_{(1)p,-p-q(-q)}\\&-L_{(1)q,p+k(p+q+k)}L_{(1)p,k(p+k)}-L_{(1)p+q,k(p+q+k)}L_{(1)q,-p-q(-p)}=0\,.\end{aligned} \tag{7.10}$$

Equation (7.6) for $n=0$ gives

$$L_{(0)p,-p}V_{(2)-p,p}-L_{(0)-p,p}V_{(2)p,-p}=0\,.$$

Since $V_{(2)-p,p}=V_{(2)p,-p}$, then $L_{(0)p,-p}=\varphi(p)$, where $\varphi(p)$ is an arbitrary even function on p ($\varphi(-p)=\varphi(p)$).

Further, Eq. (7.6) for $n=1$ is

$$\begin{aligned}&L_{(1)p,-q-p(-p)}V_{(2)-q-p,q+p}+3\varphi(p)V_{(3)-q,-p,p+q}\\&+L_{(1)p,q(p+q)}V_{(2)-q,q}-(p\leftrightarrow q)=0\,,\end{aligned} \tag{7.11}$$

and for $n=2$ it is of the form

$$\begin{aligned}&3L_{(1)p,-p-q(-q)}V_{(3)-q-p,k,p+q+k}+12\varphi(p)V_{(4)-q,-p,k,p+q-k}\\&+2L_{(2)p,-p-q+k(-q,k)}V_{(2)-q-p+k,q+p-k}+2L_{(2)p,-k(-q,p+q-k)}V_{(2)-k,k}\\&+3L_{(1)p,-p+k(k)}V_{(3)-q,-p+k,p+q-k}+3L_{(1)p,q-k(p+q-k)}V_{(3)-q,q-k,k}\\&+2L_{(2)p,q(k,p+q-k)}V_{(2)-q,q}-(p\leftrightarrow q)=0\,.\end{aligned} \tag{7.12}$$

Equations (7.6) allow us to calculate all $L_{(1)}, L_{(2)}, \ldots$, for given $V_{(2)}, V_{(3)}, \ldots$. Let us start with Eq. (7.11). Taking into account the closeness and skewsymmetry conditions (7.7) and (7.8), it is not difficult to show that the relation (7.11) is equivalent to the following

$$L_{(1)pq(p+q)}=3\frac{\varphi(p)-\varphi(q)}{f_q(\omega(p+q)-\omega(p)-\omega(q))}V_{(3)-p,-q,p+q}\,, \tag{7.13}$$

where $\omega(p)\overset{\text{def}}{=}V_{(2)p,-p}/f_p$. The function $\omega(p)$ determines the dispersion law for the corresponding equation, i.e. $\frac{\partial a_p}{\partial t}=\omega(p)a_p+\ldots$.

Analogously, using (7.13), we have from (7.12)

$$\begin{aligned}L_{(2)pq(k,p+q-k)}=&6\frac{\varphi(p)-\varphi(q)}{f_q(\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q))}V_{(4)-p,-q,p+q-k}\\&-\frac{9}{2}\frac{1}{f_q(\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q))}\\&\cdot\left(\frac{\varphi(p)-\varphi(q)}{f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))}V_{(3)-p,-q,p+q}V_{(3)-q-p,k,p+q-k}\right.\\&-\frac{(\varphi(p)-\varphi(p-k))V_{(3)-p,p-k,k}V_{(3)-q,-p+k,p+q-k}}{f_{k-p}(\omega(k)-\omega(p)-\omega(k-p))}\end{aligned}$$

$$+\frac{(\varphi(q)-\varphi(q-k))V_{(3)-q,q-k,k}V_{(3)-p,-q+k,p+q-k}}{f_{k-p}(\omega(k)-\omega(q)-\omega(k-q))}$$
$$-\frac{(\varphi(p)-\varphi(q-k))V_{(3)-p,-q+k,p+q-k}V_{(3)-q,q-k,k}}{f_{q-k}(\omega(p+q-k)-\omega(p)-\omega(q-k))}$$
$$+\left.\frac{(\varphi(q)-\varphi(p-k))V_{(3)-q,-p+k,p+q-k}V_{(3)-p,p-k,k}}{f_{p-k}(\omega(p+q-k)-\omega(q)-\omega(p-k))}\right). \tag{7.14}$$

In a similar manner one can derive, from (7.6), the formulae which express $L_{(3)}$ via $V_{(3)}$, $V_{(4)}$, $V_{(5)}$, the function $L_{(4)}$, via $V_{(3)}$, $V_{(4)}$, $V_{(5)}$, $V_{(6)}$ and so on.

Thus, for a given Hamiltonian, i.e. for given functions $V_{(2)}, V_{(3)}, \ldots$, and a certain fixed even function $\varphi(p)$, we easily calculate all functions $L_{(1)pq(p+q)}$, $L_{(2)pq(\ldots)}, \ldots$, which determine the operator L_{pq}. In order that this operator L_{pq} be the recursion operator it should satisfy equations (7.5), (7.7) and (7.8). The fulfillment of (7.8) is obvious. By simple but tedious calculations, one can show that the expressions for $L_{(1)pq(p+q)}$ and $L_{(2)pq(\ldots)}$, given by the formulae (7.13), (7.14), indeed satisfy Eqs. (7.9), (7.10) and the closeness conditions (7.7) for $n=1,2$.

Thus the formulae of the type (7.13), (7.14) allow one to calculate the recursion operator L_{pq} for given $V_{(2)}, V_{(3)}, \ldots$. Emphasize that all functions $V_{(2)}, V_{(3)}, \ldots$ (i.e. the Hamiltonian of the equation) are arbitrary ones. So, any dynamical system (6.1) with any Hamiltonian of the form (7.1) possesses, at least, the formal strong recursion operator L_{pq}.

The existence of a formal strong recursion operator for any Hamiltonian system of the form (6.1), (7.1) becomes obvious if one takes into account the following three circumstances. The first one is: any system of equations (7.5)–(7.8) is invariant under the general transformations of "coordinates" a_p and, therefore, the existence of the recursion operator for this system is independent of the choice of variables a_p. Secondly, any nonlinear system with a Hamiltonian of the form (7.1) can be linearized by a suitable canonical transformation [32, 33]. The third point is: any linear equation $\frac{\partial a_p}{\partial t}=\omega(p)a_p$ with an odd function $\omega(p)$ possesses the recursion operator of the form $L_{pq}=\varphi(p)\delta(p+q)$, where $\varphi(p)$ is an arbitrary even function.

Indeed, we have the dynamical system with Hamiltonian (7.1). Let us linearize this system (i.e. reduce the Hamiltonian H to the form $H=\int dp_1dp_2\delta(p_1+p_2)\tilde{V}_{(2)p_1p_2}b_{p_1}b_{p_2}$) by the canonical transformation $a_p\to b_p$:

$$a_p=b_p+\sum_{n=2}^{\infty}\int dp_2\ldots dp_n\delta(p-p_1-\ldots-p_n)R_{(n)p(p_2\ldots p_n)}b_{p_2}\ldots b_{p_n}. \tag{7.15}$$

Using the condition of the canonical character of the transformation (7.15), i.e.

$$\{a_p,a_q\}_b=\int dk\frac{\delta a_p}{\delta b_k}\frac{1}{f_k}\frac{\delta a_q}{\delta b_{-k}}=\frac{1}{f_p}\delta(p+q)=\{a_p,a_q\}_a, \tag{7.16}$$

we, in particular, have [32]

$$R_{(2)p+q(pq)}=-\frac{3}{2f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))}V_{(3)p,q,-p-q}. \tag{7.17}$$

The linear equation $\frac{\partial \mathscr{b}_p}{\partial t}=\omega(p)\mathscr{b}_p$, which appears after this canonical transformation, possesses the recursion operator $L^{\text{linear}}_{pq}=\varphi(p)\delta(p+q)$, where $\varphi(p)$ is an arbitrary even function. Let us now perform the inverse canonical transformation $\mathscr{b}_p \to a_p$ into the initial nonlinear system. The recursion operator is transformed under this transformation as follows:

$$\begin{aligned} L^{(\text{linear})}_{pq} \to L_{pq} &= \int dk_1 dk_2\, S_{-p,-k_1} L^{(\text{linear})}_{k_1 k_2} \tilde{S}_{k_2,q} \\ &= \int dk S_{-p,k}\varphi(k)\tilde{S}_{k,q}, \end{aligned} \tag{7.18}$$

where $S_{p,q} \overset{\text{def}}{=} \frac{\delta \mathscr{b}_q}{\delta a_p}$, $\tilde{S}_{pq}=\frac{\delta a_q}{\delta \mathscr{b}_p}$. In particular, formula (7.18) gives

$$L_{(1)pq(p+q)} = -2(\varphi(q)-\varphi(p))R_{(2)q(-p,p+q)}. \tag{7.19}$$

Substitution of (7.17) into (7.19) gives exactly the expression (7.13). In a similar manner one can obtain the expression for $L_{(2)pq(k,p+q-k)}$ of the form (7.14) and analogous formulae for $L_{(3)}, L_{(4)}, \ldots$. So, formulae (7.18) yields the strong recursion operator for an arbitrary initial nonlinear system with Hamiltonian (7.1).

VIII. Regular Recursion Operator

In the previous section it has been shown that any Hamiltonian equation possesses the formal strong recursion operator. However, in the general case such a recursion operator is a singular one due to the denominators of the forms

$$\omega(p+q)-\omega(p)-\omega(q), \qquad \omega(p+q-k)+\omega(k)-\omega(p)-\omega(q),$$

etc., in the expressions of the type (7.13) and (7.14). Similar denominators are contained in the expressions for higher symplectic forms and Hamiltonians which are generated by the recursion operator. For example, for the symplectic form $\Omega_{1pq} = (L\Omega_{(0)})_{pq} = L_{pq} f_{-q}$, we have

$$\begin{aligned} \Omega_{1(1)pq(p+q)} &= \frac{3\varphi(q)-\varphi(p)}{\omega(p+q)-\omega(p)-\omega(q)} V_{(3)-p,-q,p+q}, \\ \Omega_{1(2)pq(k,p+q-k)} &= \frac{6\varphi(q)-\varphi(p)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)} V_{(4)-p,-q,k,p+q-k} + \ldots . \end{aligned} \tag{8.1}$$

Using formulae (7.13) and (7.14), one can easily obtain the relations between the coefficient functions $\tilde{V}_{(3)}, \tilde{V}_{(4)}$, and $V_{(3)}, V_{(4)}$ of the pair of Hamiltonians related by the recursion operator L ($L\nabla H = \nabla\tilde{H}$). Equalizing the right-hand side of the equalities (7.13) and (7.14), taken correspondingly for $\tilde{V}_{(3)}, \tilde{V}_{(4)}$ and $V_{(3)}, V_{(4)}$ and, taking into account an obvious equality $\tilde{\omega}(p)=\varphi(p)\omega(p)$, we obtain

$$\begin{aligned} \tilde{V}_{(3)-p,-q,p+q} &= \frac{\varphi(p+q)\omega(p+q)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q)-\omega(p)-\omega(q)} V_{(3)-p,-q,p+q}, \\ \tilde{V}_{(4)-p,-q,k,p+q-k} &= \frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)} \end{aligned} \tag{8.2}$$

$$
\cdot V_{(4)-p,-q,k,p+q-k}+\frac{3}{4f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))}
$$

$$
\cdot\left(\frac{\varphi(q+p)\omega(q+p)-\varphi(q+p-k)\omega(q+p-k)-\varphi(k)\omega(k)}{\omega(q+p)-\omega(q+p-k)-\omega(k)}\right.
$$

$$
\left.-\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)}\right)
$$

$$
\cdot V_{(3)-p,-q,p+q}V_{(3)-q-p,k,p+q-k}-\frac{3}{4}\frac{\dfrac{\varphi(p)-\varphi(p-k)}{\varphi(p)-\varphi(q)}}{f_{k-p}(\omega(k)-\omega(p)-\omega(k-p))}
$$

$$
\cdot\left(\frac{\varphi(p+q-k)\omega(p+q-k)-\varphi(q)\omega(q)-\varphi(p-k)\omega(p-k)}{\omega(p+q-k)-\omega(q)-\omega(p-k)}\right.
$$

$$
\left.-\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)}\right)
$$

$$
\cdot V_{(3)-p,p-k,k}V_{(3)-q,-p+k,p+q-k}+\frac{3}{4}\frac{\dfrac{\varphi(q)-\varphi(q-k)}{\varphi(p)-\varphi(q)}}{f_{k-q}(\omega(k)-\omega(q)-\omega(k-q))}
$$

$$
\cdot\left(\frac{\varphi(p+q-k)\omega(p+q-k)-\varphi(p)\omega(q)-\varphi(q-k)\omega(q-k)}{\omega(p+q-k)-\omega(p)-\omega(q-k)}\right.
$$

$$
\left.-\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)}\right)
$$

$$
\cdot V_{(3)-q,q-k,k}V_{(3)-p,-q+k,p+q-k}-\frac{3}{4}\frac{\dfrac{\varphi(p)-\varphi(q-k)}{\varphi(p)-\varphi(q)}}{f_{q-k}(\omega(p+q-k)-\omega(p)-\omega(q-k))}
$$

$$
\cdot\left(\frac{\varphi(k)\omega(k)-\varphi(q)\omega(q)-\varphi(k-q)\omega(k-q)}{\omega(k)-\omega(q)-\omega(k-q)}\right.
$$

$$
\left.-\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)}\right)
$$

$$
\cdot V_{(3)-p,-q+k,p+q-k}V_{(3)-q,q-k,k}+\frac{3}{4}\frac{\dfrac{\varphi(q)-\varphi(p-k)}{\varphi(p)-\varphi(q)}}{f_{p-k}(\omega(p+q-k)-\omega(q)-\omega(p-k))}
$$

$$
\cdot\left(\frac{\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(k-p)\omega(k-p)}{\omega(k)-\omega(p)-\omega(k-p)}\right.
$$

$$
\left.-\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)}\right)
$$

$$
\cdot V_{(3)-q,-p+k,p+q-k}V_{(3)-p,p-k,k}\,. \tag{8.3}
$$

Recall that all functionals H_n ($L^n\nabla H=\nabla H_n$) are the integrals of motion for the initial Hamiltonian system (7.1). However, the presence of singularities (see e.g.

formulae of the type (8.2), (8.3)) in the coefficient functions $V_{(3)}$, $V_{(4)}$, ... of the higher integrals of motion H_1, H_2, ... makes all these integrals poorly defined in the general case. In order that all these H_n be well defined functionals on a_p it is necessary that the multiple of the form

$$\frac{\varphi(p+q)\omega(p+q)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q)-\omega(p)-\omega(q)}, \tag{8.4a}$$

$$\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)} \tag{8.4b}$$

would not contain the nonintegrable singularities.

We will refer to the recursion operator, which generates the family of well defined Hamiltonians H_n from the well defined initial Hamiltonian H, as the nonsingular recursion operator. It is clear that the demand of nonsingularity of the recursion operator leads to certain restrictions on the form of the functions $\omega(p)$, $V_{(3)}$, $V_{(4)}$, ... and allows only some subclass of equations from all equations of the form (7.1).

The stronger restriction on the Hamiltonian (i.e. on the functions $V_{(2)}$, $V_{(3)}$, $V_{(4)}$, ...) appears if one demands that all functions $\tilde{V}_{(3)}$, $\tilde{V}_{(4)}$, $\tilde{V}_{(5)}$, ..., would not have singularities at all. We will refer to the recursion operator which produces such a family of Hamiltonians as a regular recursion operator.

As we shall see, the properties of recursion operators crucially depends on the dimensionality d of the space. Let us consider subsequently the cases $d=1$, $d=2$, $d\geqq 3$.

In the one-dimensional space the multiple (8.4) is the simplest one. For $d=1$ we have $\omega(p)=\sum_{n=1}^{\infty}\alpha_n p^{2n+1}$ and $\varphi(p)=\sum_{n=0}^{\infty}\beta_n p^{2n}$. It is not difficult to show that for any $\omega(p)$ and $\varphi(p)$ the expressions (8.4) are polynomials on p, q and p, q, k. For example, for $\omega=p^3$, $\varphi=p^2$ one has

$$\begin{aligned} &\frac{\varphi(p+q)\omega(p+q)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q)-\omega(p)-\omega(q)} \\ &\quad=\frac{5pq(p+q)(p^2+pq+q^2)}{3pq(p+q)}=\frac{5}{3}(p^2+pq+q^2), \\ &\frac{\varphi(p+q-k)\omega(p+q-k)+\varphi(k)\omega(k)-\varphi(p)\omega(p)-\varphi(q)\omega(q)}{\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q)} \\ &\quad=\frac{5(p+q)(q-k)(p-k)(p^2+q^2+k^2-pk+pq-qk)}{3(p+q)(q-k)(p-k)} \\ &\quad=\frac{5}{3}(p^2+q^2+k^2-pk+pq-qk). \end{aligned} \tag{8.5}$$

Thus for any $\omega(p)$ and $\varphi(p)$ the function $\tilde{V}_{(3)-p,-q,p+q}$ has no singularities.

The expressions in the round brackets in (8.4) have no singularities, too. Let us choose $f_p=\text{const}\,p^{-2\gamma-1}(\gamma>0)$. It is not difficult to see that by virtue of the multiplers in front of the round brackets in (8.4) the expression for

$\tilde{V}_{(4)-p,-q,k,p+q-k}$ has poles at $p+q=0$, $p-q=0$, $p=0$ and $q=0$. However, it is easy to check that the residues of $\tilde{V}_{(4)-p,-q,k,p+q-k}$ in these poles are equal to zero independently of the form of $V_{(3)}\dots$.

So the functions $\tilde{V}_{(3)}$, $\tilde{V}_{(4)}$ have no singularities for any $\omega(p)$ and $\varphi(p)$. One can, however, show that the function $\tilde{V}_{(5)}$ has no such property. Analogously $\tilde{V}_{(6)}$, $\tilde{V}_{(7)}$, ... have singularities for general $V_{(3)}$, $V_{(4)}$, ... The requirement for the absence of singularities in the expressions for all the functions $\tilde{V}_{(5)}$, $\tilde{V}_{(6)}$, ..., leads to a certain system of equations for $V_{(3)}$, $V_{(4)}$, $V_{(5)}$, ... If these equations are satisfied, then the considered system possesses the regular recursion operator. By virtue of the cumbersome form of these equations, we omit them here.

In the general case all the functions $L_{(1)}$, $L_{(2)}$, $L_{(3)}$, ..., which determine the recursion operator, are not equal to zero, i.e. the recursion operator is the entire functional on a. However, it may occur that this infinite series is interrupted on some (N^{th}) term. We will refer to such a recursion operator as the N-linear recursion operator. It is clear that the requirement of the N-linearity of the recursion operator, i.e. the requirement $L_{(N+1)}=L_{(N+2)}=\dots=0$ leads to strong restrictions on the form of the functions $V_{(2)}$, $V_{(3)}$, $V_{(4)}$, ... For example, in order that the recursion operator L be linear on a (i.e. $L_{(2)}=L_{(3)}=\dots=0$) it is necessary that the right-hand side of (7.14) should be equal to zero, i.e.

$$\begin{aligned}
&\frac{3}{4}(\varphi(p)-\varphi(q))V_{(4)-p,-q,k,p+q-k}\\
&\quad-\frac{\varphi(p)-\varphi(q)}{f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))}V_{(3)-p,-q,p+q}V_{(3)-q-p,k,p+q-k}\\
&\quad+\frac{1}{f_{p-k}}\left(\frac{\varphi(p)-\varphi(p-k)}{\omega(k)-\omega(p)-\omega(k-p)}\right.\\
&\quad\left.+\frac{\varphi(q)-\varphi(p-k)}{\omega(p+q-k)-\omega(q)-\omega(p-k)}\right)V_{(3)-p,p-k,k}V_{(3)-q,-p+k,p+q-k}\\
&\quad-\frac{1}{f_{q-k}}\left(\frac{\varphi(q)-\varphi(q-k)}{\omega(k)-\omega(q)-\omega(k-q)}\right.\\
&\quad\left.+\frac{\varphi(p)-\varphi(q-k)}{\omega(p+q-k)-\omega(p)-\omega(q-k)}\right)V_{(3)-q,q-k,k}V_{(3)-p,-q+k,p+q-k}=0\,. \qquad (8.6)
\end{aligned}$$

In addition to (8.6), the conditions on $V_{(2)}$, $V_{(3)}$, $V_{(4)}$, ..., which mean that $L_{(3)}=L_{(4)}=\dots=0$ should satisfied.

In the case when Hamiltonian H is cubic on a, i.e. when $V_{(4)}=V_{(5)}=\dots=0$, the whole system of equations is reduced to the only equation (8.6) with $V_{(4)}=0$. Let us consider this equation in the simplest cas $f_q=-\frac{i}{p}$, $\varphi(p)=\alpha+\beta p^2$, $\omega(p)=\text{const}\,p^3$. After the trivial transformations Eq. (8.6) reduces to the following:

$$\begin{aligned}
&(q-p)V_{(3)-p,-q,p+q}V_{(3)-q-p,k,p+q-k}\\
&\quad+(k-q)V_{(3)-p,p-k,k}V_{(3)-q,-p+k,p+q-k}\\
&\quad+(p-k)V_{(3)-q,q-k,k}V_{(3)-p,-q+k,p+q-k}=0\,. \qquad (8.7)
\end{aligned}$$

It is not difficult to verify that Eq. (8.7) has the only solution $V_{(3)} = \text{const}$ within the class of polynomials $V_{(3)}\ldots$, i.e. within the class of local Hamiltonians H. Indeed, putting $p=k=0$ in (8.7), one gets $q(V_{(3)0,-q,q} - V_{(3)0,0,0})V_{(3)-q,0,q} = 0$. Therefore $V_{(3)0,-q,q} = V_{(3)0,0,0} = \text{const}$.

Thus, among the nonlinear equations with three linear local Hamiltonians and $\omega(p) \cong p^3$, only the equation with $V_{(3)} = \frac{\gamma}{3} = \text{const}$, i.e. the KdV equation, possesses the linear recursion operator. It is of the form

$$L_{pq} = (\alpha + \beta p^2)\delta(p+q) + \gamma \frac{p-q}{p} a_{p+q}, \tag{8.8}$$

where α, β, γ are arbitrary constants. For $\alpha=0$, $\beta=-1$ and $\gamma=-2$, the operator (8.8) is the recursion operator (1.2) in the momentum representation.

Another example is the bilinear recursion operator which corresponds to the equation with $\omega(p) = -ip^3$, $V_{(4)} = \frac{\gamma}{6} = \text{const}$, and $V_{(5)} = V_{(6)} = \ldots = 0$. In this case,

$$\begin{aligned} L_{(0)pq} &= (\alpha + \beta p^2)\delta(p+q), \\ L_{(2)pq(k,p+q-k)} &= \gamma \frac{(p-q)q}{pk - k(p+q-k)}, \\ L_{(1)} &= L_{(3)} = L_{(4)} = \ldots = 0, \end{aligned} \tag{8.9}$$

i.e.

$$L_{pq} = (\alpha + \beta p^2)\delta(p+q) + \gamma \sum_{-\infty}^{+\infty} dk \frac{q}{p-k} a_k a_{p+q-k},$$

where α, β, γ are arbitrary constants. For $\alpha=0$, $\beta=-1$, $\gamma=-1$, the operator (8.9) is the recursion operator for the modified KdV equation (see [11]) in momentum representation. Note that the operators (8.8) and (8.9) in coordinate representation have been calculated by another technique in [2.8].

Thus, in the one-dimensional space there exist Hamiltonian equations which possess the regular recursion operator. For a certain subclass of these equations the recursion operator is the polynomial of the finite order on the field a.

The situation changes dramatically when we transit to the two-dimensional space. It is connected with the circumstance that the expressions of the form (8.4) have no singularities except for certain special functions $\omega(p)$ and $\varphi(p)$. Let us first consider the expression (8.4a). For the absence of singularities in such expressions it is necessary that the numerator be equal to zero on the same manifold $\Gamma^{1,2}$ as the denominator. This means that the dispersion law $\omega(p)$ should be a degenerative one with respect to the decay process $1 \to 2+3$ (for the degenerative dispersion laws see [34, 35]), and the dispersion law $\varphi(p)\omega(p)$ should belong to the class of degenerative dispersion laws associated with $\omega(p)$. The second condition can be easily fulfilled if for the degenerative dispersion law $\omega(p)$ one chooses $\varphi(p) = \frac{\tilde{\omega}(p)}{\omega(p)}$, where $\tilde{\omega}(p)$ is any dispersion law associated with given dispersion law $\omega(p)$. A wide class of degenerative dispersion laws has been described in [34, 35].

In the description of the dispersion laws the dimension of the manifold $\Gamma^{n,m}$ which is defined by the equations $p_1 + \ldots + p_n = p_{n+1} + \ldots + p_{n+m}$, $\omega(p_1)$

$+\ldots+\omega(p_n)=\omega(p_{n+1})+\ldots+\omega(p_{n+m})$ plays an important role. If $\dim\Gamma^{1,2}$ and $\dim\Gamma^{2,2}$ are less than a maximal one (i.e. $\dim\Gamma^{1,2}<2d-1$, $\dim\Gamma^{2,2}<3d-1$), then the expressions $(\omega(p+q)-\omega(p)-\omega(q))^{-1}$ and $(\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q))^{-1}$ may have integrable singularities. For such $\omega(p)$ (with $\dim\Gamma^{n,m}<\max\dim\Gamma^{n,m}$) the nonsingular recursion operator may exist.

If $\dim\Gamma^{n,m}$ is a maximal one, then the corresponding nonlinear equation does not possess the nonsingular recursion operator. Indeed in this case, by virtue of the theorem proved in [35], the expressions of the form (8.4b) have nonintegrable singularities. The multiplers of the form

$$\frac{\varphi(p)-\varphi(p-k)}{\varphi(p)-\varphi(q)}(f_{k-p}(\omega(k)-\omega(p)-\omega(k-p)))^{-1}$$

in front of the round brackets in (8.4) have the nonintegrable singularities too. Moreover, the nonintegrable singularity is contained in the symplectic form $\Omega_{1pq(p+q)}$ (see Formula 8.1).

Thus, the nonlinear equations in two-dimensional space, which describe the nontrivial scattering of n waves into m waves ($n\neq m$) with $\max\dim\Gamma^{n,m}$, does not possess the nonsingular recursion operator. In particular, the well-known Kadomtsev–Petviashvili equation [1] for which $\dim\Gamma^{1,2}=3$ and $V_{(3)}=\text{const}$ has no nonsingular recursion operator.

An analogous situation takes place for three and higher-dimensional spaces ($d\geqq 3$). Since for $d\geqq 3$ there exist no degenerative dispersion laws with $\max\dim\Gamma^{n,m}$ [35], the nonlinear equations with $\max\dim\Gamma^{n,m}$ do not possess nonsingular recursion operators. Only the equations with $\dim\Gamma^{n,m}<\max\dim\Gamma^{n,m}$ may have the nonsingular recursion operator.

So we see that the regular recursion operator is a pure one-dimensional phenomenon. The proposed method of expansion over the fields (i.e. the perturbation theory method) seems to be adequate for an analysis of the problem of existence or nonexistence of the nonsingular recursion operator in the multidimensional spaces. All the results of Sects. 7 and 8 can be generalized to the case of the systems of equations (6.1).

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