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HAMILTONIAN APPROACH TO THE DESCRIPTION OF NON-LINEAR PLASMA PHENOMENA

V.E. ZAKHAROV, S.L. MUSHER, A.M. RUBENCHIK

Institute of Automation and Electrometry, Siberian Branch of the Academy of Sciences, 630090, Novosibirsk, U.S.S.R.

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Contents:

Abstract:

This review discusses the methods of description of non-linear processes. We demonstrate the usefulness of the Hamiltonian approach to these problems. We show the existence of Hamiltonian structures for a number of plasma situations.

The choice of normal variables results in a standard form of equations for all kinds of problems. The actual physics involved changes only dispersion laws and the structure of the matrix elements. This approach makes it possible to consider a number of problems in a unique way. We discuss the stability of monochromatic waves and the statistical description of a plasma. The connection between decay and modulational instability growth rates and matrix elements is demonstrated.

The standard form of the equations enables us to introduce a statistical description in a very simple way. We discuss the usual kinetic wave equations and their generalization for inhomogeneous turbulence and turbulence excited by a coherent pump.

We pay special attention to the problem of Langmuir turbulence. The average dynamical equations are deduced in a consistent way and we present a detailed discussion of the limits of this description.

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Institute of Automation and Electrometry, Siberian Branch of the Academy of Sciences, 630090, Novosibirsk, U.S.S.R.

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Introduction

The efficiency of solving some problem in theoretical physics depends on how far in a proper manner the descriptive formalism is chosen in the framework in which this problem is solved. Usually there are several calculation schemes available, which by a consistent application will lead to the same result. Theoreticians biased in favour of one of them, intuitively, resists all attempts aiming to explore another scheme claiming that they do not contribute anything new. Nevertheless, not all possible schemes can be treated equally. In medieval times in the universities of Europe there coexisted numerous algorithms for arithmetic division but all of them except a single one are nowadays of historical interest only.

And the reason is not only because of the maximum convenience of the "best" calculation scheme. The chosen scheme being adequate for the problem under solution after a period of implementation and adaptation begins to affect itself the style of physical thinking and enriches essentially the scientific language. Finally, it has effects on the way new physical problems are stated. This happened, for instance, with the Feynman diagram technique which originally seemed to be merely a simplification method in perturbation theory. Besides, a calculation scheme adequate for a particular problem usually possesses an important universality, which helps to find similarities between various physical problems. In a number of cases such a scheme is of independent interest from the viewpoint of mathematics.

The present review deals with methods for the description of non-linear phenomena in plasma. Non-linear phenomena such as the processes of modulational and decay instabilities, self-focusing, collapse, and various turbulent processes play a crucial role in contemporary plasma physics. Therefore the problem of their adequate description is of great importance.

According to our opinion a Hamiltonian formalism is the best method of description. The majority of non-linear phenomena in plasmas such as wave coupling, self-focusing, or collapse occur with conservation of the total energy and allow a Hamiltonian description. Dissipative effects can be taken into account as small corrections. This approach is being developed systematically since the late sixties by a group of Soviet physicists formed and originally working in the Novosibirsk Institute of Nuclear Physics.

The Hamiltonian approach is based upon the fundamental fact that the equations describing a collisionless plasma possess a hidden Hamiltonian structure. At the present time this fact is proved even for the Maxwell–Vlasov equations [12]. The Hamiltonian property of the Maxwell equations and the equations of two-fluid hydrodynamics has been proved by one of us in 1971 [6]. It permits simple and effective calculations of matrix elements of different mode interactions by means of which we can obtain the equations in which only the necessary degrees of freedom occur. These equations can be simplified to a few standard forms within very general assumptions. As a result, we have a set of standard equations with a high degree of universality. It is easy to understand the domain of validity of these equations and to take into account the necessary dissipative effects. (We then lose the Hamiltonian structure of the equations, but they are still relatively simple enough.) The program, formulated above, constitutes the contents of the first chapter of this review.

The second chapter deals with wave instabilities in plasmas. We hope to demonstrate here the efficiency and convenience of the Hamiltonian approach. Due to the standard form of the equations it is possible to consider from the common point of view the decay and modulational plasma wave instabilities. The transition to canonical variables is not trivial. But we need to overcome these difficulties only once. Then application of these results decreases drastically the number of calculations and the problem becomes much clearer. For example, all characteristics of processes under consideration in this article (instability growth rates, their structure, etc.) are determined by the matrix elements, calculated in the first chapter.

The transition to a statistical description of plasma turbulence is the subject of chapter 3. We demonstrate that this transition is simple and natural in canonical variables. It is possible to obtain not only the usual kinetic equations. The non-trivial generalizations for inhomogeneous turbulence and for turbulence excited by intensive coherent radiation are presented.

We demonstrate the usefulness of the Hamiltonian approach mainly by the example of isotropic Langmuir turbulence. As a result, magnetized plasmas are discussed without any details in our review. This does not imply the existence of any difficulties for wide applications of this tecnhique to the problems of drift or whistler turbulence. A solution of these problems is a matter for future research.

Our group has been using the Hamiltonian approach to plasma physics problems since the late sixties. We developed this approach mainly for pragmatic goals, as a method of solving some concrete physical problems. The crucial point of our theory is the introduction of canonical variables and an investigation of standard equations. However, the Hamiltonian structure of the hydrodynamical types of equations is of a strong independent interest. A flood of papers on this subject has been published in recent years (see, e.g., refs. [26 to 29] and references therein). The main purpose of these papers is the calculation of Poisson brackets for various physical quantities. The Poisson brackets for components of velocities, electrical and magnetic fields, distribution functions at different points and velocities were calculated. It was proved that the Poisson brackets have, practically in every case, a group theoretical origin. They are the Lie–Kostant–Kirillov type brackets on skew-adjoint representations of certain Lie groups. This outstanding mathematical fact is not very useful for physics. After calculating the Poisson brackets the introduction of the canonical variables, which are necessary for effective exploring of perturbation theory, is a non-trivial problem. On the other hand, it is not difficult to calculate Poisson brackets for any quantities with the help of the canonical variables. Details can be found in ref. [29].

1. Methods of describing non-linear phenomena in plasmas

In the physically most important situations a plasma is described by a system of kinetic equations for all kinds of particles and by Maxwell's equations. Within the framework of this description it is possible to find the plasma equilibrium in the fields of a given configuration and to study small oscillations about that equilibrium [1, 2]. However, this problem has never been trivial. Even in a homogeneous plasma placed in a constant magnetic field there are seven oscillation branches (not counting higher Bernstein modes) whose dispersion laws depend in a complicated way on the magnetic field, density, temperature, direction of propagation, and dissipation or growth rate, and also in the structure of the distribution functions.

The problem of non-linear interactions of these waves is further complicated, even if this interaction is assumed to be weak. Successive use of a kinetic description leads to cumbersome formulae which get even further complicated by attempts to describe strongly non-linear phenomena. It is some improvement to use a two-fluid hydrodynamical approach. A change to a hydrodynamical description, however, does not result in significant simplifications, although in so doing information concerning some important physical effects is lost.

The cause of these difficulties is that both kinetic and hydrodynamical approaches are universal and automatically describe all degrees of freedom of a plasma. However, in the given examples only waves of one or two types are generally excited. Therefore it is necessary to construct methods of describing a plasma which allow the needed degrees of freedom to be explicitly separated in a specific instance and their interaction to be determined in the simplest way. Certainly, the plasma models constructed in such a manner are not universal; however, this is compensated for by their adaptability and simplicity.

The first attempt to separate explicitly the plasma degrees of freedom was made at the beginning of the sixties, when the kinetic equations for the waves describing a weak plasma turbulence first appeared [3, 4, 5]. However, these equations are averaged and have a statistical character; the derivation contains the random phase approximation for interacting waves. In many cases this hypothesis is groundless. As will be shown in this review, the applicability of the weak-turbulence approach should be carefully checked in all cases; a variety of very important phenomena, such as collapse and self-focusing, generally are not described by a weak-turbulence theory.

Therefore the separation of degrees of freedom should be made before averaging at the level of a dynamical plasma description. A separation of this kind is widely used in theoretical physics. In solid state physics or in the theory of elementary particles it is common to start from an effective Hamiltonian system when only the needed degrees of freedom are taken into account. However, in plasma physics the Hamiltonian approach has still not gained wide popularity. Specialists in solid state physics have to deal with a Hamiltonian from the very beginning. In the same way, a plasma physicist begins with a system of kinetic or hydrodynamical equations, and the possibility of writing this as a system of Hamiltonian equations is not evident.

Hamiltonian variables have recently been found for a number of plasma physics problems. We hope to demonstrate this in the present review. It is convenient to choose for a description of Hamiltonian systems canonical variables, the classical analogue of quantum-mechanical Bose operators. The linear problem is trivial in these variables. Thus, for non-linear problems it is not necessary to solve the linear problem, which is often very complicated, many times. As a result, we avoid many technical difficulties, which are not inherent to our problem.

The selection of standard variables for all kinds of wave motion results in a uniformity of non-linear equations. The information about a concrete physical system is contained in the dispersion law and in the form of the interaction matrix elements.

The separation of the only important type of motion can lead to reasonable simplifications of the dynamical equations. As a result of their uniformity the simplified equations have a large physical generality. As an example, we can mention the non-linear parabolic equation, describing quasi-monochromatic wave propagation in isotropic media. Some other not so well known examples will be discussed below.

The development of a description of Langmuir turbulence takes a significant part of this chapter. The fastest processes in a plasma appear to be the Langmuir oscillations. As a result it is possible to simplify the dynamical equations to average them over a fast time period ω_p^{-1}. For a description of the turbulence it is necessary to take into account kinetic effects (interaction waves with ions and electrons).

It breaks the Hamiltonian structure of the averaged equations. But they are still very effective and powerful tools for investigating the turbulence. The Langmuir collapse analysis is the brightest example of it.

1.1. *Hamiltonian formalism in non-linear media*

The Hamiltonian formalism for continuous media, including plasmas, represents a natural generalization of a standard Hamiltonian formalism in classical mechanics. Some peculiarities arise due to the existence of a translation symmetry in homogeneous media. Let the medium be described by a single pair of field variables, i.e. the canonical coordinate $\boldsymbol{q}(\boldsymbol{r}, t)$ and the canonical momentum $\boldsymbol{p}(\boldsymbol{r}, t)$ obeying the canonical equations

$$\partial \boldsymbol{p}/\partial t = -\delta \mathcal{H}/\delta \boldsymbol{q}\,; \qquad \partial \boldsymbol{q}/\partial t = \delta \mathcal{H}/\delta \boldsymbol{p}\,. \tag{1.1.1}$$

Here $\mathcal{H}$ is the Hamiltonian of the system, a real functional of $\boldsymbol{p}$ and $\boldsymbol{q}$. Expand $\mathcal{H}$ in the variables $\boldsymbol{p}$ and $\boldsymbol{q}$,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \cdots .$$

The first expansion term $\mathcal{H}_0$ can always be made quadratic in $\boldsymbol{p}$ and $\boldsymbol{q}$ and is of the form

$$\mathcal{H}_0 = \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \{\tfrac{1}{2}A(\boldsymbol{r}-\boldsymbol{r}')\,\boldsymbol{p}(\boldsymbol{r})\,\boldsymbol{p}(\boldsymbol{r}') + B(\boldsymbol{r}-\boldsymbol{r}')\,\boldsymbol{p}(\boldsymbol{r})\,\boldsymbol{q}(\boldsymbol{r}') + \tfrac{1}{2}C(\boldsymbol{r}-\boldsymbol{r}')\,\boldsymbol{q}(\boldsymbol{r})\,\boldsymbol{q}(\boldsymbol{r}')\} . \tag{1.1.2}$$

The fact that the functions A, B and C in formula (1.1.2) depend on $\boldsymbol{r}-\boldsymbol{r}'$ reflects the translational invariance of the medium. The structure functions $A(\boldsymbol{x}) = A(-\boldsymbol{x})$; $C(\boldsymbol{x}) = C(-\boldsymbol{x})$ and $B(\boldsymbol{x}) = B(-\boldsymbol{x})$ describe the properties of this medium. Now let us make the Fourier transform

$$\boldsymbol{p}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int \boldsymbol{p}_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{q}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int \boldsymbol{q}_k\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}\, \mathrm{d}\boldsymbol{k} . \tag{1.1.3}$$

Due to the reality of $\boldsymbol{p}$, $\boldsymbol{q}$ we have

$$\boldsymbol{p}(-\boldsymbol{k}) = \boldsymbol{p}^*(\boldsymbol{k}); \qquad \boldsymbol{q}(-\boldsymbol{k}) = \boldsymbol{q}^*(\boldsymbol{k}) . \tag{1.1.4}$$

Substituting (1.1.3) into $\mathcal{H}_0$, we obtain

$$\mathcal{H}_0 = \int \{\tfrac{1}{2}A_k \boldsymbol{p}_k \boldsymbol{p}^*_k + B_k \boldsymbol{p}_k \boldsymbol{q}_k + \tfrac{1}{2}C_k \boldsymbol{q}_k \boldsymbol{q}^*_k\}\, \mathrm{d}\boldsymbol{k}$$

where $A_k = \int A(\boldsymbol{x}) \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x})\, \mathrm{d}x$; B_k and C_k are determined in a similar manner. It is not difficult to confirm that the Fourier transform (1.1.3) is canonical; and $\boldsymbol{p}_k$, $\boldsymbol{q}^*_k$ are the new pair of canonical variables, so that

$$\partial \boldsymbol{p}_k / \partial t = -\delta \mathcal{H} / \delta \boldsymbol{q}^*_k ; \qquad \partial \boldsymbol{q}_k / \partial t = \delta \mathcal{H} / \delta \boldsymbol{p}^*_k . \tag{1.1.5}$$

When $\mathcal{H} = \mathcal{H}_0$, we have

$$\partial \boldsymbol{q}_k / \partial t = A_k \boldsymbol{p}_k + B^*_k \boldsymbol{q}_k$$

$$\partial \boldsymbol{p}_k / \partial t = -B_k \boldsymbol{p}_k - C_k \boldsymbol{q}_k . \tag{1.1.6}$$

From the evenness of $A(\xi)$, $C(\xi)$ there follows the reality of A_k, C_k

$$A_{-k} = A_k = A^*_k ; \qquad C_{-k} = C_k = C^*_k .$$

Splitting B_k into its real and imaginary parts,

$$\begin{aligned} B_k &= B_1(k) + \mathrm{i}\, B_2(k) \\ B_1(-k) &= B_1(k) \\ B_2(-k) &= B_2(k) \end{aligned} \tag{1.1.7}$$

and assuming that p_k, $q_k \propto \exp(-\mathrm{i}\omega_k t)$, we get

$$\omega_k = B_2(k) \pm \sqrt{A_k C_k - B_1^2(k)}\,. \tag{1.1.8}$$

The medium is stable with respect to small perturbations, if

$$A_k C_k - B_1^2(k) > 0\,. \tag{1.1.9}$$

It is necessary that in this case the functions A_k and C_k have the same sign.

For ω_k we have two expressions, $\omega_{1,2}(k)$, differing in the sign in (1.1.8). Now let us show that only one of them has a physical sense. For this purpose the following substitution is made

$$\begin{aligned} p_k &= \alpha_k a_k + \alpha_{-k}^* a_{-k}^* \\ q_k &= \beta_k a_k + \beta_{-k}^* a_{-k}^* \end{aligned} \tag{1.1.10}$$

and it is required that this represents a canonical transformation, or, strictly speaking, it is required that the variable a_k obeys the equation

$$\dot{a}_k = -\mathrm{i}\, \delta\mathscr{H}/\delta a_k^*\,. \tag{1.1.11}$$

Substituting (1.1.10) into (1.1.5) and using (1.1.11), we will find the condition for α_k and β_k:

$$\begin{aligned} &|\alpha_k|^2 = |\alpha_{-k}|^2; \qquad |\beta_k|^2 = |\beta_{-k}|^2 \\ &\alpha_k \beta_k^* - \alpha_{-k}^* \beta_k = -\mathrm{i}\,. \end{aligned} \tag{1.1.12}$$

Now let us demand that a_k be a normal variable, i.e., that its time variation is according to the law $a_k \propto \exp(-\mathrm{i}\omega_k t)$ (where ω_k is one of the $\omega_{1,2k}$, it being unknown which of them). Substituting (1.1.10) into (1.1.6), we get

$$\beta_k = \frac{-A_k}{\mathrm{i}\omega_k + B_k^*}\,\alpha_k\,. \tag{1.1.13}$$

Substituting now (1.1.13) into (1.1.12) we will find that in a steady medium

$$|\alpha_k|^2 = \frac{C_k}{2\sqrt{A_k C_k - B_1^2(k)}}\,. \tag{1.1.14}$$

Thus, the sign in front of the radical in (1.1.8) must coincide with the sign of C_k (or A_k since for the steady medium $A_k C_k > 0$). Finally we obtain that

$$\omega_k = B_2(k) + \operatorname{sign} C_k \sqrt{A_k C_k - B_1^2(k)}\,. \tag{1.1.15}$$

In this case for the Hamiltonian $\mathcal{H}_0$ we have

$$\mathcal{H}_0 = \int \omega_k a_k a_k^* \,\mathrm{d}k\,. \tag{1.1.16}$$

In the majority of cases the Hamiltonian coincides in its physical sense with the wave energy in the medium; thus the formula (1.1.15) actually defines the energy wave sign. When $\omega_k > 0$, the waves have positive energy, and when $\omega_k < 0$ they have negative energy.

If the medium is invariant with respect to coordinate reflections, the condition

$$\omega_k = \omega_{-k} \tag{1.1.17}$$

should be fulfilled. This is possible only in the case when $B_2(k) \equiv 0$. The condition (1.1.17) is true for a plasma, including the case when it is contained in a magnetic field, if its distribution function is an even velocity function,

$$f(v) = f(-v)$$

but it is not true in a plasma with a current or in the presence of ion or electron beams.

When $B_2 \neq 0$ in a stable medium the coexistence of waves with both positive and negative energy is possible. When $B_2 = 0$ this is possible, as a rule, if there is an instability region $A_k C_k - B_1^2(k) < 0$ in the medium. The only exception is the case when the surfaces where the functions A_k, B_k and C_k vanish coincide with those where the frequency ω_k is equal to zero. It is evident that such a function is unstable with respect to a small perturbation of the properties of the medium. With the exception of this degenerate situation, it can be stated that in a stable medium which is invariant with respect to coordinate reflection the wave energy has the same sign in the whole of k-space.

The formula (1.1.14) defines the modulus of α_k only. It means that the phase of α_k can be chosen arbitrarily; in this case the canonical variables a_k will be determined apart from a trivial transformation. Without restricting the generality, we can put

$$\alpha_k = \left[\frac{C_k}{2\sqrt{A_k C_k - B_k^2}}\right]^{1/2}. \tag{1.1.18}$$

The next term in the expansion of the Hamiltonian $\mathcal{H}$ in powers of the a_k is cubic in the a_k and takes the form

$$\mathcal{H}^{(1)} = \tfrac{1}{2}\int \{V_{kk_1k_2} a_k^* a_{k_1} a_{k_2} + V_{kk_1k_2}^* a_k a_{k_1}^* a_{k_2}^*\}\, \delta(k - k_1 - k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2$$

$$+ \tfrac{1}{3}\int (U_{kk_1k_2} a_k a_{k_1} a_{k_2} + U_{kk_1k_2}^* a_k^* a_{k_1}^* a_{k_2}^*)\, \delta(k + k_1 + k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\,. \tag{1.1.19}$$

The coefficients $V_{kk_1k_2}$ and $U_{kk_1k_2}$ exhibit an obvious symmetry

$$V_{kk_1k_2} = V_{kk_2k_1}$$
$$U_{kk_1k_2} = U_{k_1kk_2} = U_{kk_2k_1}\,. \tag{1.1.20}$$

$\mathcal{H}^{(2)}$ contains terms of fourth order in the a_k. As a rule, only one of them is significant:

$$\mathcal{H}^{(2)} = \tfrac{1}{2}\int T_{kk_1k_2k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3}\ \delta(k + k_1 - k_2 - k_3)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,. \tag{1.1.21}$$

The coefficient $T_{kk_1k_2k_3}$ exhibits the symmetry

$$T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{kk_1k_3k_2} = T^*_{k_2k_3kk_1}\,. \tag{1.1.22}$$

If the medium is described by N pairs of canonical variables, several types of waves (not more than N) can propagate within it. Instead of the coefficients A_k, B_k, C_k there arise the matrices A_{ij}, B_{ij}, C_{ij}; in this case the problem of the diagonalization of the quadratic Hamiltonian becomes more complicated. As will be shown below, it is easily solved for a plasma.

1.2. Hamiltonian formalism in hydrodynamics

Hydrodynamics of an ideal barotropic fluid is a simple, but non-trivial example of a model of a continuous medium admitting a Hamiltonian description. We will consider even a more general model, assuming that the pressure non-locally depends on the density and requiring the following condition to be met:

$$\frac{1}{\rho}\nabla p = \nabla\frac{\delta\mathscr{E}}{\delta\rho} \tag{1.2.1}$$

where $\mathscr{E}$ is some functional of the density. For a barotropic fluid, when $p = p(\rho)$, $\mathscr{E} = \int \mathscr{E}(\rho)\,\mathrm{d}r$, so that

$$\frac{\partial^2\mathscr{E}}{\partial\rho^2} = \frac{1}{\rho}\frac{\partial p}{\partial\rho}\,. \tag{1.2.2}$$

The quantity $\mathscr{E}$ has the meaning of the internal energy density of the fluid.

The hydrodynamical equations take the form

$$\partial\rho/\partial t + \operatorname{div}(\rho v) = 0 \tag{1.2.3}$$

$$\partial v/\partial t + (v\nabla)v = -\nabla\delta\mathscr{E}/\delta\rho\,. \tag{1.2.4}$$

The equation of motion for the vorticity follows from (1.2.4):

$$\frac{\partial}{\partial t}\operatorname{curl} v - \operatorname{curl}[v \operatorname{curl} v] = 0\,. \tag{1.2.5}$$

The vorticity is transported along with the fluid. Now let us consider the function $\mu(r, t)$ satisfying the equation

$$(\nabla\mu \cdot \operatorname{curl} v) = 0 . \tag{1.2.6}$$

In this case the curl of the velocity is tangential to the μ = constant surface; hence the μ = constant surface is woven of vorticity threads. This suggests that the μ = constant surface is transported together with the fluid. Thus, the function μ satisfies the equation

$$\partial\mu/\partial t + (v\nabla)\mu = 0 . \tag{1.2.7}$$

For constructing canonical variables, let us take an advantageous formal approach suggested by B.I. Davydov in 1949 [7]. Consider the continuity equation (1.2.3) and the equation of curl v transfer (1.2.7) as additional conditions imposed on the dynamics of a particle system with potential energy $\mathscr{E}$. Then the action for the hydrodynamical equations can be written down as follows:

$$\mathscr{S}_a = \int \left\{ \rho \frac{v^2}{2} + \phi(\rho_t + \operatorname{div} \rho v) - \lambda(\mu_t + (v\nabla)\mu) \right\} \mathrm{d}r\, \mathrm{d}t - \int \mathscr{E}\, \mathrm{d}t . \tag{1.2.8}$$

Here ϕ and λ are Lagrange multipliers corresponding to the conditions (1.2.3) and (1.2.7). Assuming that $\delta\mathscr{S}_a/\delta v = 0$, we find

$$v = \frac{\lambda}{\rho} \nabla\mu + \nabla\phi , \tag{1.2.9}$$

where λ and μ are the well-known Clebsch variables (see, e.g., ref. [30]). From the conditions $\delta\mathscr{S}_a/\delta\mu = 0$ and $\delta\mathscr{S}_a/\delta\rho = 0$, it follows that

$$\frac{\partial\lambda}{\partial t} + \operatorname{div}(\lambda v) = 0 \tag{1.2.10}$$

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2 - \frac{\lambda}{\rho}(v\nabla)\mu + \frac{\delta\mathscr{E}}{\delta\rho} = 0 . \tag{1.2.11}$$

It is not difficult to check that the equations (1.2.3), (1.2.7), (1.2.10) and (1.2.11) are the Hamiltonian equations

$$\partial\lambda/\partial t = \delta\mathscr{H}/\delta\mu ; \qquad \partial\mu/\partial t = -\delta\mathscr{H}/\delta\lambda ; \qquad \partial\rho/\partial t = \delta\mathscr{H}/\delta\phi ; \qquad \partial\phi/\partial t = -\delta\mathscr{H}/\delta\rho \tag{1.2.12}$$

where $\mathscr{H} = \int \frac{1}{2}\rho v^2\, \mathrm{d}r + \mathscr{E}$ is the total fluid energy, and where v is given by (1.2.9). In the particular case of potential flow $\mu = \lambda = 0$, only one pair of variables ρ, ϕ is left. In this case (1.2.11) becomes the Bernoulli equation. For an incompressible fluid $\operatorname{div} v = 0$ and ϕ is determined from the condition

$$\nabla^2\phi = -\operatorname{div} \frac{\lambda}{\rho} \nabla\mu . \tag{1.2.13}$$

Now only one pair of variables λ and μ is left. The curl of the fluid velocity is determined from the formula

$$\operatorname{curl} \boldsymbol{v} = \left[\nabla \frac{\lambda}{\rho} \times \nabla \mu\right]. \tag{1.2.14}$$

The described procedure of constructing canonical variables can be readily transferred to the case of the equations of relativistic hydrodynamics:

$$\begin{aligned} &\partial\rho/\partial t + \operatorname{div} \rho \boldsymbol{v} = 0 \\ &\partial \boldsymbol{p}/\partial t + (\boldsymbol{v}\nabla)\boldsymbol{p} + m\nabla\, \delta\mathscr{E}/\delta\rho = 0 \\ &\boldsymbol{p} = m\boldsymbol{v}/\sqrt{1 - v^2/c}\,. \end{aligned} \tag{1.2.15}$$

Conversion to the canonical variables is expressed by the formula

$$\frac{\boldsymbol{p}}{m} = \frac{\lambda}{\rho}\nabla\mu + \nabla\phi\,.$$

The equations for them keep the form (1.2.12), where

$$\mathscr{H} = \mathscr{E} + \int \frac{\rho c^2\, \mathrm{d}\boldsymbol{r}}{\sqrt{1 - v^2/c^2}}; \tag{1.2.16}$$

the equations for ρ, λ, μ keep their previous form; the equation for ϕ takes the form

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \frac{w}{m} - \frac{\lambda}{\rho}(\boldsymbol{v}\nabla)\mu + \frac{\delta\mathscr{E}}{\delta\rho} = 0 \\ &w = mc^2/\sqrt{1 - v^2/c^2}\,. \end{aligned} \tag{1.2.17}$$

The Hamiltonian formalism can be used also for a non-barotropic (e.g., a non-uniformly heated) fluid, under the condition that the flow in it is isentropic, i.e., there is no dissipation of any form. In this case an additional pair of variables ψ, S is introduced, where S is the fluid entropy, so that

$$\boldsymbol{v} = \frac{1}{\rho}(\lambda\,\nabla\mu + \psi\,\nabla S) + \nabla\phi\,. \tag{1.2.18}$$

The internal energy must be a functional of ρ, S. The Hamiltonian formalism for a non-uniform incompressible fluid or free-boundary fluid can be obtained using the passage to the limit (1.2.18), however the discussion of these problems is beyond the scope of this review.

The simplest models of a plasma lie directly within the framework of the above-mentioned equations of generalized hydrodynamics. Now let us consider an electron plasma (ions are assumed fixed) having a not too high relativistic temperature. The kinetic calculations show that Langmuir waves with a dispersion law ω_k can propagate in such a plasma, where (non-relativistically)

$$\omega_k = \omega_p(1 + \tfrac{3}{2}k^2 r_D^2)$$
$$r_D^2 = \frac{T_e}{4\pi n e^2}. \tag{1.2.19}$$

Long-wave non-linear oscillations of this plasma are described by the system of equations [1–5]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = \nabla\left(\frac{e}{m}\varphi - \frac{3T}{m\rho}\delta\rho\right) \tag{1.2.20}$$
$$\nabla^2\varphi = \frac{4\pi e}{m}\delta\rho .$$

Here $\delta\rho = \rho - \rho_0$, ρ_0 is the electron density. It is not difficult to check that the system (1.2.20) is a special case of the system (1.2.4), so that

$$\mathscr{E} = \frac{e^2}{2m^2}\int \frac{\delta\rho(\boldsymbol{r})\,\delta\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' + \frac{3}{2}\frac{T}{m\rho_0}\int \delta\rho^2\,\mathrm{d}\boldsymbol{r}. \tag{1.2.21}$$

Now let us consider the hydrodynamics of slow motions of a non-isothermal plasma [1–5]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \boldsymbol{v} = 0$$
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{e}{M}\nabla\varphi \tag{1.2.22}$$
$$\Delta\varphi = -\frac{4\pi e}{M}(\rho - \rho_0\,\mathrm{e}^{e\varphi/T}).$$

Here ρ, $\boldsymbol{v}$ are the density and velocity of ions, M is the ion mass, T the electron temperature (the ion temperature $T_i \ll T$ does not enter into this problem); φ the electrostatic potential.

One can show that the system (1.2.22) belongs to the type (1.2.4), and

$$\mathscr{E} = \frac{1}{8\pi}\int (\nabla\varphi)^2\,\mathrm{d}\boldsymbol{r} + \frac{\rho_0 T}{M}\int \left\{\mathrm{e}^{e\varphi/T}\left(\frac{e\varphi}{T} - 1\right) + 1\right\}\mathrm{d}\boldsymbol{r}. \tag{1.2.23}$$

Calculating the variational derivative $\delta\mathscr{E}/\delta\rho$, we have

$$\frac{\delta\mathscr{E}}{\delta\rho} = \int \rho(\boldsymbol{r})\left\{-\frac{1}{4\pi}\nabla^2\frac{\delta\varphi(\boldsymbol{r}')}{\delta\rho(\boldsymbol{r})} + \frac{e^2}{MT}\,\mathrm{e}^{e\varphi/T}\frac{\delta\varphi(\boldsymbol{r}')}{\delta\rho(\boldsymbol{r})}\right\}\mathrm{d}\boldsymbol{r}'. \tag{1.2.24}$$

On the other hand, calculating $\delta\varphi(r')/\delta\rho(r)$ from the Poisson equation, we obtain

$$-\frac{1}{4\pi}\nabla^2\frac{\delta\varphi(r)}{\delta\rho(r')}+\frac{\rho_0 e^2}{T}\,\mathrm{e}^{e\varphi/T}\frac{\delta\varphi(r)}{\delta\rho(r')}=\frac{e}{M}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,. \tag{1.2.25}$$

Comparison of (1.2.24) and (1.2.25) shows that

$$\frac{\delta\mathscr{E}}{\delta\rho(r)}=\frac{e}{M}\,\varphi(r)\,. \tag{1.2.26}$$

Finally, let us consider a relativistic electron plasma interacting with an arbitrary electromagnetic field, which is not necessarily potential (the ions are assumed fixed, as before). The equations for such a plasma are of the form

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0$$

$$\left(\frac{\partial}{\partial t}+(\boldsymbol{v}\nabla)\right)\boldsymbol{p}=-e\boldsymbol{E}-\frac{e}{c}[\boldsymbol{v}\times\boldsymbol{H}]-3T\nabla\frac{\delta\rho}{\rho_0} \tag{1.2.27}$$

$$\operatorname{curl}\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t};\qquad \operatorname{div}\boldsymbol{E}=-\frac{4\pi e}{m}(\rho-\rho_0)$$

$$\operatorname{curl}\boldsymbol{H}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}-\frac{4\pi}{mc}\,e\rho\boldsymbol{v}\,.$$

Now we introduce scalar and vector potentials φ and $\boldsymbol{A}$, and take for $\boldsymbol{A}$ the Coulomb gauge:

$$\operatorname{div}\boldsymbol{A}=0\,. \tag{1.2.28}$$

Then the Poisson equation takes the form

$$\nabla^2\rho=\frac{4\pi e}{m}(\rho-\rho_0)\,.$$

It is known (see e.g. [8]), that in the Coulomb gauge the vector potential $\boldsymbol{A}$ is a canonical variable. Canonically conjugated to it is the vector

$$\boldsymbol{B}=\frac{1}{4\pi c}\left(\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}+\nabla\varphi\right)=-\frac{\boldsymbol{E}}{4\pi c}\,. \tag{1.2.29}$$

Substitution into Maxwell's equation gives

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{4\pi}\nabla^2 \boldsymbol{A} - \frac{e\rho c}{m}\frac{\boldsymbol{p}}{w}$$
$$\frac{\partial \boldsymbol{A}}{\partial t} = 4\pi c^2\left(\boldsymbol{B} - \frac{1}{4\pi c}\nabla\varphi\right). \tag{1.2.30}$$

We rewrite Euler's equation in the form

$$\frac{\partial \boldsymbol{p}}{\partial t} + \nabla w - [\boldsymbol{v}\,\mathrm{curl}\,\boldsymbol{p}] = -e\boldsymbol{E} - \frac{e}{c}[\boldsymbol{v}\times\boldsymbol{B}] - 3T\nabla\frac{\delta\rho}{\rho_0}$$

and change to the generalized momentum

$$\boldsymbol{P} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}.$$

The vector $\boldsymbol{P}$ obeys the equation

$$\frac{\partial \boldsymbol{P}}{\partial t} + \nabla w - [\boldsymbol{v}\,\mathrm{curl}\,\boldsymbol{P}] - e\nabla\varphi + 3T\nabla\frac{\delta\rho}{\rho_0} = 0.$$

The canonical variables are introduced by the formula

$$\boldsymbol{P}/m = \lambda\,\nabla\mu/\rho + \nabla\phi. \tag{1.2.31}$$

A check shows that μ, λ, ϕ in formula (1.2.31) obey the equations (1.2.7), (1.2.10), (1.2.17). The equations for λ, μ, ρ, ϕ are of the form (1.2.12), where the Hamiltonian $\mathscr{H}$ takes the form

$$\mathscr{H} = \int\left\{\rho\frac{w}{m} + \frac{3}{2}\frac{T}{m\rho_0}\delta\rho^2 + \frac{1}{8\pi}(\mathrm{curl}\,\boldsymbol{A})^2 + 2\pi c^2\boldsymbol{B}^2 + \frac{1}{4\pi}(\nabla\varphi)^2 - c(\boldsymbol{B}\,\nabla\varphi)\right\}\mathrm{d}\boldsymbol{r}. \tag{1.2.32}$$

For $\boldsymbol{B}$ and $\boldsymbol{A}$ we have

$$\partial\boldsymbol{B}/\partial t = -\delta\mathscr{H}/\delta\boldsymbol{A}; \qquad \partial\boldsymbol{A}/\partial t = \delta\mathscr{H}/\delta\boldsymbol{B}. \tag{1.2.33}$$

Taking into account Poisson's equation,

$$\mathrm{div}\,\boldsymbol{B} = \frac{1}{4\pi c}\nabla^2\varphi \tag{1.2.34}$$

it is not difficult to verify that the Hamiltonian $\mathscr{H}$ coincides with the plasma energy. As seen from the equations for λ and μ, there exists a special type of plasma motion for which $\lambda \equiv \mu \equiv 0$. Thus, even in the presence of a non-potential electromagnetic field the analogue of the theorem of conservation of vorticity is kept {curl $\boldsymbol{P} = \mathrm{curl}(\boldsymbol{p} - (e/c)\boldsymbol{A})$ is the conserved quantity}, and it is possible to find analogues of the potential oscillations.

1.3. Change to normal variables

After constructing the Hamiltonian formalism a separation of different degrees of freedom can be achieved by a change to normal variables. This procedure is here described using the hydrodynamics of an electron plasma without a magnetic field as an example. Let the Hamiltonian (1.2.23) be written in the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \cdots \tag{1.3.1}$$

where $\mathcal{H}_0$ is the quadratic part of the Hamiltonian

$$\begin{aligned} \mathcal{H}_0 &= \tfrac{1}{2}\int \left\{\rho_0 v^2 + \frac{1}{4\pi}(\mathrm{curl}\,\boldsymbol{A})^2 + 4\pi c^2 \boldsymbol{B}^2 + \frac{3T}{m\rho_0}\delta\rho^2 - \frac{1}{2\pi}\varphi\,\nabla^2\varphi - 2c(\boldsymbol{B}\,\nabla\varphi)\right\}\mathrm{d}\boldsymbol{r} \\ \boldsymbol{v} &= \nabla\phi - \frac{e}{mc}\boldsymbol{A}\,. \end{aligned} \tag{1.3.2}$$

We change to a Fourier transform in $\boldsymbol{k}$ and make a substitution of variables

$$\begin{aligned} \phi_{\boldsymbol{k}} &= -\frac{\mathrm{i}}{k}\left(\frac{\omega_{\boldsymbol{k}}}{2\rho_0}\right)^{1/2}(a_{\boldsymbol{k}} - a^*_{-\boldsymbol{k}}) \\ \delta\rho_{\boldsymbol{k}} &= k\left(\frac{\rho_0}{2\omega_{\boldsymbol{k}}}\right)^{1/2}(a_{\boldsymbol{k}} + a^*_{-\boldsymbol{k}}) \\ \boldsymbol{A}_{\boldsymbol{k}} &= c\left(\frac{2\pi}{\Omega_k}\right)^{1/2}\sum_\lambda [\boldsymbol{S}^\lambda_{\boldsymbol{k}}(b^\lambda_{\boldsymbol{k}} + b^{*\lambda}_{-\boldsymbol{k}})] \\ \boldsymbol{B}_{\boldsymbol{k}} &= -\mathrm{i}\left(\frac{\Omega_{\boldsymbol{k}}}{8\pi c^2}\right)^{1/2}\sum_\lambda [\boldsymbol{S}^\lambda_{\boldsymbol{k}}(b^\lambda_{\boldsymbol{k}} - b^{\lambda *}_{-\boldsymbol{k}})] + \frac{\mathrm{i}\boldsymbol{k}\varphi_{\boldsymbol{k}}}{4\pi c}\,. \end{aligned} \tag{1.3.3}$$

Here $\Omega_{\boldsymbol{k}} = \sqrt{\omega_p^2 + k^2c^2}$ is the dispersion law for electromagnetic waves. After the substitution (1.3.3) the Hamiltonian $\mathcal{H}_0$ takes the form

$$\mathcal{H}_0 = \int \{\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} a^*_{\boldsymbol{k}} + \Omega_{\boldsymbol{k}}(b^+_{\boldsymbol{k}} b^{+*}_{\boldsymbol{k}} + b^-_{\boldsymbol{k}} b^{-*}_{\boldsymbol{k}})\}\,\mathrm{d}\boldsymbol{k}\,. \tag{1.3.4}$$

In (1.3.3) $\boldsymbol{S}^\lambda_{\boldsymbol{k}}$ is a unit polarization vector satisfying the conditions

$$(\boldsymbol{k}\boldsymbol{S}^\lambda_{\boldsymbol{k}}) = 0; \qquad (\boldsymbol{S}^\lambda_{\boldsymbol{k}}\boldsymbol{S}^{\lambda'*}_{\boldsymbol{k}}) = \delta_{\lambda\lambda'}; \qquad \boldsymbol{S}^\lambda_{\boldsymbol{k}} = \boldsymbol{S}^{*\lambda}_{-\boldsymbol{k}}\,.$$

As is seen from (1.3.4), the variable a_k is the complex amplitude of Langmuir waves, and b_k^λ are the complex amplitudes of electromagnetic waves of different polarization.

The cubic part of the Hamiltonian,

$$\mathcal{H}_1 = \tfrac{1}{2}\int \delta\rho\, v^2\,\mathrm{d}\boldsymbol{r} = \tfrac{1}{2}\int \delta\rho\left(\nabla\phi - \frac{e}{mc}\boldsymbol{A}\right)^2 \mathrm{d}\boldsymbol{r}, \tag{1.3.5}$$

after the substitution (1.3.3) can be divided into the sum of three terms of a different physical nature

$$\mathcal{H}_1 = \mathcal{H}_1^{(1)} + \mathcal{H}_1^{(2)} + \mathcal{H}_1^{(3)}.$$

Here

$$\mathcal{H}_1^{(1)} = \frac{1}{2(2\pi)^{3/2}}\int\left(\frac{\omega_{k_1}\omega_{k_2}}{8\omega_k\rho_0}\right)^{1/2}\frac{k(\boldsymbol{k}_1\boldsymbol{k}_2)}{k_1k_2}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)(a_k + a_{-k}^*)(a_{k_1} - a_{-k_1}^*)(a_{k_2} - a_{-k_2}^*)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,; \tag{1.3.6}$$

$$\mathcal{H}_1^{(2)} = -\frac{1}{4\pi}\frac{e}{m}\int\left(\frac{\omega_{k_1}}{\omega_k\Omega_{k_2}}\right)^{1/2}\frac{k}{k_1}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)(a_k + a_{-k}^*)(a_{k_1} - a_{-k_1}^*)\sum_\lambda(\boldsymbol{k}_1\boldsymbol{S}_{k_2}^\lambda)(b_{k_2}^\lambda + b_{k_2}^{\lambda *})\,; \tag{1.3.7}$$

$$\mathcal{H}_1^{(3)} = \frac{1}{4}\frac{e^2}{m^2\pi^{1/2}}\int\left(\frac{\rho_0}{\omega_k\Omega_{k_1}\Omega_{k_2}}\right)^{1/2} k(a_k + a_{-k}^*)$$
$$\times\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\left\{\sum_\lambda \boldsymbol{S}_{k_1}^\lambda(b_{k_1}^\lambda + b_{-k_1}^{\lambda *})\times\boldsymbol{S}_{k_2}^\lambda(b_{k_2}^\lambda + b_{-k_2}^{\lambda *})\right\}\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,. \tag{1.3.8}$$

$\mathcal{H}_1^{(1)}$, $\mathcal{H}_1^{(2)}$, $\mathcal{H}_1^{(3)}$ are the first terms in the Hamiltonian of the wave interaction in a plasma

$$\mathcal{H}_{\mathrm{int}} = \mathcal{H}_1 + \mathcal{H}_2 + \cdots. \tag{1.3.9}$$

In constructing a perturbation theory, when $\mathcal{H}_{\mathrm{int}} \ll \mathcal{H}_0$, these terms will describe different elementary processes of a non-linear wave-interaction. The Hamiltonians $\mathcal{H}_1^{(2)}$, $\mathcal{H}_1^{(3)}$ give non-trivial answers even in first order of perturbation theory. $\mathcal{H}_1^{(2)}$ describes the fusion process of two Langmuir waves into one electromagnetic wave, and the inverse process of the decay of an electromagnetic wave into two Langmuir waves. The Hamiltonian $\mathcal{H}_1^{(3)}$ describes the decay of an electromagnetic wave into another electromagnetic wave (of the same or with a different polarization) and a Langmuir wave and the reverse process. The Hamiltonian $\mathcal{H}_1^{(1)}$ corresponds to the fusion process of two plasma waves into a third one. This process is forbidden by the conservation laws, and it makes a contribution only in the second order of perturbation theory.

The Hamiltonian $\mathcal{H}_1$ is a single term in $\mathcal{H}_{\mathrm{int}}$ which exists when neglecting relativistic effects. All other terms have a relativistic origin and arise from the expansion of the expression $c^2\int(\rho_0 + \delta\rho)(\sqrt{1 - v^2/c^2})^{-1}\,\mathrm{d}\boldsymbol{r}$. Thus

$$\mathcal{H}_2 = \frac{\rho_0}{2c^2}\int v^4\,\mathrm{d}\boldsymbol{r}. \tag{1.3.10}$$

Substituting (1.3.3) into (1.3.10), it is not difficult to find all fourth order terms in the interaction Hamiltonian, which, together with the third order terms, are responsible for the four-wave interactions.

Let there now be a uniform magnetic field $\boldsymbol{H}_0$ in the plasma. In this case the procedure transforming to normal variables becomes more complicated, by virtue of the fact that the corresponding potential is a linear function of the coordinates. Let us choose it in the form

$$\boldsymbol{A}_0 = \tfrac{1}{2}(-\boldsymbol{i}y + \boldsymbol{j}x)H_0; \qquad \boldsymbol{H}_0 = H_0\boldsymbol{k} \tag{1.3.11}$$

and make the canonical transformation to new variables in two stages. In the first stage let us change to the symmetric variables ϕ', λ', μ'

$$\lambda = \frac{\sqrt{\rho}}{\sqrt{2}}(\lambda' + \mu'); \qquad \mu = \frac{1}{(2\rho)^{1/2}}(\mu' - \lambda'); \qquad \phi = \phi' + \frac{\lambda'^2 - \mu'^2}{4\rho}. \tag{1.3.12}$$

In the second stage let us eliminate the constant potential component. Change to the variables λ'', μ'', ϕ''

$$\begin{aligned} &\lambda' = \lambda'' + (\omega_{\mathrm{H}}\rho)^{1/2}y; \qquad \mu' = \mu'' - (\omega_{\mathrm{H}}\rho)^{1/2}x \\ &\phi' = \phi'' - (\omega_{\mathrm{H}}/\rho)^{1/2}(x\lambda'' + y\mu'') . \end{aligned} \tag{1.3.13}$$

In this case

$$\tilde{\boldsymbol{A}} = \boldsymbol{A}'' = \boldsymbol{A} - \boldsymbol{A}_0 .$$

Here $\omega_{\mathrm{H}} = eH_0/mc$ is the Larmor electron frequency. In the new variables, which will be written below without primes like initial ones, we obtain

$$\frac{\boldsymbol{p}}{m} = -\left(\frac{\omega_{\mathrm{H}}}{\rho}\right)^{1/2}(\boldsymbol{i}\lambda + \boldsymbol{j}\mu) + \nabla\phi - \frac{e}{mc}\tilde{\boldsymbol{A}} + \frac{1}{2\rho}(\lambda\,\nabla\mu - \mu\,\nabla\lambda) . \tag{1.3.14}$$

It is not difficult to check that both substitutions of variables are canonical; and the equations for them are prescribed by formulae (1.2.12), (1.2.33) with the same Hamiltonian (1.3.32). The difference lies only in expressing the velocity $\boldsymbol{v}$ through the canonical variables. We expand the velocity in powers of the canonical variables

$$\begin{aligned} &\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1 \\ &\boldsymbol{v}_0 = -\left(\frac{\omega_{\mathrm{H}}}{\rho_0}\right)^{1/2}(\boldsymbol{i}\lambda + \boldsymbol{j}\mu) + \nabla\phi - \frac{e}{mc}\tilde{\boldsymbol{A}} \\ &\boldsymbol{v}_1 = \frac{1}{2}\left(\frac{\omega_{\mathrm{H}}}{\rho_0}\right)^{1/2}(\boldsymbol{i}\lambda + \boldsymbol{j}\mu)\frac{\delta\rho}{\rho_0} + \frac{1}{2\rho_0}(\lambda\,\nabla\mu - \mu\,\nabla\lambda) . \end{aligned} \tag{1.3.15}$$

The Hamiltonian $\mathcal{H}$ expanded in powers of the canonical variables is an infinite series even for a non-relativistic plasma. The cubic term $\mathscr{H}_1$ is of the form

$$\mathscr{H}_1 = \tfrac{1}{2}\int \delta\rho\, v_0^2\, \mathrm{d}\boldsymbol{r} + \int \rho_0(\boldsymbol{v}_0\boldsymbol{v}_1)\, \mathrm{d}\boldsymbol{r}. \tag{1.3.16}$$

For further calculations only the linearized equations for the new canonical variables are needed. We will write down these equations in $\boldsymbol{k}$-space assuming that all variables vary as $\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\boldsymbol{r})$. It is not difficult to check that they have the following form:

$$\begin{aligned}
&\mathrm{i}\omega \boldsymbol{B} = \frac{k^2\tilde{\boldsymbol{A}}}{4\pi} + \frac{e\rho_0}{mc}\boldsymbol{v}_0 \\
&-\mathrm{i}\omega\tilde{\boldsymbol{A}} = 4\pi c^2\left(\boldsymbol{B} - \frac{\mathrm{i}\boldsymbol{k}}{4\pi c}\varphi\right) \\
&-\mathrm{i}\omega\lambda = -(\omega_{\mathrm{H}}\rho_0)^{1/2} v_{0y}, \qquad -\mathrm{i}\omega\mu = (\omega_{\mathrm{H}}\rho_0)^{1/2} v_{0x} \\
&-\mathrm{i}\omega\phi = \frac{e\varphi}{m} - 3T\frac{\delta\rho}{m\rho_0} \\
&-\mathrm{i}\omega\,\delta\rho = -\mathrm{i}\rho_0(\boldsymbol{k}\boldsymbol{v}_0) \\
&-k^2\varphi = \frac{4\pi e}{m}\delta\rho, \qquad (\boldsymbol{k}\boldsymbol{A}) = 0 \\
&\boldsymbol{v}_0 = -\sqrt{\frac{\omega_{\mathrm{H}}}{\rho_0}}(\boldsymbol{i}\lambda + \boldsymbol{j}\mu) + \mathrm{i}\boldsymbol{k}\phi - \frac{e}{mc}\tilde{\boldsymbol{A}}.
\end{aligned} \tag{1.3.17}$$

The characteristic equation of the system (1.3.17) is of fourth order in ω^2 and describes four types of waves in the electron plasma with the dispersion laws $\omega_i(\boldsymbol{k})$ $(i = 1, 2, 3, 4)$. It is clear that $\omega_i(\boldsymbol{k}) > 0$. To each of the oscillation branches there corresponds a set of quantities $A_i(\boldsymbol{k})$, $B_i(\boldsymbol{k})$, $\lambda_i(\boldsymbol{k})$, $\mu_i(\boldsymbol{k}), \ldots,$ which are determined apart from multiplication by an arbitrary function $f_i(\boldsymbol{k})$. Now let us introduce the complex wave amplitudes $a_i(\boldsymbol{k})$ making the substitution

$$\begin{aligned}
\lambda(\boldsymbol{k}) &= \sum_{i=1}^{4} f_i(\boldsymbol{k})\{\lambda_i(\boldsymbol{k})\, a_i(\boldsymbol{k}) + \lambda_i^*(-\boldsymbol{k})\, a_i^*(-\boldsymbol{k})\} \\
\mu(\boldsymbol{k}) &= \sum_{i=1}^{4} f_i(\boldsymbol{k})\{\mu_i(\boldsymbol{k})\, a_i(\boldsymbol{k}) + \mu_i^*(-\boldsymbol{k})\, a_i^*(-\boldsymbol{k})\},
\end{aligned} \tag{1.3.18}$$

and so on, into the quadratic Hamiltonian $\mathscr{H}_0$ (1.3.2). The functions $f_i(\boldsymbol{k})$ are defined now from the condition that $\mathscr{H}_0$ takes the form

$$\mathscr{H}_0 = \sum_i \int \omega_i(\boldsymbol{k})\, a_i(\boldsymbol{k})\, a_i^*(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}. \tag{1.3.19}$$

Further the substitution of (1.3.18) into (1.3.15)–(1.3.16) makes it possible to express the Hamiltonian

$\mathscr{H}_1$ in the form

$$\mathscr{H}_1 = \int \Big\{ \sum_{ijl} [(V^{ijl}_{kk_1k_2} a^*_i(k)\, a_j(k_1) a_l(k_2) + \text{c.c.})\, \delta_{k-k_1-k_2} + (U^{ijl}_{kk_1k_2} a_i(k)\, a_j(k_1)\, a_l(k_2) + \text{c.c.})\, \delta_{k+k_1+k_2}] \Big\}\, dk\, dk_1\, dk_2 \,. \quad (1.3.20)$$

The quantities $V^{ijl}_{kk_1k_2}$, $U^{ijl}_{kk_1k_2}$ represent the matrix elements of different types of wave interactions. The above-mentioned method of calculating them is much simpler than the use of kinetic equations.

It is of principal importance that when using the Hamiltonian approach, there is no necessity to check the symmetry of the matrix elements.

In reality different oscillation branches are those which significantly differ in frequencies, this appreciably alleviates the calculations. For example, let us consider short-wave oscillations $kc \gg \omega_p$ in the magneto-active plasma. In this case the oscillations can be separated into potential and electromagnetic ones. The electromagnetic oscillation frequency is large, as compared with the plasma one, and they propagate practically in the same manner as in vacuum. To calculate normal variables for potential oscillations in accordance with the above-mentioned scheme, it should be noted that in the equations (1.3.17) we have

$$\tilde{A} = 0, \qquad B = \frac{1}{4\pi c} \nabla\varphi \,.$$

The quadratic Hamiltonian $\mathscr{H}_0$ is a sum of the kinetic energy of the particles, an electrostatic energy and the thermal energy of the electrons

$$\mathscr{H}_0 = \int dr \left(\rho_0 \frac{v^2}{2} + \frac{(\nabla\varphi)^2}{8\pi} + \frac{3}{2} \frac{T}{m} \frac{(\delta\rho)^2}{\rho_0} \right). \quad (1.3.21)$$

The solvability condition for the system (1.3.17) leads to a dispersion equation describing two branches of high-frequency potential oscillations:

$$\omega^4 - \omega^2(\omega_L^2(k) + \omega_H^2) + \omega_L^2(k)\, \omega_H^2 \cos^2\Theta = 0 \,. \quad (1.3.22)$$

Here Θ is the angle between the wave vector and the magnetic field, $\omega_L^2(k) = \omega_p^2 + \frac{3}{2}\kappa^2 r_D^2 \omega_p^2$.

The dependence of the wave frequency on the propagation angle is illustrated in fig. 1.1. In a weak magnetic field the upper branch corresponds to Langmuir oscillations; in a strong field the lower branch corresponds to magnetized Langmuir oscillations $\omega \simeq \omega_p |\cos\Theta|$. It should be noted also that (1.3.22) is valid for describing the lower branch only when $\cos\Theta > \sqrt{m/M}$; in the opposite case the ion motion should be taken into account.

Introducing the frequencies of the upper and the lower branches $\omega^\pm$, we obtain from (1.3.17) a relation connecting v_0 and $\delta\rho/\rho_0$, $v_0^\pm = u_k^\pm\, \delta\rho_k^\pm/\rho_0$

$$u_k^\pm = \frac{\omega_p^2}{k^2(\omega_\pm^2 - \omega_H^2)} \left\{ \omega_\pm(k) - \left[\omega_\pm \left(k - \frac{\omega_H^2}{\omega_\pm^2} k_z h \right) - i\omega_H h k \right] \right\}, \quad (1.3.23)$$

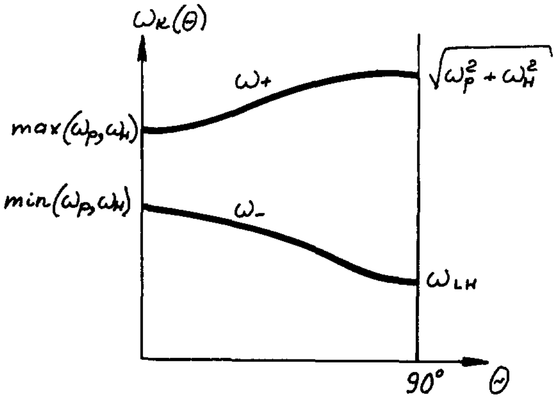

Fig. 1.1. The angular dependence of the frequencies of the potential oscillations in a magnetized plasma.

where $\boldsymbol{h}$ is a unit vector directed along the magnetic field. The relation (1.3.23) becomes evident if it is remembered that $\delta\rho/\rho_0 = -(mk^2/4\pi e\rho_0)\varphi$, and, hence, $\boldsymbol{u}_k$ is the well-known expression for the particle velocity in a constant magnetic field under the action of the electric field $\boldsymbol{E} = \mathrm{i}\boldsymbol{k}\varphi$ [1, 3, 4]. Introducing normal variables $\boldsymbol{a}_k$

$$\delta\rho_{\boldsymbol{k}}^{\pm}/\rho_0 = \alpha_{\boldsymbol{k}}^{\pm}(a_{\boldsymbol{k}}^{\pm} + a_{-\boldsymbol{k}}^{\pm *}) \tag{1.3.24}$$

and substituting into (1.3.24), we determine the multiplying factor:

$$\alpha_{\boldsymbol{k}}^{\pm} = \frac{k\omega_{\pm}^{1/2}}{\omega_{\mathrm{p}}(2\rho_0)^{1/2}} \left|\frac{\omega_{\pm}^2 - \omega_{\mathrm{H}}^2}{\omega_{+}^2 - \omega_{-}^2}\right|^{1/2} . \tag{1.3.25}$$

The connection between the hydrodynamical potential ϕ and $\delta\rho$ is readily determined:

$$\mathrm{i}\phi_{\boldsymbol{k}}^{\pm} = \frac{\omega_{\mathrm{L}}^2}{k^2\omega_{\pm}} \frac{\delta\rho_{\boldsymbol{k}}^{\pm}}{\rho_0} . \tag{1.3.26}$$

The connection between $\delta\rho$ and λ and μ can be found projecting the last of the equations (1.3.17) on the x- and y-axes

$$\lambda_{\boldsymbol{k}}^{\pm} = -\left(u_{k_x}^{\pm} - \frac{k_x}{k^2}\frac{\omega_{\mathrm{L}}^2}{\omega_{\pm}}\right)\left(\frac{\rho_0}{\omega_{\mathrm{H}}}\right)^{1/2} \frac{\delta\rho_{\boldsymbol{k}}^{\pm}}{\rho_0}$$
$$\mu_{\boldsymbol{k}}^{\pm} = -\left(u_{k_y}^{\pm} - \frac{k_y}{k^2}\frac{\omega_{\mathrm{L}}^2}{\omega_{\pm}}\right)\left(\frac{\rho_0}{\omega_{\mathrm{H}}}\right)^{1/2} \frac{\delta\rho_{\boldsymbol{k}}^{\pm}}{\rho_0} . \tag{1.3.27}$$

Expressing the interaction Hamiltonian $\mathscr{H}_1$ (1.3.16) through the normal variable, using (1.3.24)–(1.3.27), we calculate the specific form of the matrix elements $V_{\boldsymbol{k}k_1k_2}^{ijl}$, describing the interaction between potential high-frequency oscillations of a magneto-active plasma. They are listed in tables 1.1–1.3.

Above we considered only an electron plasma. It is not difficult to take into account the ion motion,

Table 1.1
Connection between the density variation δn_k and the normal variables for the basic potential plasma modes $\delta n_k = \alpha_k (a_k + a^*_{-k})$

Oscillation	α_k
ω^ℓ	$k\sqrt{n_0/2m\omega_p}$
ω^+	$\frac{k}{\omega_p}\sqrt{\frac{\omega^+}{2mn_0}\left\|\frac{\omega_+^2-\omega_H^2}{\omega_+^2-\omega_-^2}\right\|}$
ω^-	$\frac{k}{\omega_p}\sqrt{\frac{\omega^-}{2mn_0}\left\|\frac{\omega_H^2-\omega_-^2}{\omega_+^2-\omega_-^2}\right\|}$
ω_s	$\sqrt{kn_0/2MC_s}$
ω_{LH}	$k\frac{\omega_H}{\omega_p}\frac{\omega_{pi}^2}{\omega_{LH}^2}\sqrt{\frac{n_0\omega_{LH}}{2m(\omega_p^2+\omega_H^2)}}$
Ω_n	$\sqrt{(\Omega-n\omega_{Hi})n_0/T_e}$
$\Omega_\pm$	$\sqrt{\frac{\Omega_+ n_0}{2T_e}\left\|\frac{\Omega_+^2-\omega_{Hi}^2}{\Omega_+^2-\Omega_-^2}\right\|}$

ω^ℓ, Langmuir waves; ω_+, ω_-, high-frequency potential oscillations of magnetized plasma; ω_s, ion-sound waves; ω_{LH}, oscillations with frequencies close to the lower-hybrid one; Ω_n, ion-cyclotron waves with frequencies close to $n\omega_{Hi}$; $\Omega_\pm$, low-frequency oscillations of a magnetized plasma. δn_k is the Fourier transform of the electron density variation for the first three modes, for others δn_k is the ion density variation. The electromagnetic wave field is connected with the canonical variables by the relation (1.3.3)

$$E_k = \mathrm{i}(8\pi\omega_k)^{1/2}\sum_\lambda^{1,2} S_k^\lambda (b_k^\lambda - b_{-k}^{\lambda *}).$$

The S_k^λ are unit polarization vectors defined by the conditions

$$(kS_k^\lambda)=0;\qquad (S_k^\lambda, S_k^{\lambda'})=\delta_{\lambda\lambda'};$$
$$(S_k^\lambda S_k^{\lambda *})=1;\qquad S_k^\lambda = S_{-k}^{\lambda *}.$$

For obtaining the matrix elements of the interaction with sound it is necessary to substitute instead of the fluctuation $\delta\rho$ in (1.3.7), (1.3.8) the ion density variation expressed in terms of the canonical variables.

introducing additional canonical variables (ρ, ϕ, λ and μ). Thereby the number of inherent oscillations and possible non-linear processes increases. They can be considered in a similar manner, and a series of the most important examples will be presented below.

1.4. Various simplifications

We have obtained a sufficient set of Hamiltonians for the interactions between different types of

Table 1.2
Matrix elements of the three-waves coupling in isotropic plasma

Process	$V^{ijl}_{kk_1k_2}$
$\omega^{t}_{k} \to \omega^{t}_{k_1} + \omega^{\ell}_{k_2}$	$\dfrac{1}{2\sqrt{2\pi}}\dfrac{e^2\rho_0^{1/2}k_2(\boldsymbol{S}^{\lambda^*}_{k}, \boldsymbol{S}^{\lambda'}_{k_1})}{m^2[\omega^{\ell}_{k_2}\cdot\omega^{t}_{k_1}\cdot\omega^{t}_{k}]^{1/2}}$
$\omega^{t}_{k} \to \omega^{\ell}_{k_1} + \omega^{\ell}_{k_2}$	$\dfrac{e}{16\pi m\omega^{+}}\dfrac{(\boldsymbol{k}, \boldsymbol{k}_1 - \boldsymbol{k}_2)(\boldsymbol{k}_1 - \boldsymbol{k}_2, \boldsymbol{S}^{\lambda^*}_{k})}{k_1k_2}$
$\omega^{t}_{k} \to \omega^{t}_{k_2} + \Omega_{sk_2}$	$\dfrac{1}{2\sqrt{2\pi}}\dfrac{e^2}{m^2}\left(\dfrac{mn_0\Omega_s}{\omega^{t}_{k}\omega^{t}_{k_1}}\right)^{1/2}\sqrt{\dfrac{m}{T}}\,(\boldsymbol{S}^{\lambda^*}_{k}, \boldsymbol{S}^{\lambda_1}_{k_1})$
$\omega^{t}_{k} \to \omega^{\ell}_{k_1} + \Omega_{sk_2}$	$\dfrac{e}{4\pi m}\left(\dfrac{\Omega_s\omega_p}{\omega^{+}}\right)^{1/2}\sqrt{\dfrac{m}{T}}\dfrac{(\boldsymbol{k}_1, \boldsymbol{S}^{\lambda}_{k})}{k_1}$
$\omega^{\ell}_{k} \to \omega^{\ell}_{k_1} + \Omega_{sk_2}$	$\dfrac{\omega_p}{(2\pi)^{3/2}}\dfrac{\sqrt{k_2}}{2\sqrt{2}(Mn_0c_s)^{1/2}}\dfrac{(\boldsymbol{k}\boldsymbol{k}_1)}{kk_1}$
$\Omega_s(k) \to \Omega_s(k_1) + \Omega_s(k_2)$	$\dfrac{c_s}{16(\pi^3Mn_0)^{1/2}}\left\{\dfrac{(\boldsymbol{k}\boldsymbol{k}_1)k_2^{1/2}}{\sqrt{kk_1}} + \dfrac{(\boldsymbol{k}\boldsymbol{k}_2)\sqrt{k_1}}{\sqrt{kk_2}} + \dfrac{(\boldsymbol{k}_1\boldsymbol{k}_2)\sqrt{k}}{\sqrt{k_1k_2}} - \sqrt{kk_1k_2}\right\}$

plasma waves. Now let us consider the possibilities for a simplification of these Hamiltonians. Assume that we have a cubic Hamiltonian for coupling the same type of waves with amplitude a_k and dispersion law ω_k, such that for the waves under consideration the resonance equations for the three wave processes

$$\begin{aligned} &\omega_k = \omega_{k_1} + \omega_{k_2}, \qquad \omega_k + \omega_{k_1} + \omega_{k_2} = 0 \\ &\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \qquad \boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0 \end{aligned} \tag{1.4.1}$$

have no solutions. A general form of such a Hamiltonian is expressed by (1.1.19). The absence of solutions of (1.4.2) means that the quadratic non-linearities result in the existence of induced non-resonance oscillations and may be eliminated. For this purpose, it is convenient to make a transformation from the variable a_k to a new variable c_k according to the formula

$$\begin{aligned} a_k = c_k &- \int \frac{V_{kk_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,c_{k_1}c_{k_2}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \\ &+ 2\int \frac{V^{*}_{k_2kk_1}c_{k_1}c_{k_2}}{\omega_{k_2} - \omega_k - \omega_{k_1}}\,\delta(\boldsymbol{k}_2 - \boldsymbol{k} - \boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 - \int \frac{U_{kk_1k_2}c^{*}_{k_1}c^{*}_{k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}\,\delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,. \end{aligned} \tag{1.4.2}$$

The transformation (1.4.2) is canonical, accurate up to third-order terms in c_k and, after substitution into the Hamiltonian (1.1.19), it gets rid of the cubic terms. In so doing, there appear fourth-order terms in the c_k; among them only the term

$$\mathcal{H}_3 = \tfrac{1}{2}\int T_{kk_1k_2k_3}c^{*}_{k}c^{*}_{k_1}c_{k_2}c_{k_3}\delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \tag{1.4.3}$$

Table 1.3
Matrix elements of the three-wave coupling in a magnetized plasma

Process	$V^{ijl}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}$
$\omega^+_k \to \omega^+_{k_1} + \omega^-_{k_2}$	$\frac{1}{(2\pi)^{3/2}} \frac{\delta^+_k \delta^+_{k_1} \delta^-_{k_2}}{2\omega_p (2mn_0)^{1/2}} \frac{k_1 k_2}{k} \left(\frac{\omega^+_{k_1}\omega^-_{k_2}}{\omega^+_k}\right)^{1/2}$ $\times \left\{ \boldsymbol{k}(\boldsymbol{u}^+_{k_1} + \boldsymbol{u}^-_{k_2}) + \frac{k^2}{\omega_p^2} [(\boldsymbol{u}^+_k \boldsymbol{u}^+_{k_1})(\boldsymbol{k}_1 \boldsymbol{u}^-_{k_2}) + (\boldsymbol{u}^+_k \boldsymbol{u}^-_{k_2})(\boldsymbol{k}_2 \boldsymbol{u}^+_{k_1})] \right\}$
	$\boldsymbol{u}^\pm_k = \frac{\omega_p^2}{k^2(\omega_\pm^2 - \omega_H^2)} \left\{ \omega_\pm \left(\boldsymbol{k} - \frac{\omega_H^2}{\omega_\pm} k_z \boldsymbol{h} \right) + \mathrm{i}\omega_H [\boldsymbol{h}\boldsymbol{k}] \right\}$
	$\delta^\pm_k = \left| \frac{\omega^{\pm 2} - \omega_H^2}{\omega_+^2 - \omega_-^2} \right|^{1/2}$
$\omega^+_k \to \omega^-_{k_1} + \omega^-_{k_2}$	$\frac{1}{(2\pi)^{3/2}} \frac{\delta^+_k \delta^-_{k_1} \delta^-_{k_2}}{4\omega_p (2n_0 m)^{1/2}} \frac{k_1 k_2}{k} \left(\frac{\omega^-_{k_1}\omega^-_{k_2}}{\omega^+_k}\right)^{1/2}$ $\times \left\{ \boldsymbol{k}(\boldsymbol{u}^-_{k_1} + \boldsymbol{u}^-_{k_2}) + \frac{k^2}{\omega_p^2} [(\boldsymbol{u}^{+*}_k \boldsymbol{u}^-_{k_1})(\boldsymbol{k}_1 \boldsymbol{u}^-_{k_2}) + (\boldsymbol{u}^{+*}_k \boldsymbol{u}^-_{k_2})(\boldsymbol{k}_2 \boldsymbol{u}^-_{k_1})] \right\}$
$\omega^\pm_k \to \omega^\pm_{k_1} + \Omega^s_{k_2}$	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\Omega^s_{k_2}}{8n_0 T_e}}\, \delta^\pm_{k_1} \delta^\pm_k \frac{k_1}{k} (\boldsymbol{k}\boldsymbol{u}^\pm_{k_1})$
$\omega^-_k \to \omega^-_{k_1} + \omega^-_{k_2}$	$\frac{1}{(2\pi)^{3/2}} \frac{\delta^-_k \delta^-_{k_1} \delta^-_{k_2}}{4\omega_p (2mn_0)^{1/2}} \frac{k_1 k}{k_2} \left(\frac{\omega_k \omega_{k_1}}{\omega_{k_2}}\right)^{1/2}$ $\times \left\{ \boldsymbol{k}_2(\boldsymbol{u}^{-*}_k + \boldsymbol{u}^{-*}_{k_1}) + \frac{k^2}{\omega_p^2} [(\boldsymbol{u}^{-*}_{k_2} \boldsymbol{u}^{-*}_k)(\boldsymbol{k}\boldsymbol{u}^*_{k_1}) + (\boldsymbol{u}^{-*}_{k_2} \boldsymbol{u}^{-*}_{k_1})(\boldsymbol{k}\boldsymbol{u}^{-*}_{k_2})] \right\}$
If the ions are magnetized and two low frequency modes exist	
$\omega^\pm_k \to \omega^\pm_{k_1} + \Omega^\pm_{k_2}$	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\Omega^\pm_{k_2}}{8n_0 T_e}}\, \Delta^\pm(k_1)\, \delta^\pm_{k_1} \delta^\pm_{k_2} \frac{k_2}{k} (\boldsymbol{k}\boldsymbol{u}^\pm_{k_1})$
	$\Delta^\pm = \left| \frac{\Omega^{\pm 2}_k - \omega^2_{k_1}}{\Omega^{+2} - \Omega^{-2}} \right|$

should be left. In an endeavour to eliminate this term using a canonical transformation of the type (1.4.2) containing cubic terms there arise expressions having denominators which vanish on the surface determined by the equations

$$\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3} = 0, \qquad \boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 = 0 .$$

The denominators in the other fourth-order terms have singularities on the surfaces determined by the equations

$$\omega_k + \omega_{k_1} + \omega_{k_2} - \omega_{k_3} = 0, \qquad \boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 = 0 \tag{1.4.4}$$

$$\omega_k + \omega_{k_1} + \omega_{k_2} + \omega_{k_3} = 0, \qquad \boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0 . \tag{1.4.5}$$

As a rule, these equations cannot have solutions if the equations (1.4.1) have no solutions. Therefore the above-mentioned fourth-order terms are unimportant. The matrix element $T_{kk_1k_2k_3}$ is given by the formula

$$T_{kk_1k_2k_3} = -\frac{U_{-(k_2+k_3)k_2k_3}U_{-(k+k_1)kk_1}}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}} - \frac{U^*_{-(k+k_1)kk_1}U_{-(k_2+k_3)k_2k_3}}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}$$
$$-\frac{V^*_{k_2+k_3,\,k_2k_3}V_{k+k_1,\,kk_1}}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}} - \frac{V^*_{k+k_1,\,kk_1}V_{k_2+k_3,\,k_2k_3}}{\omega_{k+k_1}-\omega_k-\omega_{k_1}} - 2\frac{V_{kk_2,\,k-k_2}V^*_{k_3k_1,\,k_3-k_1}}{\omega_{k_3-k_1}+\omega_{k_1}-\omega_{k_3}}$$
$$-2\frac{V_{k_1k_3,\,k_1-k_3}V^*_{k_2k,\,k_2-k}}{\omega_{k_2-k}+\omega_k-\omega_{k_2}} - 2\frac{V_{k_1k_2,\,k_1-k_2}V_{k_3k,\,k_3-k}}{\omega_{k_3-k}+\omega_k-\omega_{k_3}} - 2\frac{V_{kk_3,\,k-k_3}V^*_{k_2k_1,\,k_2-k_1}}{\omega_{k_2-k_1}+\omega_{k_1}-\omega_{k_2}} \tag{1.4.6}$$

and possesses obviously the symmetry properties (1.1.22).

The equations of motion for c_k are of the form

$$\frac{\partial c_k}{\partial t} + \mathrm{i}\omega_k c_k = -\mathrm{i}\int T_{kk_1k_2k_3}c^*_{k_1}c_{k_2}c_{k_3}\delta(k+k_1-k_2-k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,. \tag{1.4.7}$$

It follows from (1.4.7) and (1.1.22) that (1.4.7) possesses the integral of motion

$$I = \int |c_k|^2\,\mathrm{d}k\,. \tag{1.4.8}$$

The integral I is, strictly speaking, an adiabatic invariant with respect to the full initial system. It will be called the number of quasi-particles or the wave action integral.

The above-described case takes place for the interaction between Langmuir oscillations in an electron plasma. In this case (see (1.3.6)) we have

$$V_{kk_1k_2} = U_{kk_1k_2} = \frac{\bar{1}}{4(2\pi)^{3/2}\rho_0^{1/2}}\left[\left(\frac{\omega_{k_1}\omega_{k_2}}{2\omega_k}\right)^{1/2}\frac{k(k_1k_2)}{k_1k_2} + \left(\frac{\omega_k\omega_{k_1}}{2\omega_{k_2}}\right)^{1/2}\frac{k_2(kk_1)}{kk_1} + \left(\frac{\omega_k\omega_{k_2}}{2\omega_{k_1}}\right)^{1/2}\frac{k_1(kk_2)}{kk_2}\right]. \tag{1.4.9}$$

Up to $k^2r_D^2(\ll 1)$ order terms in (1.4.6), it can be assumed in (1.4.9) that $\omega_k = \omega_p$. The obtained expression for $T_{kk_1k_2k_3}$ has a unique property. Namely,

$$T_{kk_1k_2k_3} = 0 \qquad \text{if} \qquad k\|k_1\|k_2\|k_3\,. \tag{1.4.10}$$

The validity of this equation can be confirmed by direct calculations. The property (1.4.10) results in the fact that the interaction of Langmuir waves in an electron plasma (electron non-linearities) is anomalously weak for spectrally narrow wave packets.

If Langmuir waves with a characteristic value of the electric field E_0, a wavevector k and an angular width $\Delta\Theta$ are prescribed in the plasma, the characteristic times of electron non-linearities are of the order

$$1/\tau \sim \omega_p \max(\Delta\Theta^2 k^2 r_D^2 E_0^2/8\pi nT, (kr_D)^4 E_0^2/8\pi nT) \tag{1.4.11}$$

and are so large that they usually can be neglected.

There is another possibility to simplify Hamiltonians when we consider the problem of coupling high-frequency waves with amplitude a_k and dispersion law ω_k with low-frequency waves with amplitude b_k and dispersion law Ω_k. In this case, among the terms in the cubic Hamiltonian describing such an interaction, one needs only retain the following one:

$$\mathcal{H}_{int} = \int (V_{kk_1k_2} b_k a_{k_1} a^*_{k_2} + \text{c.c.})\, \delta(k + k_1 - k_2)\, dk\, dk_1\, dk_2\,, \tag{1.4.12}$$

as the other terms of the three-wave coupling in the Hamiltonian are fast oscillating terms.

The corresponding equations of motion are of the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int \{V_{k_1k_2k} b_{k_1} a_{k_2} \delta(k - k_1 - k_2) + V^*_{k_1kk_2} b^*_{k_1} a_{k_2} \delta(k + k_1 - k_2)\}\, dk_1\, dk_2 \tag{1.4.13}$$

$$\partial b_k/\partial t + i\Omega_k b_k = -i \int V^*_{kk_1k_2} a^*_{k_1} a_{k_2} \delta(k + k_1 - k_2)\, dk_1\, dk_2\,. \tag{1.4.14}$$

1.5. Equations for envelopes

If the system of interacting waves consists of narrow packets, a further simplification of the wave interaction description takes place. Let us consider the interaction of three spectrally narrow packets with typical wave vectors k_1, k_2, k_3 lying on the surface

$$\omega_{k_1} - \omega_{k_2} - \omega_{k_3} = 0, \qquad k_1 - k_2 - k_3 = 0\,. \tag{1.5.1}$$

Imagine a_k as $a_1(k_1 + \kappa_1) + a_2(k_2 + \kappa_2) + a_3(k_3 + \kappa_3)$; $\kappa \ll k$. The substitution of this expression into the Hamiltonian (1.1.19), using the smallness of κ and neglecting unimportant terms, gives

$$\mathcal{H}_{int} = V \int [a_1^*(\kappa_1)\, a_2(\kappa_2)\, a_3(\kappa_3) + \text{c.c.}]\, \delta(\kappa_1 - \kappa_2 - \kappa_3)\, d\kappa_1\, d\kappa_2\, d\kappa_3$$
$$V \equiv V_{k_1k_2k_3}\,. \tag{1.5.2}$$

Then, using the narrowness of the envelopes, we expand the dispersion law in the quadratic Hamiltonian in powers of κ:

$$\mathcal{H}_0 = \int \omega_k\, a_k\, a_k^*\, dk = \sum_i \left[\omega(k_i) + \left.\frac{\partial\omega}{\partial k}\right|_{k=k_i} \kappa_i\right] a_i(\kappa_i)\, a_i^*(\kappa_i)\, d\kappa\,.$$

We change to the variables $c_i = a_i \exp\{i\,\omega_i(\boldsymbol{k})\,t\}$, which corresponds to a variation of the zero of the frequencies. In this variable we have

$$\tilde{\mathscr{H}} = \mathscr{H} - \sum_i \int \omega(\boldsymbol{k}_i)\, c_i c_i^*\, \mathrm{d}\boldsymbol{k} = \sum_i \int (\boldsymbol{\kappa}_i \boldsymbol{v}_i)\, c_i(\boldsymbol{\kappa}_i)\, c_i^*(\boldsymbol{\kappa}_i)\, \mathrm{d}\boldsymbol{\kappa}_i$$

$$+ V \int \{c_1^*(\boldsymbol{\kappa}_1)\, c_2(\boldsymbol{\kappa}_2)\, c_3(\boldsymbol{\kappa}_3) + \text{c.c.}\}\, \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3)\, \mathrm{d}\boldsymbol{\kappa}_1\, \mathrm{d}\boldsymbol{\kappa}_2\, \mathrm{d}\boldsymbol{\kappa}_3 \qquad (1.5.3)$$

$$\boldsymbol{v}_i = \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}=\boldsymbol{k}_i}.$$

Now let us perform the inverse Fourier transform $\psi_i = 1/(2\pi)^{3/2} \int c_i \exp(\mathrm{i}\boldsymbol{\kappa}_i \boldsymbol{r})\, \mathrm{d}\boldsymbol{\kappa}_i$. We obtain

$$\tilde{\mathscr{H}} = \sum_i \frac{\mathrm{i}}{2} \boldsymbol{v}_i \int \left[(\psi_i \nabla \psi_i^* - \text{c.c.})\, \mathrm{d}\boldsymbol{r} + \tilde{V} \int (\psi_1^* \psi_2 \psi_3 + \text{c.c.})\, \mathrm{d}\boldsymbol{r} \right]$$

$$\tilde{V} = V(2\pi)^{3/2}. \qquad (1.5.4)$$

Varying the Hamiltonian (1.5.4), we obtain the well-known equations describing the resonance three-wave interaction

$$\begin{aligned} \partial\psi_1/\partial t - (\boldsymbol{v}_1 \nabla)\psi_1 &= -\mathrm{i}\, \tilde{V} \psi_2 \psi_3 \\ \partial\psi_2/\partial t - (\boldsymbol{v}_2 \nabla)\psi_2 &= -\mathrm{i}\, \tilde{V} \psi_3^* \psi_1 \\ \partial\psi_3/\partial t - (\boldsymbol{v}_3 \nabla)\psi_3 &= -\mathrm{i}\, \tilde{V} \psi_2^* \psi_1 . \end{aligned} \qquad (1.5.5)$$

It should be underlined that the Hamiltonian $\tilde{\mathscr{H}}$ of (1.5.4) immediately gives a non-trivial integral of the system (1.5.5).

Besides the Hamiltonian $\tilde{\mathscr{H}}$, the well-known Manley–Rowe relations

$$m_1 = \int (|\psi_1|^2 + |\psi_2|^2)\, \mathrm{d}\boldsymbol{r}; \qquad m_2 = \int (|\psi_1|^2 + |\psi_3|^2)\, \mathrm{d}\boldsymbol{r}$$

are integrals of motion of (1.5.5), as well as the momentum of the wave system

$$\boldsymbol{P} = \sum \boldsymbol{p}_l; \qquad \boldsymbol{p}_l = \frac{\mathrm{i}}{2} \int (\psi_l \nabla \psi_l^* - \text{c.c.})\, \mathrm{d}\boldsymbol{r}; \qquad \tilde{\mathscr{H}} = \mathscr{H}_1 + \sum_l \boldsymbol{p}_l \boldsymbol{v}_l .$$

In the propagation of a single narrow envelope the three-wave interaction is unimportant (it must be taken into account in higher-order perturbation theory), and the interaction is described by the Hamiltonian (1.4.6).

Now let us analyse how the dynamical equations describing monochromatic wave propagation are simplified.

If the carrier wavevector is denoted by $\boldsymbol{k}_0$, it can be assumed that

$$\omega(\boldsymbol{k}) \cong \omega(\boldsymbol{k}_0) + (\boldsymbol{\kappa v}) + \tfrac{1}{2} \frac{\partial^2 \omega}{\partial k_\alpha\, \partial k_\beta}\bigg|\, \kappa_\alpha \kappa_\beta \tag{1.5.6}$$

$$T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} = T_{\boldsymbol{k}_0\boldsymbol{k}_0\boldsymbol{k}_0\boldsymbol{k}_0} = \tilde{T}(2\pi)^{-3}.$$

Introducing, as before, the envelope of the quasi-monochromatic wave ψ

$$\psi = \exp\{-\mathrm{i}\, \omega(\boldsymbol{k}_0)\, t + \mathrm{i}\boldsymbol{k}_0\boldsymbol{r}\} \int a(\boldsymbol{k}_0 + \boldsymbol{\kappa}) \exp(\mathrm{i}\boldsymbol{\kappa r})\, \mathrm{d}\boldsymbol{\kappa}/(2\pi)^{3/2}$$

and making the inverse Fourier transform in (1.4.7) we obtain

$$\mathrm{i}\psi_t + \mathrm{i}(\boldsymbol{v}_{\mathrm{gr}}\nabla)\psi + \frac{1}{2} \frac{\partial^2 \omega}{\partial k_\alpha\, \partial k_\beta}\bigg|_{\boldsymbol{k}=\boldsymbol{k}_0} \frac{\partial^2 \psi}{\partial \kappa_\alpha\, \partial \kappa_\beta} - \tilde{T}\, |\psi|^2\, \psi = 0\,. \tag{1.5.7}$$

In an isotropic medium, when the frequency depends only on the modulus of the wavevector

$$\frac{\partial^2 \omega}{\partial k_\alpha\, \partial k_\beta}\bigg|_{\boldsymbol{k}=\boldsymbol{k}_0} \kappa_\alpha \kappa_\beta = \frac{v_{\mathrm{gr}}}{k_0} \boldsymbol{\kappa}_\perp^2 + \omega'' \kappa_\parallel^2\,; \qquad \kappa_\parallel = (\boldsymbol{\kappa k}_0)/k_0$$

and (1.5.7) is appreciately simplified:

$$\mathrm{i}\left(\psi_t + v_{\mathrm{gr}} \frac{\partial \psi}{\partial z}\right) + \frac{1}{2} \frac{v_{\mathrm{gr}}}{k_0} \nabla_\perp^2 \psi + \frac{\omega''}{2} \frac{\partial^2 \psi}{\partial z^2} - \tilde{T}\, |\psi|^2\, \psi = 0\,. \tag{1.5.8}$$

The z-axis is chosen in the direction of the wave propagation.

It should be noted that, in contrast to the case of three-wave interaction, we should expand $\omega(\boldsymbol{k})$ accurate to the κ^2 terms, since the second term in (1.5.8) can be eliminated changing to a reference system moving with the group velocity.

Let in (1.5.8) $\psi_t = 0$. Then (1.5.8) describes a stationary wavepacket with characteristic longitudinal and transverse sizes, $l_\parallel$ and $l_\perp$, respectively. Assuming that

$$v_{\mathrm{gr}} \frac{\partial \psi}{\partial z} \sim \frac{v_{\mathrm{gr}}}{2k_0} \nabla_\perp^2 \psi$$

we find $l_\parallel \sim l_\perp^2 k_0$, that is $l_\parallel \gg l_\perp$. Since $\omega'' \sim v_{\mathrm{gr}}/k_0$, the term $\omega''\, \partial^2\psi/\partial z^2$ can be neglected as compared with other terms in (1.5.8). In this case there arises the known equation of stationary self-focusing

$$\mathrm{i} \frac{\partial \psi}{\partial z} + \frac{1}{2k_0} \nabla_\perp^2 \psi - \frac{\tilde{T}}{v_{\mathrm{gr}}} |\psi|^2\, \psi = 0\,. \tag{1.5.9}$$

Of course, it takes place only if $\tilde{T} < 0$.

In deriving (1.5.8) it was assumed that the matrix element $T_{kk_1k_2k_3}$ is a continuous function when all the argument tends to k_0. It is not always so; in some cases this limit depends on the direction of the vector k relative to the direction of k_0. In these cases (1.5.8) should be changed by a more complicated equation.

As for the three-wave interaction, the fact that (1.5.8) is Hamiltonian immediately gives an integral of motion, i.e. the Hamiltonian

$$\mathcal{H} = \boldsymbol{P}\boldsymbol{v}_{\mathrm{gr}} + I_2; \qquad I_2 = \tfrac{1}{2}\int \left\{\frac{v_{\mathrm{gr}}}{k_0}(\nabla_\perp\psi)^2 + \frac{\omega''}{2}\left|\frac{\partial\psi}{\partial z}\right|^2 + \tilde{T}|\psi|^4\right\}\mathrm{d}\boldsymbol{r}. \tag{1.5.10}$$

Here $\boldsymbol{P} = \frac{1}{2}\mathrm{i}\int(\psi\,\partial\psi^*/\partial z - \mathrm{c.c.})\,\mathrm{d}\boldsymbol{r}$ is a conserved quantity, too, i.e. the momentum of the wave system.

It should be noted that from the symmetry of relations (1.1.22) it follows that the value $\tilde{T}$ is real.

Now let us consider, within the framework of the system (1.4.13)–(1.4.14), the problems of the interaction of a narrow packet of high-frequency waves with low-frequency waves [11]. Let k_0 be a mean value of the high-frequency wave number. Then in (1.4.9) one can make the substitution

$$V_{kk_1k_2} = f(k) \equiv V_{kk_0k_0}; \qquad k \ll k_0$$

simultaneously expanding the high-frequency dispersion law according to the formula (1.5.6). In (1.4.13) $k \to k_0 + \kappa$ should be substituted in the arguments of the high-frequency waves a_k, which corresponds to a change to envelopes, as in the previous example. The form of the function $f(k)$ depends on the low-frequency wave nature. This function can be simply calculated, if the low-frequency waves are sound waves $\Omega_k = c_s k$.

The sound is characterized by a variation in the density of the medium $\delta\rho$ and the velocity $\boldsymbol{v}$. By virtue of the fact that $k \ll k_0$, their values are significantly changed on a scale much greater than the high-frequency wave period. Therefore it is possible to write for the local dispersion law for the high-frequency waves $\omega(\rho_0 + \delta\rho, \boldsymbol{v}, \boldsymbol{k})$

$$\omega(\rho_0 + \delta\rho, \boldsymbol{v}, \boldsymbol{k}) = \omega(k) + \frac{\partial\omega}{\partial\rho}\delta\rho + \frac{\partial\omega}{\partial\boldsymbol{v}}\boldsymbol{v}. \tag{1.5.11}$$

In the majority of cases the oscillation frequency depends on the velocity of the medium only through the Doppler shift $\delta\omega = (\boldsymbol{k}\boldsymbol{v})$, and the last term in (1.5.11) describes the change in wave energy through the medium entrainment by high-frequency oscillations.

The local high-frequency wave energy density

$$\mathscr{E} = \omega(k_0)\int a_k a_k^*\,\mathrm{d}\boldsymbol{k}$$

also varies when there is propagation of sound in the medium

$$\delta\mathscr{E} = \int\left\{\frac{\partial\omega}{\partial\rho}\delta\rho + \frac{\partial\omega}{\partial\boldsymbol{v}}\boldsymbol{v}\right\}|\psi(\boldsymbol{r})|^2\,\mathrm{d}\boldsymbol{r} \tag{1.5.12}$$

which is clearly the same as the interaction Hamiltonian. Here $\psi(\boldsymbol{r})$ is the envelope of the high-frequency wave.

As shown above, for sound motions the variables $\delta\rho$ and a hydrodynamical potential are canonical. Introducing normal variables

$$\rho_{\boldsymbol{k}} = \sqrt{\frac{k\rho_0}{2c_s}}\, b_{\boldsymbol{k}}(b_{\boldsymbol{k}} + b^*_{\boldsymbol{k}}); \qquad v_{\boldsymbol{k}} = -\mathrm{i}\boldsymbol{k}\sqrt{\frac{c_s}{2k\rho_0}}(b_{\boldsymbol{k}} - b^*_{-\boldsymbol{k}}) \tag{1.5.13}$$

and substituting (1.5.13) into (1.5.12) it is found that

$$f(\boldsymbol{k}, \boldsymbol{k}_0) = \frac{1}{\sqrt{2(2\pi)^3}}\left(\frac{\partial\omega}{\partial\rho}\sqrt{\frac{k\rho_0}{c_s}} + \frac{(\boldsymbol{k}\boldsymbol{k}_0)\sqrt{c_s}}{\sqrt{k\rho_0}}\right). \tag{1.5.14}$$

As to order of magnitude we have $\partial\omega/\partial\rho \sim \omega/\rho_0$, and the ratio of the second order to the first one in (1.5.14) is about $\sim c_s/v_{\mathrm{ph}}$. When $c_s \ll v_{\mathrm{ph}}$ the effects of the wave entrainment by the medium can be neglected. So in some cases, for an example for potential oscillations in a magnetic field, the oscillation frequency does not in general depend on the density, and the wave entrainment by the medium is the main interaction mechanism.

Besides the interaction with sound, the intrinsic high-frequency wave non-linearity should be taken into account. Adding (1.5.12) to the interaction Hamiltonian (1.5.10) and varying, we will obtain the interaction with sound, which results in (1.5.8) in the appearance of the additional terms

$$\left(\mathrm{i}\psi_t + v_{\mathrm{gr}}\frac{\partial\psi}{\partial z}\right) + \frac{v_{\mathrm{gr}}}{2k_0}\nabla_\perp^2\psi + \frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2} - \tilde{T}|\psi|^2\psi = \left(\frac{\partial\omega}{\partial\rho}\delta\rho + k_0\frac{\partial\phi}{\partial z}\right)\psi. \tag{1.5.15}$$

Adding (1.5.12) to the quadratic Hamiltonian and varying in $\delta\rho$ and ϕ, we will obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\delta\rho + \rho_0\nabla\phi = -k_0\frac{\partial}{\partial z}|\psi|^2 \\ &\frac{\partial}{\partial t}\phi + c_s^2\frac{\delta\rho}{\rho_0} = -\frac{\partial\omega}{\partial\rho}|\psi|^2. \end{aligned} \tag{1.5.16}$$

The equations (1.5.15) and (1.5.16) represent a universal system describing the interaction of a high-frequency wave with sound in an isotropic medium.

The physics of the interaction of high-frequency waves with sound depends on the ratio of the sound velocity and the group velocity of high-frequency waves c_s/v_{gr}. If $v_{\mathrm{gr}} \ll c_s$, in (1.1.15) one can change to a static approximation, assuming that

$$\frac{\partial}{\partial t}\delta\rho = \frac{\partial}{\partial t}\phi = 0.$$

In this case

$$\delta\rho = -\frac{\rho_0}{c_s^2}\frac{\partial\omega}{\partial\rho}|\psi|^2, \qquad \rho_0\nabla^2\phi = -k_0\frac{\partial}{\partial z}|\psi|^2 .$$

If the entrainment effect can be neglected, the substitution in (1.5.8) will be the only result of the interaction:

$$\tilde{T} \to \tilde{T} - \frac{\rho_0}{c_s^2}\left(\frac{\partial\omega}{\partial\rho}\right)^2 . \tag{1.5.17}$$

It is remarkable that the correction sign for $\tilde{T}$ due to the interaction with sound is always negative and favours self-focusing. Taking into account the entrainment effect the change to (1.5.8) is possible, if all the quantities depend only on the variable $\xi = z - n_1x - n_2y$. Due to the above-mentioned effect, there arises a correction for $\tilde{T}$,

$$\delta\tilde{T} = -\frac{k_0^2}{\rho_0}(1 + n_1^2 + n_2^2)^{-1} \tag{1.5.18}$$

depending on n_1 and n_2. In this case (1.5.9) is replaced by an equation of the type (1.4.7), the matrix element $T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}$ being a discontinuous function as $\boldsymbol{k}_i \to \boldsymbol{k}_0$.

It should be noted further that the above-mentioned procedure is also convenient for the interaction of narrow high-frequency packets with other low-frequency waves whose presence changes any of the parameters (e.g., a magnetic field, temperature, etc.) entering into the dispersion law for the high-frequency waves.

1.6. Averaged dynamical equations [9]

As shown in the previous section, the description of interacting waves is markedly simplified if they have close wavevectors. It stands to reason that in this case they have close frequencies. The inverse is not true, generally speaking. Thus, all Langmuir waves with the dispersion law

$$\omega_{\boldsymbol{k}} = \omega_{\mathrm{p}}(1 + \tfrac{3}{2}k^2r_{\mathrm{D}}^2)$$

when $k^2r_{\mathrm{D}}^2 \ll 1$ have close frequencies although their wavevectors can differ by several orders. This narrowness of the frequency spectrum of Langmuir waves can be used as a small parameter significantly simplifying the description of non-linear interactions. In this case the interaction with ions should be taken into account.

As mentioned above, in a hydrodynamical description the ions may be obviously included into the Hamiltonian approach scheme. In so doing, however, some important kinetic effects are lost which are connected with the Landau damping on the ions of forced harmonics of Langmuir waves.

Therefore, in the scheme, described below, of averaging over a fast Langmuir frequency it is expedient to retain the kinetic ion description. In the cases when the ions can be described hydrodynamically, we will automatically change to simplified Hamiltonians of the type (1.4.10) or (1.4.3).

The averaging method is based on the fact that in a plasma with a weak magnetic field $\omega_H \ll \omega_p$ the harmonic oscillations with frequency ω_p are the quickest type of motion.

The plasma motions can be divided into two types: high-frequency electron oscillations and low-frequency ones involving ions. Below we will confine ourselves to the consideration of long-wave oscillations, $kr_D \ll 1$. This makes it possible to consider low-frequency motions as quasi-neutral and to describe in the terms of hydrodynamics high-frequency motions whose phase velocities considerably exceed thermal ones. The interaction of high-frequency oscillations will be neglected, which allows us to describe them using the linearized hydrodynamical equations for an electron gas

$$\begin{aligned} &\frac{\partial}{\partial t}\delta n_e + \operatorname{div}(n_0 + \delta n)\boldsymbol{v}_e = 0 \\ &\frac{\partial}{\partial t}\boldsymbol{v}_e + 3v_{T_e}^2 \nabla \frac{\delta n_e}{n_0} = -\frac{e}{m}\boldsymbol{E}. \end{aligned} \tag{1.6.1}$$

These equations can be complemented by Maxwell's equations from which the magnetic field

$$\partial^2 \boldsymbol{E}/\partial t^2 + c^2 \operatorname{curlcurl} \boldsymbol{E} - 4\pi e(n_0 + \delta n)\, \partial \boldsymbol{v}_e/\partial t = 0 \tag{1.6.2}$$

is eliminated.

In (1.6.1), (1.6.2) the electron density is imagined in the form

$$n = n_0 + \delta n_e + \delta n, \qquad \delta n_e, \delta n \ll n_0 .$$

Here δn and δn_e are the density variations connected with low-frequency and high-frequency motions, respectively. In (1.6.1) and (1.6.2) the terms of the order $(\delta n_e/\delta n)v/v_e$ are eliminated. From the continuity equation it is seen that as to order of magnitude this is the ratio of the phase velocities of the low- and high-frequency motions $c_s k/\omega_p \sim kr_D\sqrt{m/M} \ll 1$. Before making further considerations, it should be noted that in the non-linear terms and the terms describing the thermal dispersion, the linear relations can be used for connecting δn_e, $\boldsymbol{v}_e$. Taking this into account, it is not difficult to reduce (1.6.1) and (1.6.2) to the equation

$$\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\boldsymbol{E} + \operatorname{curlcurl} \boldsymbol{E} - \frac{3v_{T_e}^2}{c^2}\nabla \operatorname{div} \boldsymbol{E} + \frac{\omega_p^2}{c^2}\frac{\delta n}{n_0}\boldsymbol{E} = 0 . \tag{1.6.3}$$

In the linear approximation, when $\delta n = 0$, it describes Langmuir and electromagnetic waves with the dispersion laws

$$\omega_L^2 = \omega_p^2 + 3k^2 v_{T_e}^2; \qquad \omega_t^2 = \omega_p^2 + k^2 c^2 .$$

Now let us consider oscillations with a frequency close to the plasma one (for the Langmuir oscillations this means $kr_D \ll 1$, and for electromagnetic ones $kc \ll \omega_p$) and imagine the electric field in the form

$$\boldsymbol{E} = (\tilde{\boldsymbol{E}} \exp(-\mathrm{i}\omega t) + \text{c.c.}) . \tag{1.6.4}$$

Here $\tilde{E}$ is a slowly varying quantity $\partial\tilde{\boldsymbol{E}}/\partial t \ll \omega_p\tilde{\boldsymbol{E}}$. Substituting (1.6.4) into (1.6.3) and neglecting the second derivative, finally the following expression is obtained [31]:

$$-2\mathrm{i}\omega_p \frac{\partial \tilde{\boldsymbol{E}}}{\partial t} + c^2 \,\mathrm{curlcurl}\, \tilde{\boldsymbol{E}} - 3v_{T_e}^2 \nabla \,\mathrm{div}\, \tilde{\boldsymbol{E}} + \frac{\omega_p^2}{n_0}\delta n\, \tilde{\boldsymbol{E}} = 0\,. \tag{1.6.5}$$

Equation (1.6.5) is convenient for describing oscillations with a frequency close to the plasma frequency. Taking into account the intrinsic electron non-linearity in (1.6.1), (1.6.2) could lead to the excitation of oscillations at double plasma and zero frequencies which could lead, in turn, to the appearance of terms of the type $r_D^2 \nabla^2 EE/nT$ in (1.6.5). They are negligibly small if the characteristic time of the non-linear processes following from (1.6.5) satisfies a rather soft condition

$$\frac{1}{\tau} \gg \omega_p \frac{\overline{\tilde{E}^2}}{mnv_{ph}^2} \sim \omega_p \frac{\tilde{E}^2}{8\pi nT}(kr_D)^2$$

(v_{ph} is a characteristic phase velocity).

Besides, it should be noted that in (1.6.5) the quantity $(v_{T_e}/c)^2$ is a small parameter allowing the separation of potential and non-potential oscillations. Assuming that $\tilde{\boldsymbol{E}} = \nabla\psi$ and taking the divergence of both parts of (1.6.5) we obtain

$$\nabla^2\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{3}{2}\frac{v_{T_e}^2}{\omega_p}\nabla^2\right)\psi = \omega_p \,\mathrm{div}\, \frac{\delta n}{2n_0}\nabla\psi\,. \tag{1.6.6}$$

Equation (1.6.6) conserves the integral $I = \int |\nabla\psi|^2 \,\mathrm{d}\boldsymbol{r}$ coinciding, apart from a multiplying factor, with the number of Langmuir plasmons (see section 1.4). Equation (1.6.5) conserves the analogous integral $\int |\tilde{\boldsymbol{E}}|^2 \,\mathrm{d}\boldsymbol{r}$ having the meaning of the total number of Langmuir and electromagnetic plasmons.

To close (1.6.5) it is necessary to find another connection between δn and $\tilde{\boldsymbol{E}}$. For this purpose it should be noted that the phase velocities of the electrons taking part in low-frequency motions are considerably less than the thermal velocities, and they can be described in hydrodynamical terms and considered stationary:

$$\overline{(\boldsymbol{v}_e\nabla)\boldsymbol{v}_e} = \frac{e}{m}\nabla\varphi_{el} = -\frac{e}{mc}\overline{[\boldsymbol{v}\times\boldsymbol{H}]} + \frac{T_e}{m}\frac{\nabla n}{n_0}\,. \tag{1.6.7}$$

Here the bar means averaging over time, and φ_{el} is the electrostatic potential of the low-frequency motions. Using the identity $(\boldsymbol{v}\nabla)\boldsymbol{v} = \frac{1}{2}\nabla v^2 - [\boldsymbol{v}\times \mathrm{curl}\,\boldsymbol{v}]$ and Maxwell's equation $(1/c)\partial\boldsymbol{H}/\partial t = -\mathrm{curl}\,\boldsymbol{E}$, we obtain

$$\overline{(\boldsymbol{v}_e\nabla)\boldsymbol{v}_e + \frac{e}{mc}[\boldsymbol{v}_e\times\boldsymbol{H}]} = \tfrac{1}{2}\nabla\overline{v_e^2} = \frac{e^2}{4m^2\omega_p^2}\nabla|\tilde{\boldsymbol{E}}|^2 = \frac{1}{m}\nabla\phi\,. \tag{1.6.8}$$

Thus, it is evident that high-frequency oscillations lead to the appearance of force having a potential ϕ (Miller's force), and pushing out the electrons from the region of the electric field localization. It should be underlined that this force acts on electrons only (the corresponding force acting on ions is m/M

times smaller). As regards (1.6.7), it described the Boltzmann distribution of electrons,

$$\frac{\delta n}{n_0} = \frac{1}{T_e}(e\varphi_{el} - \phi) \tag{1.6.9}$$

for which a thermodynamical equilibrium has time to be established due to the slowness of low-frequency motions. The ion distribution function obeys Vlasov's equation in the potential φ_{el}:

$$\frac{\partial f_i}{\partial t} + (\boldsymbol{v}\nabla)f_i - \frac{e}{M}\left(\nabla\varphi_{el}\frac{\partial f_i}{\partial \boldsymbol{v}}\right) = 0\,. \tag{1.6.10}$$

The quasi-neutrality conditions

$$\delta n_i = \int f_i\, d\boldsymbol{r} - n_0 = \delta n = \frac{n_0}{T_e}(e\varphi_{el} - \phi)$$

allow φ_{el} to be determined and thus the system of equations (1.6.5), (1.6.10) to be closed.

The equation (1.6.10) takes into account a non-linear interaction of low-frequency waves which in the majority of cases can be neglected. After linearization of (1.6.10) the variation of the density δn can be expressed linearly by the high-frequency force potential $\phi(\boldsymbol{r}, t)$. This connection can be expressed in terms of the dielectric tensor; however, it is more convenient to introduce a plasma Green function $G_{\kappa\Omega}$, defining it by the relations between Fourier images

$$\delta n_{\kappa\Omega} = \frac{n_0}{T_e} G_{\kappa\Omega}\phi_{\kappa\Omega} \simeq \frac{n_0}{T_e}\phi_{\kappa\Omega}\left(\frac{\varepsilon_e}{\varepsilon} - 1\right). \tag{1.6.11}$$

Here ε is the longitudinal part of a dielectric tensor, and ε_e is the electron contribution to it. For $G_{\kappa\Omega}$ from (1.6.9), (1.6.10) it follows that

$$G_{\kappa\Omega} = \frac{T_e}{Mn_0}\frac{L_{\kappa\Omega}}{1 - (T_e/Mn_0)L_{\kappa\Omega}}; \qquad L_{\kappa\Omega} = \int \frac{\boldsymbol{\kappa}\, \partial f_{0i}/\partial \boldsymbol{v}}{\boldsymbol{\kappa v} - \Omega}\, d\boldsymbol{v}\,. \tag{1.6.12}$$

The Green function possesses obvious symmetry properties analogous to those of ε:

$$G_{\kappa\Omega} = G^*_{\kappa-\Omega} = G_{-\kappa\Omega}\,.$$

What is more, since it is expressed through ε, $G_{\kappa\Omega}$ it is also analytical in the upper half-space of the variable Ω.

In some cases the system of equations (1.6.5)–(1.6.11) can be considerably simplified. If the characteristic times of all the processes are rather great $\tau^{-1} \ll kv_{T_i}$, the ion distribution in the low-frequency electric field can be considered as a Boltzmann distribution:

$$\delta n/n_0 = -e\varphi_{el}/T_i \ll 1\,.$$

With the help of a quasi-neutrality condition from (1.6.8) it follows

$$\frac{\delta n}{n_0} = -\frac{\phi}{T_e + T_i} = \frac{|\tilde{E}|^2}{16\pi n_0(T_e + T_i)}.$$

In the potential case the equation (1.6.6), within the framework of the above-mentioned "static" approximation, is of the form

$$\nabla^2(\mathrm{i}\psi_t + \tfrac{3}{2}\omega_p r_D^2 \nabla^2\psi) + \frac{\omega_p}{32\pi n_0(T_e + T_i)} \operatorname{div}|\nabla\psi|^2 \nabla\psi = 0. \tag{1.6.13}$$

From this equation the following estimate follows:

$$1/\tau \sim \omega_p W/nT \sim \omega_p k^2 r_D^2; \qquad W \sim \tilde{E}^2/8\pi.$$

From this the applicability conditions for (1.6.13) follow:

$$\frac{W}{nT} \ll \frac{m}{M}\frac{T_i}{T_e}, \qquad (kr_D)^2 \ll \frac{m}{M}\frac{T_i}{T_e}.$$

In the opposite limiting case $\tau^{-1} \gg kv_{T_i}$ for low-frequency motions the following hydrodynamical description is valid:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\delta n = \frac{1}{16\pi M}\nabla^2|\tilde{E}|^2; \qquad c_s^2 = \frac{T_e + \frac{5}{3}T_i}{M}. \tag{1.6.14}$$

In a non-isothermal plasma $T_e \gg T_i$ (1.6.14) is applicable at all amplitudes of the field; in the long-wave limit $k^2 r_D^2 < (m/M)T_i/T_e$ for small intensive oscillations $W/nT < (m/M)T_i/T_e$ the statistic equation (1.6.13) follows from (1.6.14). In an isothermal plasma $T_i \sim T_e$ equation (1.6.14) is valid for describing turbulence with a high noise level $W/nT > (m/M, k^2 r_D^2)_{\max}$, when the plasma motion becomes supersonic under the pressure of a high-frequency field. In this case the term $c_s^2\nabla^2\delta n$ in (1.6.14) can be neglected. The simple asymptotics $G_{\kappa\Omega}$ correspond to the simplified equations (1.6.13), (1.6.14). First of all, it should be noted that $G_{\kappa\Omega}$ is a function of the parameter $\xi = \Omega/\kappa v_{T_i}$. In the limit $\xi \ll 1$ or $\Omega \ll \kappa v_{T_i}$ we have

$$G_{\kappa\Omega} = -T_e/(T_e + T_i). \tag{1.6.15}$$

In the hydrodynamical limit $\xi \gg 1$ or $\Omega \gg \kappa v_{T_i}$ $G_{\kappa\Omega}$ has a pole corresponding to ion-sonic waves. Expanding (1.6.12), we obtain

$$G_{\kappa\Omega} = \frac{\kappa^2 c_s^2}{\Omega^2 - \kappa^2 c_s^2 + 2\mathrm{i}\gamma_s \cdot \Omega}. \tag{1.6.16}$$

When compared with (1.6.14), the Green function accounts for the sonic wave damping

$$\gamma_s = \kappa c_s \left(\frac{\pi}{8}\frac{m}{M}\right)^{1/2} . \qquad (1.6.17)$$

In (1.6.16) only the Landau ion damping is directly taken into account; however, within the framework of the above-mentioned scheme it is not difficult to account for the Landau electron damping. Presented in fig. 1.2 is the plot of the real and imaginary parts of $G_{\kappa\Omega}$ at arbitrary ξ. The plots are presented for

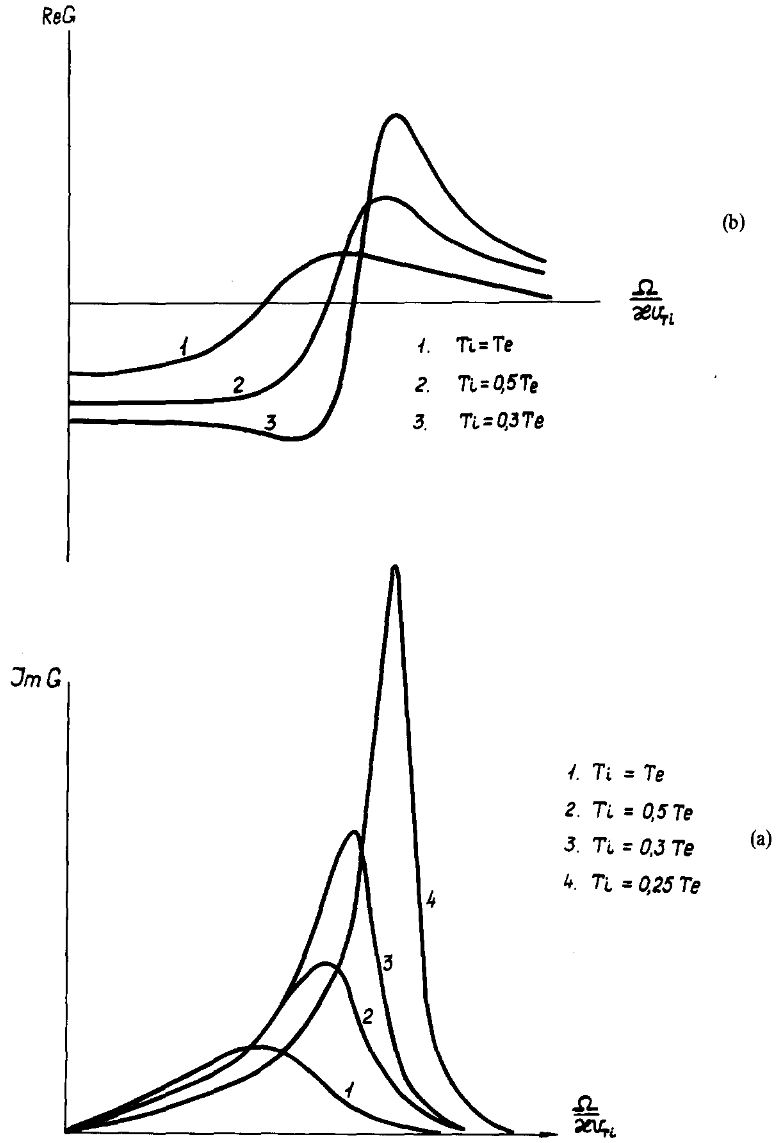

Fig. 1.2. A plot of the Green function of the real and imaginary parts for different T_e/T_i. (a) Green function (imaginary part) Im $G(x) = -\text{Im } G(-x)$; $x = \Omega/kv_{T_i}$. (b) Green function (real part) Re $G(x) = \text{Re } G(-x)$.

different ratios of the electron and ion temperatures, which, as can be seen, influence significantly the form of the Green function when $\xi \sim 1$.

Finally, in the last variant of the simplifications of the dynamical equations valid for a sufficiently strong damping of ion-sonic oscillations we can consider low-frequency motions as forced.

The relation (1.6.12) can be rewritten in the form

$$\delta n_{\kappa\Omega} = \frac{n_0 e^2 (2\pi)^{-2}}{4m\omega_p^2 T_e} \int G_{k_1-k_2,\Omega}(\tilde{E}_{k_1,\omega_1}\tilde{E}^*_{k_2,\omega_2})\,\delta(k_1-\kappa-k_2)\,\delta(\omega_1-\omega_2-\Omega)\,dk_1\,dk_2\,d\omega_1\,d\omega_2\,. \tag{1.6.18}$$

It is obvious that at a low level of non-linearity we have $\tilde{E}_{k,\omega} \simeq \tilde{E}_k \delta(\omega-\omega_k)$, ω_k is the law of wave dispersion reckoned from the plasma frequency. With this accuracy the inverse Fourier time transform can be made in (1.6.18):

$$\delta n_\kappa(t) = \frac{(2\pi)^{-3/2}}{16\pi n T_e} \int G_{k_1-k_2,\,\omega_1-\omega_2}(\tilde{E}_{k_1}\tilde{E}^*_{k_2})\,\delta(k_1-\kappa-k_2)\,dk_1\,dk_2\,. \tag{1.6.19}$$

Considering the oscillations to be potential, let us introduce the variable

$$a_k = \mathrm{i}(8\pi\omega_p)_k^{-1/2}\psi_k, \qquad E_k = -\mathrm{i}k\psi_k \tag{1.6.20}$$

determined in such a manner that the value

$$\int \omega_k |a_k|^2\,dk$$

would coincide with the total energy of Langmuir oscillations. Substituting (1.6.19) into (1.6.6), we obtain finally

$$\mathrm{i}\frac{\partial a_k}{\partial t} + (\omega_k + \mathrm{i}\gamma_k)\,a_k = \mathrm{i}\int T_{kk_1k_2k_3} a^*_{k_1} a_{k_2} a^*_{k_3}\,\delta(k+k_1-k_2-k_3)\,dk_1\,dk_2\,dk_3$$

$$T_{kk_1k_2k_3} = \frac{\omega_p^2}{(2\pi)^3 4nT_e}\left[\frac{(kk_2)(k_1k_3)\,G((\omega_1-\omega_3)/|k_1-k_3|) + G((\omega_1-\omega_2)/|k_1-k_2|)(kk_3)(k_1k_2)}{kk_1k_2k_3}\right]. \tag{1.6.21}$$

The plasma oscillation damping which can be considered to be collisional $\gamma_k \simeq \nu_{ei}$ is included into (1.6.21). The matrix element $T_{kk_1k_2k_3}$ in (1.6.21) possesses the symmetry properties following from the symmetry relations for the Green function:

$$T_{kk_1k_2k_3} = T^*_{k_2k_3kk_1}$$

when

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}\,.$$

It should be noted that in the above-considered static approximation

$$T_{kk_1k_2k_3} = \frac{\omega_p^2}{(2\pi)^3 4n(T_e + T_i)} \left\{ \frac{(kk_2)(k_1k_3) + (kk_3)(k_1k_2)}{kk_1k_2k_3} \right\} \tag{1.6.22}$$

satisfies the symmetry relations (1.1.22). Thus, (1.6.13) is a Hamiltonian of the type (1.4.7). Certainly, this can easily be checked. If, simultaneously with the substitution (1.6.20), we change to the normal variables (1.5.13) for the sound oscillations, the system of equations (1.6.16), (1.6.14) is reduced to the form (1.4.13), (1.4.14), where the matrix interaction element is

$$V_{kk_1k_2} = \frac{1}{(2\pi)^{3/2}} \frac{\omega_p}{2\sqrt{2Mnc_s}} \sqrt{k} \frac{(k_1k_2)}{k_1k_2}. \tag{1.6.23}$$

As mentioned above, the real and imaginary parts of the Green function $G_{\kappa\Omega}$ quickly decreases if $\Omega \gg \kappa v_{T_i}$. Therefore, when $kr_D \gg \sqrt{m/M}$ (1.6.21) shows that only the oscillations with close wavevectors interact with one another. The condition $(\omega_{k_1} - \omega_{k_3})/|k_1 - k_3| \sim 1$ gives $|k_1| - |k_3| \sim r_D^{-1}\sqrt{mT_i/MT_e} \equiv k_{dif}$. Here the quantity $k_{dif} \equiv r_D^{-1}\sqrt{mT_i/MT_e}$ is introduced which has the meaning of a characteristic size of the matrix interaction element.

For the validity of (1.6.21) it is necessary that the non-linear corrections in the arguments of the Green function would be negligibly small. In the region of the spectrum $k^2r_D^2 < m/M$ when the Langmuir oscillations cannot excite sound, this condition is of the form

$$1/\tau \sim \omega_p\, W/nT \sim (kr_D)^2\, \omega_p < kv_{T_i}.$$

That is, in this case (1.6.21) makes the static approximation equations more precise. When $k^2r_D^2 > m/M$, for the validity of (1.6.21) it is necessary that all the sonic oscillations would be forced, that is, all characteristic times τ would exceed the sound damping time $\gamma_s\tau > 1$.

Using the expression (1.6.16) for the Green function in a hydrodynamical approximation, from (1.6.21) we obtain for a characteristic time of a non-linear process

$$\tau^{-1} \sim \omega_p \frac{\tilde{W}}{nT} \frac{\omega_s}{\gamma_s}.$$

Here $\tilde{W}$ is the energy density within the interval of wavevectors of the order of the Green function size. If the noise density is uniformly distributed over the scale k, then $\tilde{W} \simeq Wk_{dif}/k$, the applicability condition takes the form

$$\frac{W}{nT} \frac{k_{dif}}{k} \ll kr_D \sqrt{\frac{m}{M}} \left(\frac{\gamma_s}{\omega_s}\right)^2. \tag{1.6.24}$$

In particular, in the isothermal plasma, where $\gamma_s \sim \omega_s$, the condition (1.6.24) is in the form

$$W/nT < k^2r_D^2.$$

Listed in the table are the values of the ratio γ_s/ω_s for different ratios of electron and ion temperatures:

T_e/T_i	2.1	1.4	0.9	0.8	0.64	0.5	0.36	0.25	0.16	0.1
γ_s/ω_s	0.65	0.52	0.39	0.36	0.3	0.24	0.17	0.1	5×10^{-2}	4.2×10^{-2}

Above, when deriving (1.6.21), the electric field was considered to be potential. In actual fact, the plasma inhomogeneity, arising from high-frequency pressure, mixes polarizations, that results in converting plasma oscillations into electromagnetic ones with close frequency $\omega_k - \omega_p \ll \omega_p$ and vice versa. To obtain the equations describing the above-mentioned process, the electric field can be represented as

$$\boldsymbol{E}_k = \mathrm{i}(8\pi\omega_p)^{1/2} \sum_\lambda \boldsymbol{S}_{k_\lambda} a_{k_\lambda} \tag{1.6.25}$$

where $\boldsymbol{S}_{k_\lambda}$ are the unit polarization vectors, for the Langmuir waves $\boldsymbol{S}_{k_\lambda} = \boldsymbol{k}/k$, and for the electromagnetic waves $\boldsymbol{S}_{k_\lambda}$ satisfy the conditions

$$\boldsymbol{S}^\lambda_{\boldsymbol{k}} = \boldsymbol{S}^{*\lambda}_{-\boldsymbol{k}}; \qquad \boldsymbol{S}_{k_\lambda}\boldsymbol{S}^*_{k_{\lambda'}} = \delta_{\lambda\lambda'}; \qquad (\boldsymbol{k}\boldsymbol{S}_{k_\lambda}) = 0\,.$$

Then, substituting (1.6.25) into (1.6.5), we obtain

$$\frac{\partial a_{k_\lambda}}{\partial t} + \mathrm{i}\Omega_{k_\lambda} a_{k_\lambda} = \sum_{\lambda'} \int (\boldsymbol{S}^\lambda_{\boldsymbol{k}} \boldsymbol{S}^{*\lambda'}_{\boldsymbol{k}_1})\, \delta n_{\boldsymbol{\kappa}}\, a_{k'_{\lambda'}}\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{\kappa})\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{\kappa}\,.$$

In these equations the fundamental frequencies are expressed as:

$$\Omega_{k_{1,2}} \equiv \Omega_k = \tfrac{1}{2}k^2c^2/\omega_p; \qquad \Omega_{k_3} \equiv \omega_k = \tfrac{3}{2}\omega_p k^2 r_D^2\,.$$

Eliminating δn_{κ}, it is possible to obtain an equation generalizing (1.6.21). The conditions of its applicability are analogous to the conditions of applicability of (1.6.21).

1.7. Averaged description of the oscillations of magnetized plasma

Now let us consider a magneto-active plasma. The magnetic field leads to the appearance of new oscillation branches, changes the wave dispersion law and makes the description of the interactions more complicated. Nevertheless, here too the separation of a definite high-frequency oscillation branch significantly simplifies the description of non-linear effects and allows changing to the averaged description.

It is not difficult to take into account the effect of a weak magnetic field on Langmuir and electromagnetic oscillations with a frequency close to the plasma frequency, and to elucidate how the equations obtained in the previous section are modified. For this purpose, it is enough to take into account the Lorentz force $\boldsymbol{F} = (e/c)[\boldsymbol{v}\boldsymbol{H}]$ in the electron equation of motion (1.6.1). In virtue of the smallness of the magnetic field it is possible to assume, with an accuracy up to the quadratic terms in ω_H/ω_p, that

$$F = -\frac{e}{c}[vH] \simeq -\frac{ie^2}{mc\omega_p}[EH]. \tag{1.7.1}$$

As a result, (1.6.5) is obtained with an addition conditioned by the magnetic field

$$\begin{aligned} &iE_t - \frac{i}{2}\frac{\omega_H}{\omega_p}[hE] + \frac{c^2}{2\omega_p}\,\mathrm{curlcurl}\,E - \tfrac{3}{2}\omega_p r_D^2\,\mathrm{grad\,div}\,E + \omega_p\frac{\delta n}{2n_0}E = 0, \\ &h = H/H. \end{aligned} \tag{1.7.2}$$

Equation (1.7.2) is suitable for the description of long-wave $kc \lesssim \omega_p$ oscillations. To describe the influence of the magnetic field on the short-wave $kc \gg \omega_p$ Langmuir oscillations, it is necessary to take into account the terms quadratic in ω_H (in eq. (1.7.1)). Assuming the electric field to be almost potential $E = \nabla\psi$, we obtain a generalization of (1.6.6) taking into account a weak magnetic field

$$\nabla^2(i\psi_t + \tfrac{3}{2}\omega_p k^2 r_D^2 \nabla^2\psi) - \frac{\omega_H^2}{2\omega_p}\nabla_\perp^2\psi - \omega_p\frac{\mathrm{div}\,\delta n\,\nabla\psi}{2n_0} = 0. \tag{1.7.3}$$

Since the magnetic field does not influence the ion motion, the equations describing the low-frequency motions are unchanged. Therefore, it is evident that the structure of (1.6.12), (1.6.19) in which the magnetic field leads only to a change in the dispersion law of the oscillations, remains unchanged. It should be noted also, that (1.7.2), (1.7.3) conserve the total number of waves.

In a strong magnetic field $\omega_H \gtrsim \omega_p$ an essential reconstruction of the oscillation spectrum takes place.

We shall restrict ourselves to the consideration of two branches of potential oscillations $kc \gg \omega_p$ being described by the dispersion equation (1.3.22). In this case also we shall consider a transverse propagation of oscillations when the frequency of a lower oscillation branch becomes closer to the lower-hybrid one, $\omega_{LH} = \omega_H^2\omega_{p_i}/(\omega_H^2 + \omega_p^2)$, and the ion motion affects the dispersion law in an essential way.

As seen from (1.3.22), the oscillation frequencies change within wide limits with a change in the wavevector direction. Therefore, as a rule, there is no necessity to take into account the corrections to the law of wave dispersion connected with the thermal motion of particles.

High-frequency oscillations are described by the linearized hydrodynamical equations

$$\begin{aligned} &\frac{\partial}{\partial t}\delta n_e + \mathrm{div}(n_0 + \delta n_e)v_e = 0 \\ &\frac{\partial}{\partial t}v_e = -\frac{e}{m}E - \frac{e}{c}[v_e H] \\ &E = -\nabla\varphi_{el} \end{aligned} \tag{1.7.4}$$

as in the case of an isotropic plasma. Here the electron non-linearities are omitted which have growth rates, as in the case of an isotropic plasma, which have an additional small factor $\propto k^2 r_D^2$. However, other processes, for example the decay processes inside the lower oscillation branch, can certainly not always be neglected. Besides, in (1.7.4) we neglect the high-frequency wave plasma entrainment, i.e. non-linear terms which incorporate the velocity of the slow motions. In accordance with the general results of chapter 1, they make an essential contribution, when the law of wave dispersion does not depend on the density (e.g., for a weak magnetic field $\omega_p > \omega_H$, $\omega \simeq \omega_H |\cos \Theta|$). In these cases the equations become more complicated although the general scheme remains the same. Appropriate calculations were carried out in a paper by B.I. Sturman [10].

Dynamical equations describing the interaction between oscillations and low-frequency motions have the simplest form in the $\boldsymbol{k}$-representation. We will confine ourselves to the investigation of the analogues of (1.6.21), since in the coordinate representation the complicated description is not justified by some interesting physical applications. The region of frequencies close to lower-hybrid one ω_{LH} is an exception. It will be analysed separately.

Now let us change to the canonical variables $a_{\boldsymbol{k}}$ introduced in section 1.3. Then it is possible to obtain the analogue of (1.6.6) from the system (1.7.5):

$$\left(\frac{\partial}{\partial t} + \mathrm{i}\omega_{\boldsymbol{k}}^{\pm}\right) a_{\boldsymbol{k}}^{\pm} = -\frac{\mathrm{i}}{2} \int \left| \frac{\omega_{\boldsymbol{k}_1}^{\pm 2} - \omega_H^2}{\omega_{\boldsymbol{k}_1}^{\pm 2} - \omega_{\boldsymbol{k}_1}^{-2}} \right| (\boldsymbol{k}\boldsymbol{u}_{\boldsymbol{k}_1}^{\pm})\, \delta n_{\boldsymbol{\kappa}}\, a_{\boldsymbol{k}_1}^{\pm} \frac{k_1}{k} \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{\kappa})\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{\kappa}\,. \tag{1.7.5}$$

It should be noted that the superscripts $\pm$ belong to the upper and lower oscillation branches, respectively, and the notation introduced in section 1.3 is used. When deriving (1.7.5), we omitted the terms containing the small frequency change during the interaction $\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'} \ll \omega_{\boldsymbol{k}}$.

Derivation of the equation describing the plasma density variation affected by ponderomotive forces is more complicated, as compared to the previous section, due to the necessity to take into account the scattering by electrons.

Let us divide the electron distribution function f_e into quickly oscillating and slowly oscillating parts ($\tilde{f}$ and $\bar{f}$ respectively). Analogously, for an electric potential $\varphi = \tilde{\varphi}_e + \bar{\varphi}$; $\boldsymbol{E} = -\nabla\varphi$ we have

$$\frac{\partial \tilde{f}}{\partial t} + (\boldsymbol{v}\nabla)\tilde{f} - \omega_H[\boldsymbol{v}\boldsymbol{h}]\tilde{f} + \frac{e}{m}\frac{\partial f_0}{\partial \boldsymbol{v}}\nabla\tilde{\varphi} = 0 \tag{1.7.6}$$

$$\frac{\partial \bar{f}}{\partial t} + (\boldsymbol{v}\nabla)\bar{f} - \omega_H[\boldsymbol{v}\boldsymbol{h}]\bar{f} + \frac{e}{m}\nabla\bar{\varphi}\frac{\partial f_0}{\partial \boldsymbol{v}} = -\frac{e}{m}\overline{\nabla\tilde{\varphi}\frac{\partial \tilde{f}}{\partial \boldsymbol{v}}}\,. \tag{1.7.7}$$

A line above the terms on the right-hand side of (1.7.7) denotes the averaging over time, and the term itself describes the action of ponderomotive forces.

The ion distribution function f_i is described by an analogous equation

$$\frac{\partial f_i}{\partial t} + (\boldsymbol{v}\nabla)f_i + \left(\omega_{H_i}[\boldsymbol{v}\boldsymbol{h}] - \frac{e}{M}\nabla\bar{\varphi}\right)\frac{\partial f_{0i}}{\partial \boldsymbol{v}} = 0\,. \tag{1.7.8}$$

The linear equations (1.7.6)–(1.7.8) are easily integrated (see [1, 2]). Assuming $kr_H \ll 1$ (r_H is the Larmor radius), it is possible to obtain, as a result of some cumbersome though simple, calculations

$$\delta n_{\boldsymbol{k}\Omega} = G_{\boldsymbol{k}\Omega}\phi_{\boldsymbol{k}\Omega} n_0 / T\,. \tag{1.7.9}$$

Here the Green function $G_{k\Omega}$ is prescribed, as previously, by the formula (1.6.11), $G_{k\Omega} = \varepsilon_e/\varepsilon - 1$ where by ε is meant the longitudinal part of the dielectric constant, and the high-frequency potential has a more complicated structure than in an isotropic plasma,

$$\phi_{k\omega} = \frac{T_e}{n_0}\int a^{\pm}_{q_1} a^{\pm *}_{q_2} \frac{(k_1 u_{k_2})}{k_1^2 r_D^2 \omega_1} \delta(q - q_1 - q_2)\,dq_1\,dq_2\, \alpha^{\pm}_{q_1}\alpha^{\pm}_{q_2}$$

$$q = (k, \omega); \qquad \alpha^{\pm}_k = \frac{k\omega_{\pm}^{1/2}}{\omega_p (2\rho_0)^{1/2}} \left|\frac{\omega_{\pm}^2 - \omega_H^2}{\omega_+^2 - \omega_-^2}\right|^{1/2}. \tag{1.7.10}$$

If the ions are not magnetized, $kv_{T_i} \gg \omega_{H_i}$, the Green function coincides with (1.6.12). In the opposite limiting case and in a non-isothermal plasma $G_{k\Omega}$ has two poles corresponding to the two low-frequency eigen oscillations:

$$\Omega_{\pm} = \tfrac{1}{2}[\omega_{H_i}^2 + \kappa^2 c_s^2 \pm \{(\omega_{H_i} + \kappa^2 c_s^2)^2 - 4\omega_{H_i}^2 \kappa^2 c_s^2\}^{1/2}]$$

$$G_{\kappa\Omega} = \frac{\kappa^2 c_s^2(\Omega^2 - \omega_{H_i}^2 \cos^2\Theta)}{(\Omega^2 - \Omega_+^2)(\Omega^2 - \Omega_-^2)}. \tag{1.7.11}$$

If $T_e \sim T_i$, we may use again (1.6.12) assuming the ion motion to be one-dimensional.

At very nearly transverse propagation of oscillations, the interaction between oscillations and ion-cyclotron waves can be significant. It is not difficult to write the appropriate expression for $G_{k\omega}$, because the expression for a dielectric constant in this case is presented in many books (see, for example, [1, 2]). It should be only noted that for ion-cyclotron waves we can also introduce canonical variables [10] and describe their interaction within the framework of the above-mentioned formalism.

For nearly transverse propagation, the induced scattering by electrons becomes significant, too. In this case, for non-magnetized ions the expression generalizing (1.6.12) is obtained:

$$G_{k\omega} = \frac{T_e L_e L_i}{T_e L_i + T_i L_e}$$

$$L_e = \int \frac{k_z v_z f_e^0\, dv_z}{k_z v_z - \omega}; \qquad L_i = \int \frac{(kv) f_{0i}\, dv}{kv - \omega}. \tag{1.7.12}$$

When the oscillation frequency approaches the lower hybrid one, it is important to take into account the ion motion, and the system (1.7.4) must be supplemented by the equations

$$\frac{\partial v_i}{\partial t} = \frac{e}{M}E = -\frac{e}{M}\nabla\varphi_e, \qquad \frac{\partial}{\partial t}\delta n_i + \operatorname{div} n_0 v_i = 0$$

$$\Delta\varphi_e = 4\pi e(\delta n_e - \delta n_i). \tag{1.7.13}$$

In the linear approximation the equations (1.7.13), (1.7.4) describe oscillations with a dispersion law

$$\omega_k^2 = \frac{\omega_H^2\omega_{pi}^2}{\omega_p^2+\omega_H^2}\left(1+\cos^2\Theta\,\frac{M}{m}\right) = \omega_{LH}^2\left(1+\cos^2\Theta\,\frac{M}{m}\right) \tag{1.7.14}$$

which pass to the branch ω^-, when $\cos\Theta > \sqrt{m/M}$.

It is not difficult to establish that in this case the connection between the canonical variables a_k and the density variation δn_i takes the form

$$a_k = \frac{\omega_p}{\omega_H}\frac{\omega_k^2}{\omega_{pi}^2}\sqrt{\frac{2n_0 m(\omega_p^2+\omega_H^2)}{\omega_k k^2}}\,\frac{\delta n_i}{n_0}\,. \tag{1.7.15}$$

The equation (1.7.5) retains its form, but, using the condition $\cos\Theta < \sqrt{m/M}$, it can be greatly simplified:

$$\left(\frac{\partial}{\partial t}+\mathrm{i}\omega_k\right)a_k = -\frac{\mathrm{i}}{2}\int\frac{\omega_H^2 k_1}{(\omega_H^2+\omega_p^2)k}(\boldsymbol{k}\boldsymbol{u}_{k_1})\frac{\delta n_\kappa}{n_0}\,a_{k_1}\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{\kappa}$$

$$\boldsymbol{u}_k \cong \mathrm{i}\frac{\omega_p^2}{k^2\omega_H}[\boldsymbol{k}\boldsymbol{h}] \tag{1.7.16}$$

and the connection δn_κ and a_k is given by the expression

$$\frac{\delta n_{k\omega}}{n_0} = G_{k\omega}\frac{\omega_k^2}{\omega_p^2+\omega_H^2}\int\frac{k_2}{k_1}\frac{\boldsymbol{k}_1\boldsymbol{u}_{k_2}}{2n_0T}\,a_{q_1}a_{q_2}\,\delta(q_1+q_2-q)\,\mathrm{d}q_1\,\mathrm{d}q_2\,. \tag{1.7.17}$$

In this case $G_{k\omega}$ is determined by the expression (1.7.12).

The relations (1.7.15)–(1.7.17) involve a full description of plasma turbulence with a frequency close to ω_{LH}; however, they are rather complicated. On the other hand, when $\cos\Theta < \sqrt{m/M}$, the dispersion law of the oscillations contains a high constant frequency ω_{LH}, that allows the analogue of (1.6.6) to be obtained by averaging over the high frequency. However, it will be more illustrative to obtain it directly from the hydrodynamics equation, and not by the simplification of (1.7.16). Let us introduce a positive frequency part of the electric potential

$$\boldsymbol{E} = \tfrac{1}{2}(\nabla\psi\exp(-\mathrm{i}\omega_{LH}t)+\text{c.c.})$$

and, taking into account that for such small frequencies the electron velocity is determined by their drift in the electric field

$$\boldsymbol{v} = c[\nabla\psi\boldsymbol{H}]/H^2$$

we obtain

$$\nabla^2\left(\mathrm{i}\psi_t - \frac{\omega_{LH}}{2}\frac{M}{m}\frac{\partial^2\psi}{\partial z^2}\right) = 4\pi e\,\operatorname{div}\delta n\,\boldsymbol{v}\,. \tag{1.7.18}$$

For the angles $\cos\Theta \ll \sqrt{m/M}$, when the angular dispersion becomes small, it is necessary to account for thermal dispersion. Then the dispersion equation takes the form (see, for example, [1, 2])

$$\omega^2 = \omega_{\mathrm{LH}}^2(1 + \tfrac{1}{2}\cos^2\Theta\, M/m + \tfrac{1}{2}k^2R^2)$$
$$k^2R^2 = \begin{cases} 3k^2 r_{\mathrm{D}}^2\, T_e/T_i, & \omega_{\mathrm{H}} > \omega_{\mathrm{p}} \\ k^2 r_{\mathrm{H}}^2(\tfrac{3}{4} + 3T_i/T_e), & \omega_{\mathrm{H}} < \omega_{\mathrm{p}}\,. \end{cases} \tag{1.7.19}$$

Taking into account the thermal dispersion in (1.7.18) is carried out in the same manner as for Langmuir oscillations, we obtain as a result,

$$\nabla^2(\mathrm{i}\psi_t + \tfrac{1}{2}R^2\nabla^2\psi) - \frac{\omega_{\mathrm{LH}}}{2}\frac{M}{m}\frac{\partial^2\psi}{\partial z^2} = 4\pi e \,\mathrm{div}\,\delta n\, \boldsymbol{v}\,. \tag{1.7.20}$$

Slow motions of electrons along a magnetic field are caused by the ponderomotive force

$$f = \overline{(\boldsymbol{v}\nabla)\,v_z} = \frac{\mathrm{i}e}{m\omega_{\mathrm{H}}\omega_{\mathrm{LH}}}\frac{\partial}{\partial z}[\nabla\psi\,\nabla\psi^*]_z = \frac{\partial}{\partial z}\phi\,.$$

It is evident that, as in an isotropic plasma, it is potential.

Now let us consider a static approximation (see section 1.6), when the distribution of electrons and ions can be considered as Boltzmannian. Then δn is simply connected with ϕ

$$\delta n = -e\phi/(T_e + T_i)\,.$$

Finally we obtain

$$\nabla^2\left(\mathrm{i}\psi_t + \omega_{\mathrm{LH}}\frac{R^2}{2}\nabla^2\psi\right) - \frac{\omega_{\mathrm{LH}}}{2}\frac{M}{m}\Delta_z\psi - \frac{e^2\omega_{\mathrm{p}}^2}{2m\omega_{\mathrm{LH}}(T_e + T_i)(\omega_{\mathrm{p}}^2 + \omega_{\mathrm{H}}^2)}\mathrm{div}([\nabla\psi\,\nabla\psi^*]_z[\boldsymbol{h}\,\nabla\psi]) = 0\,. \tag{1.7.21}$$

The properties of this equation, the conditions of its applicability and physical situations described by it are discussed in detail in [10], for example.

2. Decay and modulational instabilities

2.0. Introduction

The non-linear equations describing wave interactions which were obtained in chapter 1 are very complicated. The simplest problem which can be solved using them is the problem of a monochromatic wave instability. Its solution allows the characteristic values of non-linear interaction times and scales of exciting oscillations to be obtained. On the one hand, the description of this instability within the framework of the equations for the normal amplitudes substantially simplifies calculations, because the solution of a linear problem in these variables is trivial. On the other hand, by virtue of the fact that all physical information is contained only in specific expressions for the matrix elements and dispersion laws, it is sufficient to find only once the connection between them and the instability parameters. For instance, the instability growth rates happen to be proportional to the value of matrix elements, and, therefore, tables 1.2 and 1.3 presented in chapter 1 can serve, in addition, as a table of the growth rates of various decay processes.

As far as we know, the decay processes are developed in the regions near the surfaces in $\boldsymbol{k}$-space, on which the decay conditions, i.e. the laws of conservation of energy-momentum for waves, are fulfilled. The width of these regions is defined by the wave amplitude. It is evident that the concept of waves as interacting quasi-particles, i.e. the concept of decay processes as well are valid only until the instability growth rate exceeds the minimum frequency of the interacting waves. In the opposite case there arises the intersection of decay regions, and complex modified instabilities occur.

The type of these instabilities essentially depends on the structure of the matrix elements. In this chapter they are investigated on the basis of the interaction of high-frequency waves with sound. A very important problem of the instability of a high-amplitude Langmuir wave is also investigated in detail. Instabilities arising under the action of a homogeneous high-frequency external field on a plasma, or the so-called parametric instabilities, are considered in a separate section.

In a homogeneous plasma the threshold for the decay instabilities is defined by a linear damping of the excited oscillations, i.e., the energy flux generated by pumping must be compensated by its dissipation.

In an inhomogeneous plasma the oscillations change their wavevector and get out of resonance with the pumping. Due to the narrowness of the resonance region, it is the energy carrying-out from this region, that defines the thresholds of the decay instabilities in a lot of experiments. Therefore in concluding this chapter the authors analyse the effect of inhomogeneities on decay instabilities. It should be noted that there are many questions of importance still unclarified, therefore we tried to describe a physical picture of the phenomenon in the simplest manner and have obtained an expression for the threshold via simple estimates.

2.1. Decay instabilities

Let us consider the problem of the stability of monochromatic waves with a small amplitude in the medium described by a three-wave interaction Hamiltonian (1.1.19) where only those terms are appreciable which do not contain fast time oscillations. If we restrict ourselves to the case of waves with a positive energy, the interaction Hamiltonian is of the form

$$\mathcal{H}^{(3)} = \int \{V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} a_{\boldsymbol{k}}^* a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} + \text{c.c.}\}\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 \tag{2.1.1}$$

and the interacting oscillations are observed near the surface

$$\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2} \tag{2.1.2}$$

$$\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2\,. \tag{2.1.3}$$

The equations of motion corresponding to the Hamiltonian (2.1.1) are of the form

$$\partial a_k/\partial t + \gamma_k a_k = -\mathrm{i}\,\delta(\mathcal{H}^0 + \mathcal{H}^{(3)})/\delta a_k^* \\ = -\mathrm{i}\left\{\omega_k a_k + \int [V_{kk_1k_2}\, a_{k_1}\, a_{k_2}\, \delta(k - k_1 - k_2) + 2V^*_{k_1kk_2} a_{k_1} a^*_{k_2} \delta(k_1 - k - k_2)]\, \mathrm{d}k_1\, \mathrm{d}k_2\right\}. \tag{2.1.4}$$

A damping of the oscillations with a rate γ_k is phenomenologically introduced. In (2.1.4) the monochromatic wave

$$a_k = a_0\delta(k - k_0)\exp\{-\mathrm{i}\,\omega(k_0)\,t\}, \qquad a_0 = (2\pi)^{3/2},$$

is an approximate solution of (2.1.4). The coefficient $(2\pi)^{3/2}$ is chosen so that the energy density would be in the form $\omega_{k_0}a_0^2$. Since Va_0 enters all the results, interesting from a physical point of view, in what follows the coefficients $(2\pi)^{3/2}$ will be omitted as in the expression for matrix elements and for monochromatic waves. It is seen from (2.1.4) that a monochromatic wave can interact with a small amplitude wave occurring very close to the surfaces

$$\omega_{k_0} = \omega_p + \omega_{k_0-p} \tag{2.1.5}$$

$$\omega_{k_0} = \omega_{k_1} - \omega_{k_1-k_0}\,. \tag{2.1.6}$$

If these surfaces are far from one another, both these processes can be considered separately. Conserving the terms corresponding to the process (2.1.5) in the equation for small amplitude waves, we obtain

$$\dot{c}_k + \gamma_k c_k = -\mathrm{i}\,V^*_{k_0k,\,k_0-k}\,a_0 c^*_{k_0-k}\exp(-\mathrm{i}\,\Delta\omega\, t)$$

$$\dot{c}_{k_0-k} + \gamma_{k_0-k} c_{k_0-k} = -\mathrm{i}\,V^*_{k_0k,\,k_0-k}\,a_0 c^*_k\exp(-\mathrm{i}\,\Delta\omega\, t)$$

$$\Delta\omega = \omega_{k_0} - \omega_k - \omega_{k_0-k}, \qquad c_k = a_k\exp(\mathrm{i}\omega_k t)\,.$$

These equations have solutions of the form

$$c_k = c_0\exp(-\mathrm{i}\,\Delta\omega/2\; t - \mathrm{i}\,\omega_k\, t), \qquad c^*_{k_0-k} \sim \exp(\mathrm{i}\,\Delta\omega/2\; t - \mathrm{i}\,\omega_k\, t)\,.$$

In order to avoid complicated formulae, let us consider firstly the case $\gamma_k = \gamma_{k_0-k} = 0$. Then

$$\omega_k = \sqrt{\tfrac{1}{4}(\Delta\omega)^2 - \gamma_0^2}\,; \qquad \gamma_0^2 = |V_{k_0k,\,k_0-k}\,a_0|^2\,. \tag{2.1.7}$$

Thus, the instability with the maximum growth rate $\gamma_{\max} = \gamma_0$ proportional to the oscillation amplitude develops near the surface (2.1.5). The condition $\Delta\omega = 0$ can be considered as a law of conservation of energy for interacting waves, and the instability as a process of the decay of the wave a_{k_0} into a_k and a_{k_0-k}. Therefore the instability (2.1.7) received the name "decay instability". The waves with frequencies ω_{k_0}, ω_k and ω_{k_0-k} may belong to different parts of a spectrum, and, nevertheless, the way of considering the instabilities and the resulting formulae do not alter.

The domain of the interaction near the decay surface $\Delta\omega = 2\gamma_0$ can be estimated from the uncertainty relation. The wave frequency which increases with the growth rate γ_0 is determined accurately to within γ_0.

The decay instability was first obtained as early as 1962 by V.N. Oraevsky and R.Z. Sagdeev [13] on the example of the decay of a Langmur oscillation into Langmuir and ion-sound oscillations. From (2.1.7) it is seen that the maximum growth rate of the decay instability is universally connected with the value of the matrix interaction element and the decaying wave energy $\mathscr{E}$ ($|a_0|^2 = \mathscr{E}/\omega_k$). Therefore, the table of matrix elements for different type interactions which is presented in chapter 1 can also serve as a table of growth rates of different decay instabilities.

The decay instability has a threshold associated with wave damping. Assuming for simplicity that $\Delta\omega = 0$, we obtain

$$\omega_k = \mathrm{i}[-\tfrac{1}{2}(\gamma_k - \gamma_{k_0-k}) + \sqrt{\tfrac{1}{4}(\gamma_k + \gamma_{k_0-k})^2 - \gamma_0^2}] \tag{2.1.8}$$

and for the threshold value $\gamma_0 = |V_{k_0k,\,k_0-k}|^2\,|a_0|^2$ we have

$$\gamma_0^2 = \gamma_k\gamma_{k_0-k}\,.$$

In a plasma with a sufficiently high ion temperature for decay processes involving high-frequency waves and sound the situation is typical when $\gamma_h < \gamma_0 < \gamma_s$, where γ_h and γ_s are the damping rates of the high-frequency and the sound wave, respectively. In this case it follows from (2.1.8) that for the instability growth rate we have

$$\gamma \sim \gamma_0^2/\gamma_s\,. \tag{2.1.9}$$

The condition $\gamma_s > \gamma_0$ is the condition of the applicability of the dynamical equations which are valid in an isothermal plasma also. It will readily be seen that (1.6.21) has a solution in the form of a monochromatic wave

$$a_k = a_0 \exp(-\mathrm{i}\tilde{\omega}_{k_0}t); \qquad \tilde{\omega}_{k_0} = \omega_{k_0} + T_{k_0k_0k_0k_0}|a_0|^2\,.$$

After examining the latter for stability, we find that unsteady oscillations increase with a growth rate

$$\gamma = T_{kk_0}|a_0|^2 = \underline{T}\left(\frac{\omega_k - \omega_{k_0}}{|k = k_0|}\right)|a_0|^2$$

$$T_{kk_0} = \mathrm{Im}\; T_{kk_0kk_0} \tag{2.1.10}$$

$$T_{kk_0} = -T_{k_0k} = \frac{\omega_\mathrm{p}^2}{4\pi nT}\,\mathrm{Im}\; G\left(\frac{\omega_k - \omega_{k_0}}{|k - k_0|}\right).$$

As seen from the plot Im $G(\xi)$ presented in fig. 1.2, as a result of the instability only oscillations with $|k| < |k_0|$ are excited. In a non-isothermal plasma T_{kk_0} has a sharp maximum when $\omega_{k_0-k} = |k - k_0|c_s$ (see the expression for Im $G(\xi)$ in the hydrodynamical limit). It is evident that this relation is identical to the decay conditions (2.1.5). In an isothermal plasma the maximum of the instability growth rate is achieved when $\omega_{k_0} - \omega_k = \frac{5}{4}|k - k_0|c_s$. This relation is the analogue of the decay conditions for an isothermal plasma. Here the factor $\frac{5}{4}$ shows that together with the induced ion scattering, the decay involving strongly damped ion-sound oscillations makes a great contribution.

Now let us consider a similar problem of a spatial increase of oscillations as a result of a decay instability. We use the equations of the envelope derived in chapter 1. Assume for simplicity that damping does not take place and the decay conditions for frequencies are valid. Linearizing the equation (1.5.5) against the background of a uniform pumping wave we obtain

$$\begin{aligned} &\partial a_{k_1}/\partial t + u_1\, \partial a_{k_1}/\partial x = \mathrm{i}\gamma_0 a^*_{k_2} \exp(\mathrm{i}\, \Delta k\, x) \\ &\partial a_{k_2}/\partial t + u_2\, \partial a_{k_2}/\partial x = \mathrm{i}\gamma_0 a^*_{k_1} \exp(\mathrm{i}\, \Delta k\, x) \\ &\Delta k = k_0 - k_1 - k_2\,; \qquad \gamma_0^2 = |V_{k_0k_1k_2} a_{k_0}|^2\,. \end{aligned} \tag{2.1.11}$$

Now let us consider a stationary problem of a spatial distribution of oscillations. The solution of (2.1.11) is of the form

$$a_{k_1} \sim \exp\left(\mathrm{i}\frac{\Delta k}{2} x + \mathrm{i}\kappa x\right); \qquad a^*_{k_2} \sim \exp\left(-\mathrm{i}\frac{\Delta k}{2} x + \mathrm{i}\kappa x\right).$$

For κ we obtain an equation similar to (2.1.7),

$$\kappa^2 = (\Delta k)^2/4 - \gamma_0^2/u_1 u_2\,. \tag{2.1.12}$$

It is seen that for waves propagating in one direction, $u_1 u_2 > 0$, there exists a spatial instability: fixing the oscillation amplitudes at some point, we see that they exponentially increase as we move along x. The instability fails if

$$(\Delta k)^2 > 4\gamma_0^2/u_1 u_2\,. \tag{2.1.13}$$

If three-wave processes are not permissible by conservation laws, four-wave processes become basic. As a result of monochromatic-wave instabilities, those oscillations are excited whose wavevectors are near the surface

$$2k_0 = k_1 + k_2, \qquad 2\omega_{k_0} = \omega_{k_1} + \omega_{k_2}\,. \tag{2.1.14}$$

As mentioned above, a non-linear wave medium is described by (1.4.8) in the absence of three-wave processes. This equation has the exact solution,

$$a(k) = a_0 \exp\{\mathrm{i}\tilde{\omega}(k_0)\, t\}\, \delta(k - k_0)$$

where

$$\tilde{\omega}_{k_0} = \omega(k_0) + T_{k_0k_0k_0k_0}|a_0|^2\,.$$

Linearizing (1.4.8) against the background of this solution and further doing as in the case of decays, we find the instability growth rate. If damping is neglected, it is equal to

$$\gamma^2 = [T_{k_0k_0k_0k_0}|a_0^2|]^2 - \tfrac{1}{4}(\Delta\omega)^2$$
$$\Delta\omega = 2\tilde{\omega}_{k_0} - \tilde{\omega}_{k_1} - \tilde{\omega}_{k_2}\,. \tag{2.1.15}$$

Here all the terms are renormalized, taking into account non-linearities

$$\tilde{\omega}_{k_{1,2}} = \omega_{k_{1,2}} + 2T_{k_{1,2}k_0k_{1,2}}|a_0|^2\,.$$

The maximum growth rate is obtained when $\Delta\omega = 0$ and proportional to $|a_0|^2$: $\gamma_{\max} = T_{k_0k_0k_1k_2}|a_0|^2$. Therefore the process (2.1.14) is called an ordinary second-order decay instability.

For satisfying the decay conditions (2.1.4), it is necessary that the dispersion law should be convex $\omega_k'' < 0$. However, when $k_1, k_2 \to k_0$, we have $\Delta\omega \to 0$, as the maximum growth rate remains finite. Therefore, the process (2.1.4) occurring with a small change in k: $2\omega_{k_0} = \omega_{k_0+\kappa} + \omega_{k_0-\kappa}$, $\kappa \ll k_0$ is of great interest.

In the simplest case when the Hamiltonian coefficients are continuous as $k \to k_0$ (2.1.15) gives

$$\omega = (\kappa v) \pm \sqrt{\tilde{T}|a_0|^2\Delta + \tfrac{1}{4}\Delta^2}\,. \tag{2.1.16}$$

Here $\tilde{T} = T_{k_0k_0k_0k_0}$, $\Delta = (\partial^2\omega/\partial k_\alpha \partial k_\beta)\kappa_\alpha\kappa_\beta$. For an isotropic medium $\Delta = L(\Theta)\,\kappa^2$, where Θ is the angle between κ and k_0

$$L = \omega_k'' \cos^2\Theta + (\omega_k'/k_0)\sin^2\Theta\,.$$

Then the instability criterion takes the form

$$L(\Theta)\,\tilde{T} < 0\,.$$

This condition is often called the Lighthill criterion [14].

As seen from (2.1.16), increasing perturbations propagate with a group velocity close to the primary wave velocity; therefore they are absolute perturbations in that reference system where the wave rests. As distinguished from the first-order decay instability, it results in a group of modulations fixed with respect to the primary wave. The group velocity of the perturbations, is $\mathrm{d}\omega/\mathrm{d}\kappa \sim v_{\mathrm{gr}} + \tfrac{1}{6}\kappa^2\,\partial^3\omega/\partial\kappa^3$, and, consequently, an absolute instability character is retained

$$\kappa/k_0 \sim (\gamma/\omega)^{1/3}\,.$$

When $\omega'' < 0$, the function $L(\Theta)$ is alternating and therefore instability exists for any sign of $\tilde{T}$. Along each direction the instability region is bounded by the values $\kappa^2 < 4|\tilde{T}a_0^2/L(\Theta)|$, the maximum growth rate equal to $\tilde{T}a_0^2$ being achieved when $\kappa^2 = |2\tilde{T}a_0^2/L(\theta)|$. For the angle $\tan^2\Theta = \omega''k/v_{\mathrm{gr}}$, $L(\Theta) = 0$, and along this direction the instability region is bounded, and with increasing κ it transforms to a second-order decay instability.

The above-considered instability shows itself as an increase of long-wave modulations of an initial monochromatic wave, and therefore it is very often called modulational. It should be noted that as a result of the development of the modulational instability wave self-focusing takes place, and therefore this instability also is called self-focused. A non-linear stage of this instability will be analysed in following sections.

In conclusion it should be noted that in the cases when the matrix elements are not continuous if $\boldsymbol{\kappa} \to 0$, it can be shown that the above-mentioned formula (2.1.11) holds, however the coefficient $\tilde{T}$ becomes dependent on the angle Θ.

Thus, the canonical equations allow the decay instabilities to be systemized and described, their growth rates readily being expressed through matrix interaction elements.

2.2. Modified decay instabilities

As mentioned above, an independent consideration of different decay processes is true only at not too great initial wave amplitudes $\gamma \ll \omega_{\boldsymbol{k}}$. In the opposite case the resonance zones intersect, and there appear different combined instabilities the properties of which depend on an interaction type.

Now let us consider as an example of this effect the interaction of a narrow high-frequency oscillation packet with sound [11]. In this case two types of processes take place: the decay of a high-frequency wave into a high-frequency wave and a sound one

$$\omega_{\boldsymbol{k}_0} = \omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}} + \Omega_{\boldsymbol{\kappa}} \tag{2.2.1}$$

and a scattering of high-frequency waves on one another,

$$2\omega_{\boldsymbol{k}_0} = \omega_{\boldsymbol{k}_0+\boldsymbol{\kappa}} + \omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}\,. \tag{2.2.2}$$

It is evident that the distance between the decay surfaces is $\Omega_\kappa L\kappa^2$. In view of the smallness of the sound frequency, even if $\boldsymbol{\kappa} \sim \boldsymbol{k}_0$, the condition $\gamma > \Omega_{\boldsymbol{\kappa}}$ is fulfilled in the majority of experimental cases. Besides, even at not too great oscillation intensities the growth rate becomes larger than the sound frequency for the processes running with a small variation in a high-frequency wave number.

Interaction between high-frequency waves and sound are described by (1.5.15), (1.5.16) derived in chapter 1. Consider them in the simplified variant neglecting the intrinsic non-linearity of the high-frequency waves (assuming $T = 0$) and the entrainment effect $k_0\phi_z \ll (\partial\omega/\partial\rho)\,\delta\rho$. In this case we have

$$\mathrm{i}\left(\psi_t + v_{\mathrm{gr}}\frac{\partial\psi}{\partial z}\right) + \frac{v_{\mathrm{gr}}}{2k_0}\nabla_\perp^2\psi + \frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2} = \frac{\partial\omega}{\partial\rho}\,\delta\rho\,\psi$$

$$\boldsymbol{v}_{\mathrm{gr}} = \partial\omega/\partial\boldsymbol{k} = \omega'\,\boldsymbol{k}_0/k_0 \tag{2.2.3}$$

$$\frac{\partial^2}{\partial t^2}\,\delta\rho - c_s^2\Delta\,\delta\rho = \rho_0\frac{\partial\omega}{\partial\rho}\,\Delta|\psi|^2\,.$$

The system (2.2.3) has a trivial exact solution $\psi = a_0$.

Linearizing (1.5.15)–(1.5.16) against the background of a monochromatic wave and assuming the perturbations to be proportional to $\exp(-\mathrm{i}\Omega t + \mathrm{i}\boldsymbol{pr})$ we obtain the dispersion equation

$$\{(\Omega - pu\cos\Theta)^2 - \tfrac{1}{4}L^2(\Theta)\,p^4\}\,(\Omega^2 - c_s^2p^2) = L(\Theta)\,p^4a_0^2\beta^2\rho_0$$
$$\beta = \partial\omega/\partial\rho\,. \tag{2.2.4}$$

Here a_0 is the initial wave amplitude, $\cos\Theta = \boldsymbol{pk}_0/pk_0$

$$L(\Theta) = \omega''\cos^2\Theta + (u/k_0)\sin^2\Theta\,; \qquad u = \partial\omega/\partial k\,.$$

Assuming the wave amplitude sufficiently small, let us first consider the first-order decay instability. As far as p is small when compared with k_0, the equation of the decay surface can be re-written in the form

$$-pu\cos\Theta + \tfrac{1}{2}L(\Theta)\,p^2 + pc_s = 0\,.$$

Near this surface the equation (2.2.4) can be simplified as:

$$(\Omega - pu\cos\Theta + \tfrac{1}{2}L(\Theta)\,p^2)\,(\Omega - pc_s) = \frac{\beta^2a_0^2p\rho_0}{2c_s}\,.$$

Now the dispersion relation for the decay instability can be obtained,

$$\Omega = \tfrac{1}{2}(pu\cos\Theta + pc_s - \tfrac{1}{2}Lp^2) \pm \sqrt{\frac{\beta^2a_0^2p\rho_0}{c_s} + \left(\frac{pu\cos\Theta - pc_s - \frac{1}{2}Lp^2}{2}\right)^2}$$

coinciding with the expressions (2.1.7) with taken into account the matrix element (1.5.14). The maximum instability growth rate is achieved on the surface (2.2.1) and equals

$$\gamma_d = \beta a_0\sqrt{\frac{p\rho_0}{c_s}}$$

which also coincides with (2.1.7).

In order to obtain the second-order decay instability, it is assumed that $\Omega \simeq pu\cos\Theta$. Then (2.2.4) is reduced to the form

$$(\Omega - pu\cos\Theta)^2 - \tfrac{1}{4}L^2p^4 = -T_{\text{eff}}a_0^2Lp^2$$
$$T_{\text{eff}} = \frac{\beta^2\rho_0}{c_s^2 - u^2\cos^2\Theta}\,. \tag{2.2.5}$$

This expression is equivalent to (2.1.16) but the difference is that the non-linearity resulting from the interaction with sound is anisotropic. It is obvious that as the high-frequency wave interacts with sound there arise spontaneous modulations in a transverse direction. The maximum instability growth rate

$\gamma = T_{\rm eff} a_0^2$ is achieved at the angles determined by the condition

$$\tfrac{1}{2}Lp^2 = T_{\rm eff} a_0^2 .$$

When deriving (2.2.5), (2.2.4) the following conditions are assumed to be fulfilled:

$$Lp^2 > \gamma; \qquad \gamma < pc_s . \tag{2.2.6}$$

In the opposite case the separation of decay instabilities of the second order becomes impossible.

Combined instabilities will be considered in the most interesting case $u \gg c_s$, when the effect of low-frequency waves is significant only for almost transverse pertubations. Thus it can be assumed that $L \simeq u/k_0$. Now let us introduce the notation: $q = \beta^2 \rho_0 / c_s^2$. Then the following cases become possible:

1. $\quad q\, a_0^2/\omega_{k_0} < c_s^2/u^2 .$

In this case for $p/k_0 \gg ((c_s/u)q\, a_0^2/\omega_{k0})^{1/3}$ first-order decay instability takes place. For smaller p the first of the conditions (2.2.6) is broken, and for $p/k_0 \ll ((c_s/u)q\, a_0^2/\omega_{k0})^{1/3}$ the term Lp^2 of (2.2.4) can be neglected, and this equation can be simplified:

$$(\Omega - pu\cos^2\Theta)^2 (\Omega - pc_s) = -(u/k_0)p^3 c_s q a_0^2 . \tag{2.2.7}$$

The strongest instability takes place on the cone $\cos\Theta = c_s/u$ where

$$\operatorname{Im}\Omega = \tfrac{1}{2}\sqrt{3}\, p((u/k_0)\, c_s q a_0^2)^{1/3} . \tag{2.2.8}$$

2. $\quad c_s^2/u^2 \ll q a_0^2 \ll c_s/u .$

Now for $p/k_0 > (u/c_s)q\, a_0^2/\omega_{k0}$ the decay instability is again realized. For smaller p the second of the conditions (2.2.6) fails, and then (2.2.4) is of the form

$$\Omega^2(\Omega - pu\cos\Theta + (u/2k_0)p^2) = p^2 c_s^2 q a_0^2 . \tag{2.2.9}$$

The instability has a maximum on the surface $\cos\Theta = -\tfrac{1}{2}p/k_0$ where

$$\operatorname{Im}\Omega = \tfrac{1}{2}\sqrt{3}(p^2 c_s^2 q a_0^2)^{1/3} . \tag{2.2.10}$$

The instability (2.2.10) is called the modified decay instability. When $p/k_0 \sim ((c_s^2/u^2) q a_0^2/\omega_{k0})^{1/4}$ its growth rate is comparable with Lp^2. For smaller p (2.2.9) should be replaced by

$$\Omega^2(\Omega - pu\cos\Theta)^2 = (u/k_0)p^4 c_s^2 q a_0^2 . \tag{2.2.11}$$

The maximum growth rate

$$\operatorname{Im}\Omega = p((u/k_0)c_s^2 q a_0^2)^{1/4} \tag{2.2.12}$$

is attained when $\Theta = \pi/2$.

3. $q a_0^2/\omega_{\kappa_0} > c_s/u$.

This case differs from the preceding one in that there is no domain of first-order decay instability, and the modified instability extends to $p \sim k_0$. The maximum growth rate of the modified instability is expressed by

$$\gamma \sim (k_0^2 c_s^2 q a_0^2)^{1/3}. \tag{2.2.13}$$

It should be noted that all the instabilities, except for the modulational ones are convective in the reference frame of the HF wave. Their group velocities of the perturbations $\partial\Omega/\partial \boldsymbol{p}$ substantially differ from those of the primary waves. For the modulational instability the group velocity of the perturbations coincides with the group velocity of the primary waves with an accuracy of non-linear terms. Thus, modulational instabilities are not so sensitive to plasma inhomogeneities or to the coherence of the primary wave.

Also it should be noted that for the instabilities (2.2.12), (2.2.8) and (2.2.5) for small wave numbers the growth rate is proportional to p. With the help of the initial dispersion relation (2.2.3) it is not difficult to see that this corresponds to the neglect of the term Lp^2.

The neglect of the diffraction term is the transition to the non-linear geometric optics approximation, and therefore the instabilities can be obtained within the framework of the Vedenov–Rudakov equation (see chapter 3). The modified decay instability as well as the maximum modulational instability growth rate can be obtained taking into account diffraction effects.

2.3. Instability of Langmuir waves

The first section of this chapter deals with the first- and second-order decay instabilities of waves which can be described within the framework of a unified formalism. To describe modified decay instabilities, in the case when decay zones begin to intersect, a specific form of matrix elements is significant. In the previous section we investigated these instabilities on the basis of HF wave interaction with a long-wave sound. In this section we will consider a very important problem of the stability of Langmuir waves with finite amplitudes. On the one hand, in specific cases it is reduced to the problems considered above. On the other hand, within the framework of specific equations, we can study the problem in more detail, and then the role of the above-mentioned approximations becomes more evident.

Consider a stationary Langmuir wave,

$$\psi_0 = (A/k_0)\exp(-\mathrm{i}\omega_{k_0}t + \mathrm{i}\boldsymbol{k}_0\boldsymbol{r}); \qquad \delta n = 0.$$

Let us linearize the basic system of dynamical equations (1.6.6)–(1.6.11) on the basis of this solution and assume that

$$\delta\psi \sim \exp(-\mathrm{i}\omega_{k_0}t + \mathrm{i}\boldsymbol{k}_0\boldsymbol{r} + \mathrm{i}\boldsymbol{\kappa}\boldsymbol{r} - \mathrm{i}\Omega t)$$

$$\delta\psi^* \sim \exp(-\mathrm{i}\Omega t + \mathrm{i}\boldsymbol{\kappa}\boldsymbol{r} + \mathrm{i}\omega_0 t - \mathrm{i}\boldsymbol{k}_0\boldsymbol{r}).$$

For Ω the following dispersion relation is obtained:

$$1+\frac{\omega_p}{4T}\frac{W}{n_0^2}G(\kappa,\Omega)\left\{\frac{(\boldsymbol{k}_0,\boldsymbol{k}_0+\boldsymbol{\kappa})^2}{k_0^2|\boldsymbol{k}_0+\boldsymbol{\kappa}|^2(\omega_{\boldsymbol{k}_0+\boldsymbol{\kappa}}-\Omega-\omega_{\boldsymbol{k}_0})}+\frac{(\boldsymbol{k}_0,\boldsymbol{k}_0-\boldsymbol{\kappa})^2}{k_0^2|\boldsymbol{k}_0-\boldsymbol{\kappa}|^2(\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}+\Omega-\omega_{\boldsymbol{k}_0})}\right\}=0. \tag{2.3.1}$$

It should be noted that here ω_k is counted from the plasma frequency $\omega_k=\frac{3}{2}\omega_p k^2 r_D^2$, $W=A^2/8\pi$ is the oscillation density, and $G_{\kappa\Omega}$ is the Green function introduced in the first chapter.

The instability character significantly depends on the wave amplitude and the wavevector as well as on the temperature relation between ions and electrons. The situations under consideration are tightly connected with the simplifications of $G_{\kappa\Omega}$ analysed in the first chapter.

Let us consider the case of an isothermal plasma and not too high amplitudes. In this case the wave instability results from its induced scattering by ions. In the dispersion equation (2.3.1) the second term in square brackets plays the main role, and in the argument of the Green function Ω can be changed to $\omega_{\boldsymbol{k}_0}-\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}$. Since the width of the Green function is of the order of the sound damping rate γ_s, such a procedure is valid as long as the instability growth rate $\gamma\ll\gamma_s$.

The equation (2.3.1) takes the form

$$\Omega+\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}-\omega_{\boldsymbol{k}_0}+\frac{\omega_p}{4T}\frac{W}{n_0^2}\frac{(\boldsymbol{k}_0,\boldsymbol{k}_0-\boldsymbol{\kappa})^2}{k_0^2|\boldsymbol{k}_0-\boldsymbol{\kappa}|^2}G_{\boldsymbol{\kappa},\,\omega_{k0}-\omega_{k0-\kappa}}=0.$$

For the growth rate value the following expression is correct:

$$\gamma\simeq\frac{\omega_p(\boldsymbol{k}_0,\boldsymbol{k}_0-\boldsymbol{\kappa})^2}{4n_0^2Tk_0^2|\boldsymbol{k}_0-\boldsymbol{\kappa}|^2}W\,\mathrm{Im}\,G\left(\frac{\omega_{\boldsymbol{k}_0}-\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}}{|\boldsymbol{\kappa}|}\right). \tag{2.3.2}$$

Apart from the notation, this expression coincides with that considered in section 2.1 and is obtained directly from eq. (1.6.21).

A hydrodynamical approximation for the Green function (1.6.16) can be used in a non-isothermal plasma.

In this case eq. (2.3.1) takes the form

$$1+\frac{\omega_p}{4}\frac{W}{n_0T}\frac{\kappa^2v_T^2}{\Omega^2-\kappa^2c_s^2}\frac{m}{M}\left[\frac{(\boldsymbol{k}_0,\boldsymbol{k}_0+\boldsymbol{\kappa})^2}{k_0^2|\boldsymbol{k}_0+\boldsymbol{\kappa}|^2(-\Omega+\omega_{\boldsymbol{k}_0+\boldsymbol{\kappa}}-\omega_{\boldsymbol{k}_0})}+\frac{(\boldsymbol{k}_0,\boldsymbol{k}_0-\boldsymbol{\kappa})^2}{k_0^2|\boldsymbol{k}_0+\boldsymbol{\kappa}|^2(\Omega+\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}-\omega_{\boldsymbol{k}_0})}\right]=0. \tag{2.3.3}$$

When the wave amplitudes are small and $kr_D>\frac{1}{3}\sqrt{m/M}$, omitting the second term in the square brackets and assuming that $\Omega\simeq\Omega_\kappa$, (2.3.3) can be simplified to the form

$$(\Omega-\Omega_\kappa)(\Omega+\omega_{\boldsymbol{k}_0-\boldsymbol{\kappa}}-\omega_{\boldsymbol{k}_0})+\frac{\omega_p^3\kappa^2r_D^2}{8\;\kappa c_s}\frac{m}{M}\frac{W}{n_0T}=0. \tag{2.3.4}$$

This is a partial case of eq. (2.2.4) describing the first-order decay instability. Its maximum growth rate is expressed by

$$\gamma \simeq \frac{\omega_p}{2\sqrt{2}} 4 \sqrt{\frac{m}{M}} (\kappa r_D)^{1/2} \sqrt{\frac{W}{n_0 T}} . \tag{2.3.5}$$

It is achieved for backscattering that $\kappa = 2\kappa_0$. When $\kappa \ll k_0$, eq. (2.3.1) in the hydrodynamic limit passes to (2.2.4) which in the notation of the present section is as follows:

$$[(\Omega - 3v_T r_D(\boldsymbol{\kappa k}_0))^2 - 9\omega_p^2(\kappa r_D)^4](\Omega^2 - \kappa^2 c_s^2) = \tfrac{3}{2}\omega_p^2 \frac{W}{n_0 T} \kappa^2 c_s^2 (\kappa r_D)^2 . \tag{2.3.6}$$

In accordance with the results of the previous section, when the values of κ are rather great, but $\kappa \ll k_0$, we have from (2.3.6) the decay instability (2.3.5) and the modulational instability with maximum growth rate

$$\gamma \simeq \omega_p W/n_0 T \tag{2.3.7}$$

which is achieved when $(\kappa r_D)^2 \sim W/n_0 T$.

When

$$\kappa < \kappa_{cr} \sim k_0 \frac{W}{n_0 T} \frac{M}{m} \frac{1}{k_0 r_0} ,$$

we have the modified instabilities considered above. When $W > n_0 T > (k_0 r_D)\sqrt{m/M}$ the modified decay region extends up to $\kappa \sim k_0$, and for its growth rate we have, analogously to (2.2.13),

$$\gamma \simeq \omega_p \left(\frac{W}{M n_0} \frac{k_0^2}{\omega_p^2} \right)^{1/3} . \tag{2.3.8}$$

It should be noted that the plasma temperature does not figure in this formula. It is also valid when $T_i \sim T_e$.

The growth rate (2.3.8) is achieved when $\kappa \sim 2k_0$. The instability retains its character up to the intensities $W/n_0 T \sim (M/m)(k r_D)^4$. With great intensities the instability is developed with a growth rate of the order (2.3.8), but non-localized in the vicinity of the surface $\omega_\kappa \simeq \omega_{k_0}$. The growth rate of this instability is almost constant within the region $\kappa \simeq \kappa_0$. It is important that for such great amplitudes the instability resulting in the excitation of scales small compared with the initial wavelength possesses the greatest growth rate (as will be shown below).

The long Langmuir wave instability $k r_D < \frac{1}{3}\sqrt{m/M}$ depends qualitatively slightly on the temperature ratio of electrons and ions. For small-amplitude waves $W/n_0 T < (k r_D)^2$ a static approximation for the Green function (1.6.15) may be used. In this case a modulational-type instability takes place

$$\gamma = \sqrt{\tfrac{3}{4} q \omega_p^2 \kappa^2 r_D^2 W/n_0 T - \tfrac{9}{4}\omega_p (\kappa r_D)^4}$$
$$q = (T_i + T_e)/T_e \tag{2.3.9}$$

the maximum of which, $\gamma_{max} \sim \frac{1}{2} q \omega_p W/n_0 T$ is achieved when $(\kappa r_D)^2 \simeq \frac{1}{6} W/n_0 T$. With increasing $W/n_0 T$, the wavevector of growing perturbances becomes greater than k_0, and for intensities $W/n_0 T > k_0^2 r_D^2$ it can be assumed that $k_0 = 0$. Within a hydrodynamical limit the equation (2.3.1) is simplified to the form

$$(\Omega^2 - c_s^2 \kappa^2)(-\Omega^2 + \tfrac{9}{4}\omega_p^2 \kappa^4 r_D^4) + \tfrac{9}{4}\omega_p^4 (\kappa r_D)^4 \frac{m}{M}\frac{W}{n_0 T}\cos^2\Theta = 0\,. \tag{2.3.10}$$

Here Θ is the angle between the wavevector and the electric field of the initial Langmuir wave. It is obvious that when the condition

$$\frac{W}{n_0 T}\cos^2\Theta > (\kappa r_D)^2 \tag{2.3.11}$$

is satisfied, the instability takes place.

For waves of not too great amplitude $W/n_0 T < m/M$, Ω may be neglected when compared with the sound frequency. In fact, in this case a static limit of $G_{\kappa\Omega}$ may be used.

As is seen, there is the instability with the growth rate

$$\gamma \simeq \omega_p \sqrt{\tfrac{3}{4}(\kappa r_D)^2 (W/n_0 T)\cos^2\Theta - \tfrac{9}{4}(\kappa r_D)^4}\,. \tag{2.3.12}$$

The maximum growth rate $\gamma \simeq \frac{1}{4}\omega_p W/n_0 T$ is achieved when $(\kappa r_D)^2 = \frac{1}{6} W/n_0 T$, $\cos^2\Theta = 1$. For large amplitudes the phase velocity of the perturbations becomes more than the sound velocity, and (2.3.10) is simplified to the form

$$\Omega^2(\Omega^2 - \tfrac{9}{4}\omega_p^2 (\kappa r_D)^4) = \tfrac{9}{4}\omega_p^4 (\kappa r_D)^4 \frac{m}{M}\frac{W}{n_0 T}\cos^2\Theta\,. \tag{2.3.13}$$

When κ are small, we have

$$\gamma \simeq \omega_p (\kappa r_D)\left(\frac{3}{4}\frac{W}{n_0 T}\frac{m}{M}\cos^2\Theta\right)^{1/4}. \tag{2.3.14}$$

When $\kappa r_D \sim ((m/M)(W/n_0 T)\cos^2\Theta)^{1/2}$, the growth rate achieves the value

$$\gamma \simeq \omega_{pi}\left(\frac{W}{n_0 T}\cos^2\Theta\right)^{1/2} \tag{2.3.15}$$

which is not practically varied up to $(\kappa r_D)^2 \sim (W/n_0 T)\cos^2\Theta$ and then, on the stability boundary (2.3.11), drops to zero.

In the literature devoted to Langmuir turbulence, there exists a great terminological muddle. For example, the instability (2.3.7), (2.3.9) as well as (2.3.12) and (2.3.14) are called modulational. By this is meant that the instability (2.3.7), (2.3.9) results in an initial monochromatic wave amplitude modulation, and (2.3.12) and (2.3.14) result in the appearance of the plasma density modulation. To distinguish them (2.3.12) will be called a subsonic modulational instability (SMI-I), and (2.3.14) a supersonic modulational instability (SMI-II).

In conclusion, it is appropriate to present schematically the basic results of this section (fig. 2.1). Illustrated in this figure are the maximum growth rate of the Langmuir wave instability dependent on its amplitude and the wavevector, as well as the wavevector corresponding to this value of the growth rate.

When the amplitudes $W/n_0T < kr_D\sqrt{m/M}$ are sufficiently small in the decay spectrum region $k_0r_D > \frac{1}{3}\sqrt{m/M}$, the first-order decay instability (2.3.5) (region I) possesses the maximum growth rate. With increasing intensity the region I turns into the modified decay (2.3.8) (region III). Within the non-decay spectrum region at small intensities the modulational instability $\kappa < k_0$ (2.3.9) (region II) possesses the maximum growth rate. With increasing W/n_0T it turns to the subsonic modulational instability (2.3.12) of the Langmuir wave condensate with $k_0 = 0$ (region IV). And, finally, at great intensities a supersonic instability of the Langmuir condensate (2.3.14) (region V) is of great importance.

2.4. Parametric instabilities

Among various decay instabilities a special class stands out: the instabilities of a homogeneous external field which are often called parametric instabilities. As a rule, the problems of exciting potential plasma oscillations by electromagnetic waves belong to this class. It has been known that when a homogeneous high-frequency field is superimposed on an electronic plasma, the oscillations are not excited [15]. Only a uniform electron motion in an external field arises. Therefore, the electromagnetic wave decay into two high-frequency potential oscillations is possible only with taking into account the finiteness of the electromagnetic wave number (see table 1.2).

Parametric instabilities of a homogeneous field in a plasma arise when one of the excited oscillations is a low-frequency one, e.g., a sound oscillation involving ions. Thus, a parametric instability is an example of the interaction of low- and high-frequency waves mentioned above.

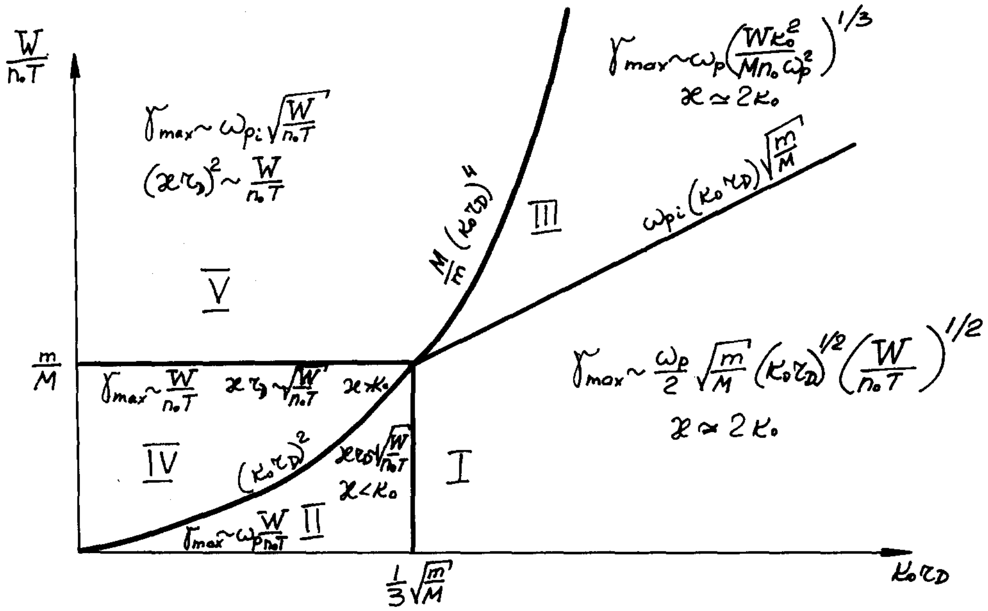

Fig. 2.1. Location of different instability types for the monochromatic Langmuir wave in an isotropic plasma. k is the wavevector corresponding to maximum growth rate. I, first-order decay instability; II, modulational instability; III, modified decay; IV, subsonic modulational instability, V, supersonic modulational instability.

The linear theory of parametric instabilities is described in detail in the book by V.P. Silin [15]; therefore the principal object of this section is to demonstrate that parametric instabilities may be investigated within the framework of a Hamiltonian formalism in a rather general form. Let a pair of waves in the medium be excited – a high-frequency wave with the amplitude a_k and a low-frequency wave with the amplitude b_k. Hamiltonian of the interaction with an external field (pumping) H_p can be obtained assuming in (1.4.12) one of the high-frequency amplitudes to be $h\exp(-i\omega_0 t)$:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_p = \int (\omega_k a_k a_k^* + \Omega_k b_k b_k^*)\,dk + \int \{hV_k a_k(b_{-k} + b_k^*)\,e^{-i\omega_0 t} + \text{c.c.}\}\,dk. \tag{2.4.1}$$

Here $V_k = V_{-k,0,k}$, h is the amplitude of the external field of frequency ω_0.

The term $a_k b_k^*$ in the Hamiltonian should be retained only in the case when the instability growth rates are comparable with the sound frequency. By virtue of the fact that $(b_{-k} + b_k^*) \sim \delta n_k$, it is shown in (2.4.1) that the high-frequency wave interaction takes place via density fluctuation scattering. In the case of the interaction with an electric field hV_k linearly depends on an electric field and therefore depends on $\boldsymbol{kE}$ only, i.e. $V_k = V_{-k}$. For example, for the decay into an electromagnetic wave and sound (see table 1.2) we have

$$V_k = \frac{(\omega_p\Omega_s)^{1/2}}{2\sqrt{2}} \sqrt{\frac{E_0^2}{8\pi n_0 T}}\,\frac{kE_0}{kE_0}.$$

The equations of motion, corresponding to (2.4.1) are:

$$\begin{aligned} \partial a_k/\partial t + i\omega_k a_k &= -ihV_k(b_k + b_{-k}^*)\,e^{-i\omega_0 t} \\ \partial b_k/\partial t + i\Omega_k b_k &= -ih[V_k^*\,e^{i\omega_0 t} a_k - V_{-k} a_{-k}^*\,e^{-i\omega_0 t}]. \end{aligned} \tag{2.4.2}$$

A zeroth-order solution of (2.4.2) can be unsteady with respect to the oscillation growth,

$$b_k, b_k^* \sim e^{-i\omega t}; \qquad a_k \sim e^{-i(\omega+\omega_0)t}; \qquad a_k^* \sim e^{i(\omega_0-\omega)t}.$$

For the complex frequency ω the following dispersion equation is obtained:

$$(\omega^2 - \delta_k^2)(\omega^2 - \Omega_k^2) = -4\Omega_k\delta_k h^2 V_k^2, \qquad \delta_k = \omega_0 - \omega_k. \tag{2.4.3}$$

It should be noted that (2.3.6) is a partial case of (2.4.3) with $\delta_k = \frac{2}{3}\omega_p k^2 r_D^2$, therefore the above-mentioned Langmuir wave instability with $k_0 = 0$ is a partial, very special case of a parametric instability.

If the external field is not too large $hV_k < \Omega$, (2.3.4) must describe two types of instabilities: first- and second-order decay instabilities. The first-order decay instability possesses the maximum growth rate. Its growth rate is maximal on the sphere

$$\omega_0 = \omega_k + \Omega_k \qquad \text{or} \qquad -\delta_k = \omega_k - \omega_0 = -\Omega_k$$

and (2.4.3) in the vicinity of this surface is simplified to the form

$$\omega \cong \Omega_k + \mathrm{i}\sqrt{h^2 V_k^2 - \tfrac{1}{4}(\delta_k - \Omega_k)^2}\,. \tag{2.4.4}$$

As is seen from (2.4.2), the sum of the phases of the exciting waves has a well-defined magnitude (it is equal to $\pi/2$ on the decay surface). This phase correlation is conserved at some non-linear stage of the instability development also, and is of importance when studying a super-threshold system behaviour (see chapter 3). Within the other region of $\boldsymbol{k}$-space there appears a second-order decay instability near the surface

$$2\omega_0 = \omega_k + \omega_{-k} \qquad \text{or} \qquad \delta_k = 0\,.$$

In this case the equation (2.4.3) takes the form

$$\omega^2 = \delta_k^2 + \frac{4h^2 V_k^2}{\Omega_k}\,\delta_k\,. \tag{2.4.5}$$

The instability takes place when $\delta_k < 0$. The maximum growth rate $\gamma = 2h^2 V_k^2/\Omega_k$ is achieved when $\delta_k = -2h^2 V_k^2/\Omega_k$. It is evident that these results are perfectly similar to those obtained in section 2.1, and therefore parametric instabilities are a special case of decay ones. The instabilities (2.4.4) and (2.4.5) were obtained independent of the works devoted to decay processes. Therefore (2.4.4) is often called a periodic instability, and (2.4.5) an aperiodic, or two-stream one.

Due to the simplicity of the dispersion equation (2.4.3), the investigation of combined instabilities when $hV_k > \Omega_k$ is simplified. We have

$$\omega^2 = \tfrac{1}{2}(\delta_k^2 + \Omega_k^2) \pm \sqrt{\tfrac{1}{4}(\Omega_k^2 - \delta_k^2)^2 - 4\Omega_k \delta_k h^2 V_k^2}\,.$$

It is evident that, when hV_k is great, aperiodic and periodic oscillation branches are observed as well.

It should be noted that if the external field frequency is less than the plasma one, only an aperiodic instability can be developed.

Now let us consider the problem of Langmuir and ion-sound wave excitation. In this case the dispersion equation takes the form

$$\omega^2 = \tfrac{1}{2}(\Omega_k^2 + \delta_k^2) \pm \sqrt{\frac{(\Omega_k^2 - \delta_k^2)^2}{4} - 4\Omega_k^2 \delta_k \omega_\mathrm{p} \frac{E_0^2}{8\pi n_0 T}\cos^2\Theta}$$

$$\cos\Theta = \boldsymbol{k}\boldsymbol{E}_0/kE_0\,; \qquad \delta_k = \omega_\mathrm{p}(\Delta - \tfrac{3}{2}k^2 r_\mathrm{D}^2)\,; \qquad \Delta = (\omega_0 - \omega_\mathrm{p})/\omega_\mathrm{p}\,.$$

Firstly let us consider the periodic instability $\delta_k > 0$. When $E_0^2/8\pi n_0 T < k r_\mathrm{D}\sqrt{m/M}$, we have an ordinary decay instability of an electromagnetic wave with maximum growth rate

$$\gamma \simeq \left(\omega_\mathrm{p}\Omega_\mathrm{s}\frac{E_0^2}{8\pi n_0 T}\cos^2\Theta\right)^{1/2}.$$

In the opposite limiting case we have $E_0^2/8\pi n_0 T > k r_\mathrm{D}\sqrt{m/M}$. When δ_k is not so great, we have for the instability growth rate

$$\gamma \simeq \sqrt{\Omega_k^2 \frac{\delta_k}{2} \omega_p \frac{E_0^2}{8\pi n_0 T}} . \tag{2.4.6}$$

It achieves its maximum value

$$\gamma_{\max} \simeq \omega_p \left(\frac{m}{M} \Delta^2 \frac{E_0^2}{16\pi n_0 T} \cos^2 \Theta \right)^{1/4} \tag{2.4.7}$$

when

$$k^2 r_D^2 = \tfrac{1}{3}\Delta ; \qquad \delta_k = \tfrac{1}{2}\omega_p \Delta ; \qquad \cos^2 \Theta = 1 .$$

This expression holds true if the following conditions are fulfilled:

$$\delta_k \omega_p \frac{E_0^2}{16\pi n_0 T} > \Omega_k^2, \qquad \Omega_k \frac{E_0^2}{16\pi n_0 T} > \delta_k^3 . \tag{2.4.8}$$

The first condition is valid for not too small densities of the external field $E_0^2/16\pi n_0 T > m/M$. The second condition is valid for not too great mismatches Δ only: $\Delta^2 < (E_0^2/16\pi n_0 T) m/M$. With increasing mismatch the expression (2.4.6) for the growth rate holds true, and its maximum is determined from the condition $\omega_p \Omega_s^2 E_0^2/8\pi n_0 T \sim \delta_k^3$. For the maximum growth rate the following expression is valid:

$$\gamma_{\max} \simeq \omega_p \left(\frac{m}{M} k^2 r_D^2 \frac{E_0^2}{8\pi n_0 T} \right)^{1/3} = \omega_p \left(\frac{m}{M} \frac{E_0^2}{8\pi n_0 T} \Delta \right)^{1/3} . \tag{2.4.9}$$

This expression holds as long as the exciting oscillations do not fall within the strong Landau damping region. This takes place for a Maxwell distribution function when $\Delta \simeq 0.3$. In this case the characteristic growth rate width is about $\delta k/k_0 \sim \gamma/\omega_{k_0}$, where k_0 is the characteristic wavevector of the exciting oscillations defined by the condition $\Delta \simeq 3(k_0 r_D)^2$. Figure 2.2 shows the maximum growth rate of a periodic instability as a function of mismatch under the condition that we are within the modified decay instability region

$$E_0^2/8\pi n_0 T > k_0 r_D \sqrt{m/M} .$$

Now let an aperiodic instability be considered. When the external field intensities are small, (2.4.6) is reduced to (2.4.5) and describes the second-order decay instability with growth rate

$$\gamma \simeq \omega_p (E_0^2/8\pi n_0 T) . \tag{2.4.10}$$

For large $E_0^2/8\pi n_0 T$ an aperiodic instability is investigated similar to a periodic one, and its maximum growth rate coincides with (2.4.9) as to order of magnitude. Presented in fig. 2.3 is the wavevector dependence of the parametric instability growth rate in the most interesting case of great intensities and

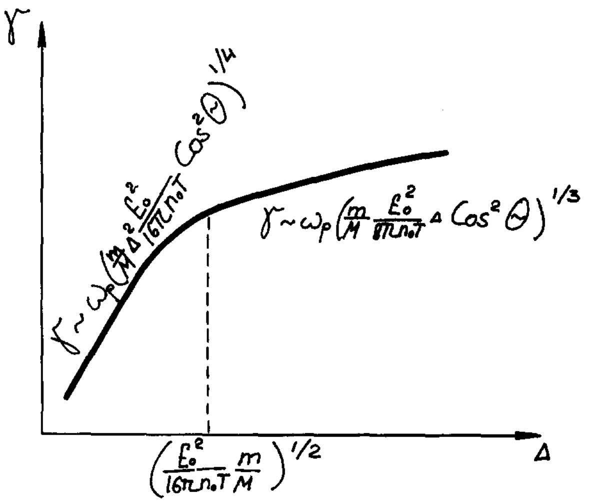

Fig. 2.2. Maximum growth rate of periodical instability as a function of mismatch $\Delta = (\omega - \omega_p)/\omega_p$.

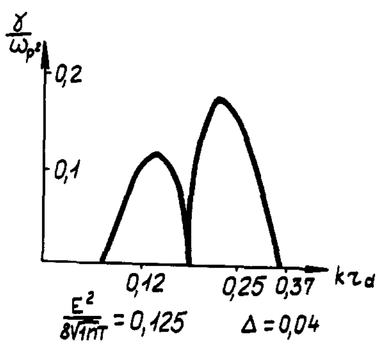

Fig. 2.3. Growth rate of parametric instability for great intensities and mismatches as a function of the excitation wavenumber.

mismatches. It is evident that numerically the aperiodic instability growth rate occurs to be greater than the periodic one.

It should be noted that (2.4.9) does not depend on temperature, and when $v_{osc} \sim v_{ph}$ ($v_{osc} = eE_0/m\omega_0$) it transforms to the well-known result by V.P. Silin [15] for a cold plasma. The maximum possible growth rate of a parametric instability is expressed by

$$\gamma_{max} \simeq \omega_{p_e}(m/M)^{1/3} .$$

In the present section the oscillation damping which defines the instability thresholds is ignored, but in some cases it influences the growth rate structure. The detailed analysis of the dispersion equation (2.4.3), taking into account dissipations, is presented in the book by V.P. Silin [15].

2.5. The effect of inhomogeneity on decay instabilities

In a real plasma the decay instability threshold is often determined not by the wave damping, but by the plasma inhomogeneity or that of the initial wave. In these cases the wave interaction region is restricted by the pump wave localization region or the scales on which the wave propagating in an inhomogeneous plasma change their wavevector and go out of resonance.

The departure of waves from the interaction region is just the factor defining the instability thresholds.

In the description of such effects the inhomogeneity scale is assumed large compared to the wavelength. The interaction of three waves a_0, a_1, a_2 will be described using the envelope equation generalizing the equations obtained in chapter 1:

$$\begin{aligned}
&\partial a_0/\partial t + u_0\ \partial a_0/\partial x + i\omega_0(x)\, a_0 = i V a_2^* a_1^* \\
&\partial a_2/\partial t + u_2\ \partial a_2/\partial x + i\,\omega_2(x)\, a_2 = i V a_0 a_1^* \qquad (2.5.1)\\
&\partial a_1/\partial t + u_1\ \partial a_1/\partial x + i\,\omega_1(x)\, a_1 = i V a_0 a_2^* .
\end{aligned}$$

Here V is the interaction matrix element $V_{k_0k_1k_2}$ in which the inhomogeneity effect can be neglected. The frequencies ω_i in (2.5.1) correspond to the wavevectors $k_i(x_0)$, $\omega_i(x, k_i(x_0))$ taken at a point x_0, where the decay conditions are fulfilled:

$$\sum_i k_i(x_0) = 0 .$$

It is convenient to write down eq. (2.5.1) as follows. Let at a point x_0 the decay conditions for the wave frequencies $\sum_i \omega_i(k_i(x_0), x_0) = 0$ be valid. Then, since when propagating in an inhomogeneous medium the wave frequency does not change, the decay conditions for the frequencies will be fulfilled over the whole space.

Let us expand the frequencies in (2.5.1) in a series and choose the origin at $x = x_0$. Then we have

$$\omega_i(x, k_i(0)) = \omega_i(0, k_i(0)) + \frac{\mathrm{d}\omega_i}{\mathrm{d}x} x_i .$$

From the condition $\omega(k, x) = \text{constant}$ it follows that

$$\mathrm{d}\omega_i/\mathrm{d}x = -u_i \, \mathrm{d}k_i/\mathrm{d}x .$$

Substituting variables and assuming the pump wave amplitude prescribed, eq. (2.5.1) takes a more conventional form:

$$\begin{aligned} &\partial a/\partial t + u_1 \, \partial a/\partial x = \mathrm{i}\, \gamma_0(x)\, b^* \, \mathrm{e}^{\mathrm{i}\phi} \\ &\partial b/\partial t + u_2 \, \partial b/\partial x = \mathrm{i}\, \gamma_0(x)\, a^* \, \mathrm{e}^{\mathrm{i}\phi} \\ &\phi = \frac{x^2}{2} \frac{\mathrm{d}}{\mathrm{d}x}(k_0 - k_1 - k_2) = \frac{k'}{2} x^2 . \end{aligned} \tag{2.5.2}$$

Here $\gamma_0(x) = V a_0(x)$ is the instability growth rate in a homogeneous medium. Now let us consider the case when secondary waves propagate in one direction. Oscillations begin to increase on the instability region boundary, and during the time of their prolongation they reach their final level. It can be determined using a simple estimate. Let us consider the case of inhomogeneous pumping. Let the instability be localized within a region of size L. Then, according to (2.1.12) the oscillations, after passing the interaction region, will achieve the value

$$A = A_0 \exp\left\{\frac{\gamma_0 L}{(u_1 u_2)^{1/2}}\right\} = A_0 \, \mathrm{e}^K .$$

When the condition $K > \Lambda$ (Λ is the Coulomb logarithm) is valid, the oscillation level is determined by non-linear effects. Therefore the growth rate value $\gamma_0 \simeq \sqrt{u_1 u_2 \Lambda}/L$ is naturally called the threshold.

In the case when the threshold is defined by the inhomogeneity, the instability zone size can be determined from the following considerations. Propagating in an inhomogeneous medium, the oscillations change their wavevector, and the decay conditions are broken. The interaction region boundary is determined from (2.1.13)

$$\frac{2\gamma_0}{\sqrt{u_1 u_2}} = \Delta k_{\mathrm{th}} = x\frac{\mathrm{d}}{\mathrm{d}x}(k_0 - k_1 - k_2) = \frac{k'x}{2}$$

i.e., the instability region size is $L \sim 4\gamma_0/\sqrt{u_1 u_2 k'}$, and for the amplification coefficient we have

$$K = \frac{4\gamma_0^2}{u_1 u_2 k'}$$

which is in good agreement with the result $K = \pi\gamma_0^2/u_1 u_2 k'$ obtained by an exact calculation.

Let us consider waves propagating in opposite directions and discuss first the uniform pumping wave excitation in the linear inhomogeneity medium. In this case the solution (2.5.2) is expressed by parabolic cylinder functions and has been investigated in [16, 17]. The instability is of a convective character, i.e. the wave after passing the interaction region amplifies up to the finite value

$$A = A_0 \exp(\pi\gamma_0^2/u_1 u_2 k').$$

If the density profile is smooth, the instability becomes absolute (for a quadratic profile see [15, 17]), i.e. when the critical value is increased by pumping, there are no stationary solutions, and the oscillation level is determined by non-linear effects. To understand the characteristic peculiarities of absolute instabilities, let us consider the wave excitation within the layer. Let us consider the stationary solutions of (2.5.2)

$$u_1\,\partial a/\partial x = \mathrm{i}\gamma_0 b^*, \qquad \gamma_0(x) = \begin{cases}\gamma_0, & 0<x<l\\ 0\,. & \end{cases}$$
$$-u_2\,\partial b/\partial x = \mathrm{i}\gamma_0 a^*, \tag{2.5.3}$$

Natural boundary conditions are that the amplitudes of entering into the layer are small, on the level of thermal noises, and it can be assumed that $a(0)=0$, $b(l)=0$. Then the solution (2.5.3) is

$$a(x) = \sin\kappa x; \qquad b(x) = \cos\kappa x.$$

This solution exists only when $\kappa^2 = \gamma_0^2/u_1 u_2 = l^{-2}(\frac{1}{2}\pi + m\pi)^2$. For such stationary solutions the energy flow to the system from the pumping wave is compensated by the departure from the region of its localization. The basic solution $m=0$ corresponds to the threshold pumping value, when $\gamma_0^2 > \frac{1}{4}u_1 u_2(\pi^2/l^2)$ the noise level is restricted by non-linear effects. $m = 1, 2, \ldots$ correspond to stationary solutions quickly oscillating in space, and therefore leading to a great energy departure from a layer. Probably they have no particular physical sense, because they are unstable with respect to small parameter changes.

An estimate of the inhomogeneity effect on the induced scattering by particles is of particular interest. For correctness let us confine ourselves to the consideration of the electromagnetic wave conversion to the Langmuir one by ions. In a homogeneous plasma the growth rate of this process is

$$\gamma_k \sim \operatorname{Im} G\left(\frac{\omega_0 - \omega_k}{k}\right)\gamma_0$$

where γ_0 is the maximum growth rate. Propagating through the inhomogeneity, the oscillation changes its wavevector, and Im G decreases. As was mentioned above, the width of Im $G \sim kv_{T_i}$; therefore, in order for the growth rate to be markedly changed, it is necessary to change the wavevector by the magnitude of it itself. For Langmuir oscillations it takes place over the scale $l \sim L(kr_D)^2$; $L^{-1} = \mathrm{d} \ln n(x)/\mathrm{d}x$. In so doing the oscillations increase by a factor $\exp(\gamma L/v_{gr})$, as a result of instability (the instability is convective), and the condition $\gamma L/u > \Lambda$ can be written down as a threshold.

So far we have ignored the wave damping. Taking this into account and transforming from dissipationless thresholds to instability thresholds is not trivial. The point is that the dissipation changes the interaction zone width, therefore the calculations become complicated (see review [15]), and it is not expedient to do it in a general form.

It follows from the results described above that the instability thresholds are minimum for those waves which group velocities tend to zero. In this case $\partial k/\partial x$ transforms to infinity. If group velocities do not vanish simultaneously, an anomalous instability threshold decrease does not take place, since $(\partial\omega/\partial k)\partial k/\partial x = -\partial\omega/\partial x$. So, when an electromagnetic wave decays into a Langmuir wave and an ion sound one, in a plasma there appears just such a situation. If the turning points coincide, the instability thresholds really decrease. In this case the instabilities become absolute also for a non-linear density profile. However, our quasi-classic description has not already been applicable. A detailed investigation of the problems belonging to the class under consideration may be found in the review [15]. It should be noted only that the situation with the coincidence of two turning points is not exceptional. This often can be achieved by selecting a transverse value of the wavevector. Because a minimum threshold value is of greatest interest, the consideration of such a situation is very important.

3. Statistical description of wave interactions

3.1. Introduction

In many physical situations the interaction of such a great number of monochromatic waves takes place that it is necessary to describe these phenomena statistically. In this description the information on interacting wave phases is lost and the wave field is described using the language of mean quadratic amplitudes. These values can be determined as follows. Let the wave field characterized by the complex amplitude a_k be statically uniform. Then for the correlation function $\langle a_k a_k^* \rangle$ we have*

$$\langle a_k a_{k'}^* \rangle = (2\pi)^3 n_k \delta_{k-k'} . \tag{3.1.1}$$

The value n_k enters an infinite set of equations for correlation functions following from dynamical equations for a_k. The statistical description problem is a problem of obtaining a closed equation for n_k. Such an equation, if it can be obtained, is called a kinetic equation.

To derive a kinetic equation it is necessary to make some assumptions about the properties of higher correlation functions. If the wave field is a Gaussian stochastic process, for the fourth correlation function we have [3 to 5]

$$\langle a_k^* a_{k_1}^* a_{k_2} a_{k_3} \rangle = n_k n_{k_1} (\delta_{k-k_2} \delta_{k_1-k_3} + \delta_{k-k_3} \delta_{k_1-k_2}) . \tag{3.1.2}$$

* The factor $(2\pi)^3$ will be omitted together with $(2\pi)^{-3/2}$ in matrix elements of three-wave interaction and with $(2\pi)^{-3}$ in those of four-wave interaction; the final results are the same.

A similar property is fulfilled for even correlation functions of a higher order. Odd correlation functions turn to zero. For the applicability of a kinetic equation it is necessary that the wave field be close to Gaussian.

The Gaussian stochastic process (3.1.2) is compatible only with linear equations for $a_k(t)$. Therefore for the applicability of a kinetic equation the requirement of a small level of interacting wave non-linearity is necessary that will be assumed below. Sufficient applicability conditions for kinetic equations are more detailed and depend on a detailed structure of the function n_k and the interaction character as well.

In a conservative medium where first-order decays (three-wave processes) can take place, a kinetic equation contains terms quadratic in n_k. If three-wave processes are forbidden, the terms quadratic in n_k describe only the self-consistent field type effects, a relative frequency shift of various waves. In a statistically uniform situation these effects do not lead to energy transfer between waves. In a non-conservative medium the self-consistent field effects also lead to a mutual renormalizing of the wave damping, that already mean interaction.

Such an interaction takes place, for example, for the induced scattering of Langmuir waves by plasma ions. In a conservative inhomogeneous plasma (or in a homogeneous medium with the function n_k dependent on the coordinates) the self-consistent field effects lead to an interaction, since non-linear frequency shifts change the interacting wave packet trajectories. The corresponding theory will be called below the collisionless wave kinetics.

In the cases when for a statistical description of a wave field a kinetic equation is applicable, we will say that we have a weak wave turbulence. Below the kinetic equations of a weak turbulence for basic physical situations will be derived. A more rigorous derivation as well as a calculation of the next approximations requires the usage of a diagram technique described in [18].

3.2. Kinetic equation for decay processes

Let in a medium admitting the wave propagation of a single type with amplitude $a_k(t)$ and the dispersion law ω_k which permit the three-wave interaction (2.1.3). Such a medium is described by an interaction Hamiltonian (2.1.1). The equations for a_k are of the form (2.1.4). Let us multiply this equation by a_k^*, add to it the complex-conjugated one and average the equation using formula (3.1.1). Now we have the equation

$$\frac{\partial n_k}{\partial t} + \gamma_k n_k - 2\,\mathrm{Im} \int \mathrm{d}k_1\, \mathrm{d}k_2 \{ V_{kk_1k_2} I_{kk_1k_2}\, \delta(k - k_1 - k_2) + 2\, V^*_{k_1kk_2} I^*_{k_1kk_2}\, \delta(k_1 - k - k_2)\} = 0\,. \qquad (3.2.1)$$

Here $I_{kk_1k_2} = \langle a_k^* a_{k_1} a_{k_2} \rangle$ is the third-order correlation function. For a Gaussian stochastic process $I_{kk_1k_2} = 0$. In our case $I_{kk_1k_2}$, though small, differs from zero. For it the equation must be written which can be obtained by the same method as used for (3.2.1). In so doing $I_{kk_1k_2}$ is then expressed in terms of fourth-order correlation functions. Considering the wave field to be close to a Gaussian one let us assume the hypothesis (3.1.2) for fourth-order correlators. The equation for the fourth-order correlator takes the form

$$\frac{\partial I_{kk_1k_2}}{\partial t} - \mathrm{i}(\omega_k - \omega_{k_1} - \omega_{k_2}) I_{kk_1k_2} = -2\mathrm{i}\{ V_{kk_1k_2} n_{k_1} n_{k_2} - V_{k_2kk_1} n_k n_{k_1} - V_{k_1kk_2} n_k n_{k_2}\}\,. \qquad (3.2.2)$$

Let us neglect $(\partial/\partial t)I_{kk_1k_2}$ as compared with the characteristic frequency difference in the packets. Assuming that the waves have a small damping γ_k, let us use the well-known formula $\mathrm{Im}_{\varepsilon\to 0}(x+\mathrm{i}\varepsilon)^{-1} = \pi\,\delta(x)$. Then the finally known kinetic equation for waves is obtained [3 to 5]

$$\partial n_k/\partial t + \gamma_k n_k = \int (R_{kk_1k_2} - R_{k_1kk_2} - R_{k_2kk_1})\,\mathrm{d}k_1\,\mathrm{d}k_2$$
$$R_{kk_1k_2} = 2\pi|V_{kk_1k_2}|^2 (n_{k_1}n_{k_2} - n_k n_{k_1} - n_k n_{k_2})\,\delta(\omega_k - \omega_{k_1} - \omega_{k_2})\,\delta(k - k_1 - k_2)\,. \tag{3.2.3}$$

Thus it is evident that the kernel of the kinetic equation is simply expressed through the Hamiltonian coefficients.

Estimating a characteristic non-linear time from the equation (3.2.3), we have

$$\frac{1}{\tau} \simeq \frac{|V|^2}{\omega}\int n_k\,\mathrm{d}k \simeq \frac{\Gamma^2}{\omega}$$

where Γ is the characteristic growth rate of the decay instability of a monochromatic wave.

Let n_k consist of three wave packets with mean frequencies and wavevectors which satisfy the resonance conditions (2.1.2), (2.1.3). Let the characteristic packet widths in k-space be Δk. Then

$$\frac{1}{\tau} \simeq \frac{|V|^2}{\Delta\omega}\int n_k\,\mathrm{d}k, \qquad \Delta\omega \simeq \Delta u\,\Delta k\,,$$

where Δu is the group velocity difference of interacting waves. It is evident that the time during which n_k changes must be much larger than the inverse decay instability growth rate $1/\tau \gg \Gamma$. Now we can obtain finally the applicability criterion for the kinetic equation

$$\Gamma \ll \Delta\omega\,. \tag{3.2.4}$$

This criterion has a simple physical sense. Each triple of interacting waves is in resonance during a time of the order of $\tau \simeq 1/\Delta\omega$. For the applicability of the kinetic equation it is necessary that during this time the decay instability leading to a full correlation of phases of the generated secondary waves could not develop.

When the wave propagates in a weakly dispersive medium

$$\omega_k = ck(1+\varepsilon k^2); \qquad \varepsilon k^2 \ll 1; \qquad \varepsilon > 0 \tag{3.2.5}$$

the resonance conditions are satisfied for waves having almost parallel wavevectors. In this case the group velocity difference $\Delta u \simeq \varepsilon\,\Delta k$ is small.

For the applicability condition of the kinetic equation the rigid criterion is obtained:

$$\Gamma \ll \omega''(\Delta k)^2 \simeq \varepsilon kc(\Delta k)^2\,. \tag{3.2.6}$$

From (3.2.6) it follows that in a medium with a linear dispersion law the decay kinetic equation is inapplicable.

A more rigorous criterion (3.2.4) can be derived using the Wyld diagram technique [18]; however, the detailed discussion of this problem is outside the framework of the present paper.

The case when in the medium high-frequency waves with the dispersion law ω_k and low-frequency waves with the dispersion Ω_k interact is of great importance. In this case the process described by the resonance condition

$$\omega_k = \omega_{k_1} + \Omega_{k_2}; \qquad k = k_1 + k_2 \tag{3.2.7}$$

is realized.

Such a situation takes place, for example, in the interaction of Langmuir and ion-sound waves in non-isothermal plasmas. In this case the interaction Hamiltonian is described by the formula (1.4.12). Introducing the averaged values

$$\langle a_k a_{k'}^* \rangle = N_k \delta(k - k'); \qquad \langle b_k b_{k'}^* \rangle = n_k \delta(k - k')$$

we obtain the kinetic equations

$$\begin{gathered}\partial N_k / \partial t + \gamma_k N_k = \int (T_{k_2|kk_1} - T_{k_2|k_1k})\, \mathrm{d}k_1\, \mathrm{d}k_2 \\ \partial n_k / \partial t = \Gamma_k n_k = - \int T_{k|k_1k_2}\, \mathrm{d}k_1\, \mathrm{d}k_2 \\ T_{k_2|kk_1} = 2\pi\, |V_{k_2kk_1}|^2 (N_{k_1} n_{k_2} - N_k n_{k_1} - N_{k_1} N_{k_2})\, \delta(k - k_1 - k_2)\, \delta(\omega_k - \omega_{k_1} - \Omega_{k_2})\,.\end{gathered} \tag{3.2.8}$$

It should be noted that the resonance conditions (3.2.7) do not change if a constant value is added to the frequency ω_k.

For applicability of the equations (3.2.8) the criterion (3.2.4) may turn out to be insufficient. At a great frequency difference of interacting waves the decay instability growth rate (in this case the instability becomes modified) may become greater than the low-frequency Ω_k. In this case the criterion (3.2.4) must take the form

$$1/\tau < \min(\Gamma, \Omega_k)\,. \tag{3.2.9}$$

In conclusion it should be noted that when deriving the kinetic equation, the imaginary parts of the frequencies in (3.2.2) may be conserved for the third-order correlator $I_{kk_1k_2}$. In this case in the kinetic equation the broadening of the δ-function over frequencies up to a width of the order of γ_k takes place. When $\Gamma > \gamma_k$, taking into account of this broadening goes beyond the accuracy of kinetic equations. In an isothermal plasma when the characteristic growth rates are less than the sound damping, this broadening is rather essential.

3.3. Kinetic equation in the non-decay case

It is known (see chapter 1) that in the case when three-wave interactions are forbidden, a non-linear medium is described by the equation (see (1.6.21))

$$\mathrm{i}\frac{\partial a_k}{\partial t}+(-\omega_k+\mathrm{i}\gamma_k)a_k=\mathrm{i}\int T_{kk_1k_2k_3}a^*_{k_1}a_{k_2}a_{k_3}\delta(k+k_1-k_2-k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,. \tag{3.3.1}$$

The equation (3.3.1), even if the damping γ_k is absent, does not necessarily describe a conservative medium. The medium is conservative if the function $T_{kk_1k_2k_3}$ satisfies the symmetry relations

$$T_{kk_1k_2k_3}=T_{k_1kk_2k_3}=T^*_{k_2k_3kk_1}\,. \tag{3.3.2}$$

Multiplying (3.3.1) by a^*_k and subtracting the complex-conjugate expression, we obtain

$$\frac{\partial n_k}{\partial t}+2\gamma_k n_k=\mathrm{Im}\int T_{kk_1k_2k_3}\langle a^*_k a^*_{k_1}a_{k_2}a_{k_3}\rangle\,\delta(k+k_1-k_2-k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,. \tag{3.3.3}$$

Making the correlation decoupling according to the formula (3.1.2), we obtain

$$\partial n_k/\partial t+2n_k\left(\gamma_k-\int T_{kk'}n_{k'}\,\mathrm{d}k'\right)-f_k=0 \tag{3.3.4}$$

where

$$T_{kk'}=\mathrm{Im}\,T_{kk'kk'}\,. \tag{3.3.5}$$

The matrix elements for some important processes are presented in table 3.1 (p. 365).

From (3.3.2) it follows that in a conservative medium $T_{kk'}=0$. In an isothermal plasma it follows from the Green function symmetry (see section 1.6) that

$$T_{kk'}=-T_{k'k}\,. \tag{3.3.6}$$

From (3.3.4) we have

$$\partial N/\partial t+2\int\gamma_k n_k\,\mathrm{d}k=0;\qquad N=\int n_k\,\mathrm{d}k \tag{3.3.7}$$

that is a balance condition for the total quasi-particle number described by eq. (3.3.5). When γ_k equals zero, $N=\text{constant}$. Thus, a non-linear term in (3.3.5) conserves the total number of quasi-particles. Applied to an isothermal plasma, eq. (3.3.5) describes the induced Langmuir plasmon scattering by ions. It is the simplest one of kinetic equations possessing many interesting properties.

For many problems it is necessary to take into account small terms which were omitted when deriving (3.3.4). Therefore a small thermal noise source f_k induced by thermodynamical fluctuations and four-plasmon processes has been included in (3.3.4). As a rule, the noises conditioned by the second cause are more important, the properties of the noises f_k being determined self-consistently with the excited oscillation distribution. These processes make also a small contribution to the wave damping which can usually be ignored.

Now let us discuss the applicability condition (3.3.4). Except the applicability conditions of the initial dynamic equations, it is necessary that the condition (3.2.4) should be valid. In the given case it has the form

$$\int T_{kk'} n_{k'} \, dk' < \Delta\omega . \tag{3.3.8}$$

Then for the most interesting case of isotropic Langmuir turbulence we have

$$\omega_p \frac{W}{nT} \frac{k_{dif}}{k} < \omega_s .$$

Besides, in eq. (3.3.5) the four-plasmon collisional term is neglected. For the validity of doing this it is sufficient that Re $T_{kk_1k_2k_3} \simeq$ Im $T_{kk_1k_2k_3}$. It should be noted that when $f_k = 0$ in reality the equation (3.3.4) has a wider applicability region than ordinary kinetic equations. It can be given a sense, when the wave phases are not random. Let, for example, the wave field represent a set of monochromatic waves

$$a_k = \sum_m c_m \delta(k - k_m) . \tag{3.3.9}$$

Substituting (3.3.9) into the initial dynamic equation, we have

$$\mathrm{i}\frac{\partial c_m}{\partial t} + \mathrm{i}\gamma_m c_m - \omega_{k_m} c_m = 2c_m \sum_{m'} T_{k_m k_{m'} k_m k_{m'}} |c_{m'}|^2 \qquad \text{for} \qquad n_m = |c_m|^2$$

$$\frac{1}{2}\frac{\partial n_m}{\partial t} + \gamma_m n_m = 2n_m \sum_{m'} T_{mm'} n_{m'}$$

i.e. eq. (3.3.4) when $f_k = 0$.

The prominent property of (3.3.4) when $f_k = 0$ is that, in spite of the presence of dissipation, it is Hamiltonian [19]. From the sense of the value n_k it is evident that $n_k > 0$. Let us introduce a new variable $P_k = \ln n_k$ determined along all of the real axis. Then eq. (3.3.4) can be rewritten in the form

$$\int R_{kk'} \frac{\partial P_k}{\partial t} \, dk' + 2(\tilde{\Gamma}_k - \exp(P_k)) = 0$$
$$\tilde{\Gamma}_k = \int R_{kk'} \gamma_{k'} \, dk' \tag{3.3.10}$$

where $R_{kk'}$ is the kernel of the operator inverse to the operator with the kernel $T_{kk'}$. It is evident that $R_{kk'} = -R_{k'k}$.

The equation (3.3.10) is Hamiltonian, i.e. it can be written in the form

$$\int R_{kk'} \frac{\partial P_{k'}}{\partial t} \, dk' = \frac{\delta \mathcal{H}}{\delta P_k^*} \tag{3.3.11}$$

where the Hamiltonian $\mathscr{H}$ takes the form

$$\mathscr{H} = \int \mathrm{d}\boldsymbol{k}\, (\exp(P_{\boldsymbol{k}}) - \tilde{\Gamma}_{\boldsymbol{k}} P_{\boldsymbol{k}}) .$$

With the help of (3.3.11) it is easy to make sure that $\mathscr{H}$ is an integral of motion. When $\gamma_{\boldsymbol{k}} = 0$, it transforms to the well-known law of conservation of the number of quanta which is valid as was mentioned in chapter 1, yet within the framework of a dynamic description. When $\gamma_{\boldsymbol{k}}$ is not equal to zero, the Hamiltonian $\mathscr{H}$ is not calculated constructively, because of the difficulties of the inversion of $T_{\boldsymbol{k}\boldsymbol{k}'}$. However, its existence allows important conclusions to be made with respect to wave dynamics. For example, it follows that (3.3.11) has no asymptotic steady stationary solutions. In reality, in a stationary state the Hamiltonian $\mathscr{H}$ differs, generally speaking, from that calculated from the initial data. Thus, the relaxation process to a stationary state (if it takes place) occurs only due to a small noise term.

There exists one more group of physical problems when the self-consistent field approximation, i.e. a direct splitting up of fourth-order correlators described non-trivial physical phenomena. Let us consider the oscillating excitation by a homogeneous external field in a medium with a non-decay dispersion law. The Hamiltonian of the wave interaction with pumping is as follows (see chapter 1)

$$\mathscr{H}_{\mathrm{p}} = \tfrac{1}{2} \int (V_{\boldsymbol{k}} \exp(2\mathrm{i}\omega_0 t) a_{\boldsymbol{k}} a_{-\boldsymbol{k}} + \mathrm{c.c.})\, \mathrm{d}\boldsymbol{k} \tag{3.3.12}$$

and leads to the dynamical equations

$$\frac{\partial a_{\boldsymbol{k}}}{\partial t} + \mathrm{i}\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} = \mathrm{i}\, V_{\boldsymbol{k}} a^*_{-\boldsymbol{k}} \exp(-2\mathrm{i}\omega_0 t) + \int T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} a^*_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3}\, \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3 . \tag{3.3.13}$$

As was mentioned above, the sum of the phases of the oscillations with opposite wavevectors has a well-defined value. It is to be expected that at a non-linear instability stage an anomalous correlator, $\langle a_{\boldsymbol{k}} a_{\boldsymbol{k}_1} \rangle = \sigma_{\boldsymbol{k}} \delta(\boldsymbol{k} + \boldsymbol{k}')$ will differ from zero. It should be noted that in this case the individual phases are random.

Now let us present one more argument explaining the introduction of an anomalous correlator $\sigma_{\boldsymbol{k}}$. The energy flux to a plasma from an external field is easily expressed through the Hamiltonian (3.3.12). Averaging over individual random phases, we obtain

$$Q = \partial \mathscr{H} / \partial t = 2\, \mathrm{Im} \int V_{\boldsymbol{k}} \sigma^*_{\boldsymbol{k}}\, \mathrm{d}\boldsymbol{k} . \tag{3.3.14}$$

Thus, the energy transform to collective freedom degrees necessarily leads to the anomalous correlator arising.

Multiplying (3.3.13) by $a^*_{\boldsymbol{k}}$ and by $a_{-\boldsymbol{k}}$ and averaging over phases, the equations will be obtained for $\sigma_{\boldsymbol{k}}$ and $n_{\boldsymbol{k}}$, respectively. Splitting the fourth-order correlators in terms of the pair ones:

$$\langle a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4}\rangle = n_{k_1} n_{k_2}[\delta(k_1 - k_3)\,\delta(k_2 - k_4) + \delta(k_1 - k_4)\,\delta(k_2 - k_3)] + \sigma^*_{k_1}\sigma_{k_3}\,\delta(k_1 + k_2)\,\delta(k_3 + k_4)$$

$$\langle a^*_{k_1} a_{k_2} a_{k_3} a_{k_4}\rangle = n_{k_1}[\sigma_{k_2}\,\delta(k_1 - k_3)\,\delta(k_2 + k_4) + \sigma_{k_3}\,\delta(k_1 - k_4)\,\delta(k_2 + k_3) + \sigma_{k_4}\,\delta(k_1 - k_2)\,\delta(k_3 + k_4)]\,, \tag{3.3.15}$$

we obtain

$$\dot n_k = 2n_k[-\gamma_k + \mathrm{Im}\, P^*_k \sigma_k]$$
$$\dot\sigma_k = 2\sigma_k[-\mathrm{i}(\tilde\omega_k - \omega_0) + \gamma_k] + \mathrm{i} P_k(n_k + n_{-k})\,. \tag{3.3.16}$$

These equations differ from the linear ones only by the renormalization of the frequencies and the pumping interaction, i.e., they are the self-consistent field equations,

$$\tilde\omega_k = \omega_k + 2\int T_{kk'} n_{k'}\, \mathrm{d}k'$$
$$P_k = V_k \sigma^*_k + \int S_{kk'}\sigma_{k'}\, \mathrm{d}k' \tag{3.3.17}$$
$$T_{kk'} = T_{kk'kk'}; \qquad S_{kk'} = T_{k,-k',k,-k'}\,.$$

The equations (3.3.16) are usually called the equations of the S-theory due to an essential role of the coefficient $S_{kk'}$ [20]. From (3.3.16) we have the relations

$$(\partial/\partial t + 4\gamma_k)(|\sigma_k|^2 - n_k n_{-k}) = 0$$
$$(\partial/\partial t + 2\gamma_k)(n_k - n_{-k}) = 0$$

which point to the fact that during the time of the order of γ_k^{-1} the arbitrary initial conditions relax to the state (it is not necessarily stationary) for which

$$n_k = n_{-k}\,; \qquad |\sigma_k| = n_k$$

is valid.

The last equality shows that the wave phases in pairs a_k, a_{-k} are completely correlated. The complete phase correlation in pairs makes it possible to change to new variables, i.e. to the wave phase sum in pair ϕ_k and their number n_k. In these variables the energy flux to the plasma is equal to

$$Q = 2\int V_k n_k \sin\phi_k\, \mathrm{d}k\,.$$

Now a physical sense of the effects considered becomes evident. At a linear instability stage, as was shown in chapter 2, the oscillations with $\phi_k = \frac{1}{2}\pi$ possess the maximum growth rate. Non-linear effects lead to a deviation of ϕ_k from $\frac{1}{2}\pi$ which worsens the coupling with the pumping and thus stabilizes the instability. Below, this problem will be considered in more detail.

The simplest generalization of the equations (3.3.16) is a consideration of excited oscillations belonging to two different spectrum branches, e.g., Langmuir and ion-sound waves [21]. In this case an analogous system of equations for three correlators n^s_k, N^ℓ_k and $\sigma_k\,\delta(k-k') = \exp(\mathrm{i}\omega_0 t)\langle a_k b_{-k'}\rangle$ is obtained. A relation similar to (3.3.17) shows that the wave phases in pairs a_k, b_{-k} are also completely correlated

$$|\sigma_k|^2 = n^s_k N^\ell_k\,.$$

When two types of oscillations are excited, there exists a situation when the wave dampings significantly differ from one another, and the relation

$$\gamma_1 > V_k > \gamma_2$$

is valid.

Just such a situation arises for a parametric excitation of Langmuir and ion-sound waves in an isothermal plasma. In this case the wave with damping γ_1, is a forced oscillation and can be eliminated from the equation. It is not difficult to be sure that in this case the sum of the excited wave phases is equal to $\frac{1}{2}\pi$ and does not change even when taking into account non-linear effects. In this case $n_1 < \sigma < n_2$ and, consequently, for a parametric wave excitation in an isothermal plasma such correlation effects are insignificant.

The other important generalization of eq. (3.3.16) is a consideration of pumping with a wavevector differing from zero [22], e.g. the electromagnetic wave decay into two plasmons. If the pumping wave is written in the form $h\exp(-2\mathrm{i}(\omega_0 - \kappa r))$, as a result of instability the wave pairs with $a_{\kappa+k}$, $a_{\kappa-k}$ are excited. The equations of S-theory are similar to the case of the excitation of two different oscillation types. They include three correlators $n^{\mp}_k\,\delta(k-k') = \langle a_{\kappa\pm k} a^*_{\kappa\pm k'}\rangle$; $\langle a_{\kappa+k} a_{\kappa-k'} \exp(2\mathrm{i}\omega_0 t)\rangle = \sigma_k\,\delta(k-k')$.

Also the relation $|\sigma_k|^2 = n^+_k n^-_k$ holds.

The equations (3.3.16) are written for the case of a monochromatic pumping excitation. The criterion of its spectrum narrowness is analogous to (3.2.4): $V_k > \Delta\omega$. In the opposite limiting case the role of anomalous correlators decreases, but because S-model terms are quadratic in n_k and for a non-coherent pumping there exists such a parameter region when the collision term can be ignored.

It was proposed above that for exiting waves the decay processes are insignificant. In practice such situations are very rare. For instance, for Langmuir oscillations it is possible only for the long-wavelength part of spectrum, $kr_D < \sqrt{m/M}$. In the opposite case, as is seen from (3.2.3), the phase correlation effects make the same contribution in the order of magnitude as the processes of a spectrum cascading. Some examples will be considered below.

If the amplitudes of the fields inducing oscillations are so large that $V > \omega_k$, we turn to the region of modified decay instabilities. In this case, as was shown in section 3.2, there arise some second- and third-order anomalous correlators, and it is impossible to construct a self-consistent description.

If the interaction effect on correlation properties of oscillations is taken into account, the collision term describing the oscillations scattering one upon another will be obtained in the next approximation with respect to n_k. Now let us illustrate the derivation of the four-plasmon collision term, counting for simplicity that anomalous correlations are insignificant.

Let us write down the equation for the fourth-order correlator, where the sixth-order correlators are expressed in terms of the pair ones:

$$\frac{\partial I_{kk_1k_2k_3}}{\partial t} + \mathrm{i}(\tilde{\omega}_{k_2} + \tilde{\omega}_{k_3} - \tilde{\omega}_k - \tilde{\omega}_{k_1}) I_{kk_1k_2k_3}$$
$$= 2T_{kk_1k_2k_3}(n_k n_{k_2} n_{k_3} + n_{k_1} n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_3} - n_k n_{k_1} n_{k_2}) . \tag{3.3.18}$$

Here ω_k are the frequencies renormalized due to the wave interaction which are the same as in (3.3.16). Besides, for simplicity, in the right-hand part non-conservative corrections were not considered. Also ignoring time derivatives (as in chapter 2), we obtain

$$I_{kk_1k_2k_3} = n_k n_{k_1}[\delta(k - k_2)\,\delta(k_1 - k_3) + \delta(k - k_3)\,\delta(k - k_2)]$$
$$+ \frac{2\mathrm{i}\, T_{kk_1k_2k_3}}{\Delta\omega} n_k n_{k_1} n_{k_2} n_{k_3} \left(\frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}}\right) . \tag{3.3.19}$$

Here the first term is the solution of the uniform equation (3.3.18) corresponding to purely Gaussian fluctuations. Substituting (3.3.19) into (3.3.3) and using the relation $\mathrm{Im}_{\varepsilon\to 0}(x + \mathrm{i}\varepsilon)^{-1} = \pi\delta(x)$, the well-known kinetic equation is obtained for the waves:

$$\partial n_k/\partial t + \gamma_k n_k = 2\pi \int |T_{kk_1k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)$$
$$\times\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) n_k n_{k_1} n_{k_2} n_{k_3} \left(\frac{1}{n_{k_1}} + \frac{1}{n_k} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}}\right) \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 . \tag{3.3.20}$$

As seen from (3.3.20), the characteristic non-linear growth rate $\gamma_{\mathrm{nl}} \sim (T^2/\Delta\omega) \int n_k\, \mathrm{d}k$ and the applicability condition (3.2.4) for the case of Langmuir turbulence gives

$$W/nT < (kr_{\mathrm{D}})^2$$

which, in the case of narrow spectra, transforms to

$$W/nT < (\Delta k\, r_{\mathrm{D}})^2 .$$

3.4. Collisionless wave kinetics

1. If the medium inhomogeneities as well as the wave distribution inhomogeneities are taken into account, a wave interaction mechanism, which is new in principle, arises. Let us consider a kinetic equation in an inhomogeneous medium. It is in the form

$$\frac{\partial n_k}{\partial t} + \frac{\partial \omega_k}{\partial k}\frac{\partial n_k}{\partial r} - \frac{\partial \omega_k}{\partial r}\frac{\partial n_k}{\partial k} = 0 . \tag{3.4.1}$$

Here ω_k is the wave dispersion law.

A similar equation can be obtained for the medium with the interaction described by (3.3.1) when $\gamma_k = 0$ and when the condition (3.3.2) is valid.

After the Fourier transformation it is not difficult to be sure that from the above-mentioned equality $\langle a_k^* a_{k'} \rangle = n_k \delta(k - k')$ it follows that the density of the oscillation energy is uniform in space. Therefore the medium inhomogeneity leads to a delta-function broadening, and $n_{kk'}$ depends on both arguments. However, if the inhomogeneity is quasi-classical, the dependence on the difference of arguments is sharper than on the sum. Let us introduce the following notations:

$$n_{kk'} = n_{k_+, k_-}; \qquad k^+ = \tfrac{1}{2}(k + k'); \qquad \kappa = k - k'.$$

Let us consider a Hamiltonian medium with a non-decay dispersion law. The equation for a pair correlator will be written by expressing fourth-order correlation functions in terms of the pair ones:

$$I_{kk_1k_2k_3} = n_{kk_2} n_{k_1k_3} + n_{kk_3} n_{k_1k_2}$$

$$\left[\frac{\partial}{\partial t} + \mathrm{i}(\omega_k - \omega_{k'})\right] n_{kk'} = 2\mathrm{i} \int \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3\, \{T_{kk_1k_2k_3} n_{kk_2} n_{k_1k_3}\, \delta(k + k_1 - k_2 - k_3) - T^*_{k_1kk_2k_3} n^*_{kk_2} n^*_{k_1k_3}\, \delta(k + k_1 - k_2 - k_3)\}. \tag{3.4.2}$$

In a homogeneous medium the right-hand side of the equation turns to zero, which corresponds to the case when the interaction in the self-consistent field approximation only renormalizes the oscillation frequency uniformly over space. Let for (3.4.2) make integration in k_2 and introduce new variables $\tilde{k}^+ = \frac{1}{2}(k_1 + k_3)$, $\tilde{\kappa} = k_1 - k_3$. Then (3.4.2) takes the form

$$\left[\frac{\partial}{\partial t} + \mathrm{i}(\omega_{k^+ + \kappa/2} - \omega_{k^+ - \kappa/2})\right] n_{k^+\kappa} = -2\mathrm{i} \int \mathrm{d}\tilde{k}\, \mathrm{d}\tilde{\kappa}\, n_{\tilde{k}\tilde{\kappa}} \times \{T_{k^+ - \kappa/2,\, \tilde{k} + \kappa/2,\, k^+ - \kappa/2 + \tilde{\kappa},\, \tilde{k} - \tilde{\kappa}/2}\, n_{k^+ + \kappa/2,\, \kappa - \tilde{\kappa}} - T_{k^+ + \kappa/2 - \tilde{\kappa},\, \tilde{k} + \tilde{\kappa}/2,\, k + \kappa/2,\, \tilde{k} - \tilde{\kappa}/2}\, n_{k^+ - \tilde{\kappa}/2,\, \kappa - \tilde{\kappa}}\}. \tag{3.4.3}$$

We expand in series the matrix elements and correlators in κ, $\tilde{\kappa}$ confining ourselves to the approximation lowest in $\tilde{\kappa}$. We then introduce the oscillation density $n(r)$ slowly varying in space

$$n(r) = \frac{1}{(2\pi)^{3/2}} \int n_{k\kappa} \exp(\mathrm{i}\kappa r)\, \mathrm{d}\kappa.$$

After making the Fourier transform, we obtain

$$\frac{\partial n_k}{\partial t} + \frac{\partial \tilde{\omega}_k}{\partial k} \frac{\partial n_k}{\partial r} - \frac{\partial \tilde{\omega}_k}{\partial r} \frac{\partial n_k}{\partial k} = 0. \tag{3.4.4}$$

Here $\tilde{\omega}_k$ is the frequency renormalized due to the interaction $\tilde{\omega}_k = \omega_k + 2 \int T_{kk'} n_{k'}\, \mathrm{d}k'$. The derivatives of the matrix element in (3.4.4) are to be understood in the following manner:

$$\frac{\partial T_{kk'}}{\partial k} = \frac{\partial}{\partial k_1} (T_{k_1, k'kk'} + T_{kk'k_1k'})\big|_{k_1 = k}.$$

It is not difficult to derive (3.4.4) even when the medium is non-conservative. In this case terms are added to (3.4.4) which describe linear and non-linear damping (3.3.4).

Equation (3.4.4) formally coincides with (3.4.1), however, in (3.4.4) the self-consistent non-linear frequency shift of the interacting waves is taken into account.

It is significant that in the medium described by (3.4.4) an inhomogeneity can arise spontaneously as a result of the development of an instability analogous to the modulational instability of monochromatic waves.

The equation (3.4.4) is significantly simplified when considering narrow wave packets. Expanding ω_k in the vicinity of the packet centre

$$\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}_0} + (\boldsymbol{\kappa} \boldsymbol{v}_{\mathrm{gr}}) + \tfrac{1}{2}\omega'' \kappa_{\parallel}^2 + \frac{v_{\mathrm{gr}}}{2k_0}\kappa_{\perp}^2$$

in the system moving with a group velocity we can obtain, for simplicity confining ourselves to the case of a positive dispersion and introducing dimensionless variables [23],

$$\frac{\partial n_{\boldsymbol{\kappa}}}{\partial t} + \boldsymbol{\kappa}\frac{\partial n_{\boldsymbol{\kappa}}}{\partial \boldsymbol{r}} - 2T\frac{\partial N}{\partial \boldsymbol{r}}\frac{\partial n_{\boldsymbol{\kappa}}}{\partial \boldsymbol{\kappa}} = 0$$
$$T = T_{\boldsymbol{k}_0, \boldsymbol{k}_0}; \qquad N = \int n_{\boldsymbol{k}}\, \mathrm{d}\boldsymbol{k}. \tag{3.4.5}$$

This equation resembles the Vlasov kinetic equation for charged particles moving in a self-consistent potential. The condition $N = \int n_{\boldsymbol{k}}\, \mathrm{d}\boldsymbol{k}$ is the analogue of the Poisson equation.

When deriving (3.4.5), (3.4.4), the four-plasmon collision term is ignored. The criterion of correctness of this action for the simplest case when the spectrum has a single scale in $\boldsymbol{k}$-space, k_0, is

$$\frac{1}{k_0 L} > \frac{W}{nT}(k_0 r_{\mathrm{D}})^{-2}.$$

The most interesting region of applicability of the collisionless kinetics is Langmuir turbulence. In a non-isothermal plasma this application is difficult because of the sound excitation.

In an isothermal plasma when ion-sound motions can be considered forced, the Langmuir turbulence is described by the equation (1.6.21) and, consequently, the obtained results can be directly applied. The self-consistent field equations describing a weakly inhomogeneous Langmuir turbulence in a homogeneous medium take the form

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} + \frac{\partial \tilde{\omega}_{\boldsymbol{k}}}{\partial \boldsymbol{k}}\frac{\partial n_{\boldsymbol{k}}}{\partial \boldsymbol{r}} - \frac{\partial \tilde{\omega}_{\boldsymbol{k}}}{\partial \boldsymbol{r}}\frac{\partial n_{\boldsymbol{k}}}{\partial \boldsymbol{k}} = \left(-\gamma_{\boldsymbol{k}} + \int T_{\boldsymbol{k}\boldsymbol{k}'} n_{\boldsymbol{k}'}\, \mathrm{d}\boldsymbol{k}'\right) n_{\boldsymbol{k}}$$
$$\tilde{\omega}_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} + 2\int \tilde{T}_{\boldsymbol{k}\boldsymbol{k}'} n_{\boldsymbol{k}'}\, \mathrm{d}\boldsymbol{k}', \qquad \tilde{T}_{\boldsymbol{k}\boldsymbol{k}'} = \frac{\omega_{\mathrm{p}}^2}{2nT}\operatorname{Re} G. \tag{3.4.6}$$

In the homogeneous case eq. (3.4.6) transforms into (3.3.4) describing induced scattering by ions.

As was mentioned in chapter 1, eq. (1.6.21) is applicable to an isothermal plasma when $kr_D < \frac{1}{3}\sqrt{m/M}$. In the decay part of spectrum when $kr_D > \sqrt{m/M}$, eq. (1.6.21) has poles corresponding to decay processes, and therefore (3.4.6) can be valid only for describing exotic initial conditions for which decay processes are impossible or for describing a small intensity turbulence satisfying a rigid condition:

$$\omega_p \frac{W}{nT} \frac{k_{\mathrm{dif}}}{k} \ll \gamma_s \simeq \sqrt{m/M}\,\Omega_s .$$

2. The idea that the modulational spectrum instability of Langmuir turbulence can be described using the Vedenov–Rudakov equations is widespread. These equations obtained in 1964 are of the form [24]

$$\frac{\partial n_k}{\partial t} + \frac{\partial \tilde{\omega}_k}{\partial \boldsymbol{k}} \frac{\partial n_k}{\partial \boldsymbol{r}} - \frac{\partial \tilde{\omega}_k}{\partial \boldsymbol{r}} \frac{\partial n_k}{\partial \boldsymbol{k}} = 0 . \tag{3.4.7}$$

Here $\tilde{\omega}_k = \omega_p(\frac{3}{2}k^2 r_D^2 + \delta n/n)$ is the dispersion law of Langmuir oscillations. A slow quasi-classic density variation under the action of ponderomotive forces is described by the equation

$$\frac{\partial^2}{\partial t^2}\delta n - c_s^2 \nabla^2 \delta n = \frac{1}{4\pi M} \nabla^2 \int n_k \,\mathrm{d}\boldsymbol{k} . \tag{3.4.8}$$

The physical meaning of (3.4.7), (3.4.8) is evident – Langmuir oscillations lead to a plasma density re-distribution that changes their trajectories.

It should be noted that in the Vedenov–Rudakov equations oscillations with different $\boldsymbol{k}$ interact only via a mutual semi-classical modulation of the plasma density. Let us show that in (3.4.7), (3.4.8) the essential non-linear effects are omitted. Consider, for simplicity, one-dimensional Langmuir turbulence described by (1.6.6), (1.6.14):

$$\mathrm{i}\frac{\partial E}{\partial t} + E_{xx} = -\delta n\, E, \qquad \frac{\partial^2 n}{\partial t^2} - c_s^2 \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2} |E|^2 . \tag{3.4.9}$$

The Vedenov–Rudakov equations are obtained when averaging (3.4.9), assuming $\langle \delta n\, EE^* \rangle \simeq \delta n \langle EE^* \rangle$.

Let Langmuir oscillations consist of two narrow packets with wavevectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$

$$E = E_1 + E_2 ; \qquad E_{1,2} \sim \exp\{\mathrm{i}(\boldsymbol{k}_{1,2}\boldsymbol{r} - \omega(k_{1,2})\, t)\} . \tag{3.4.10}$$

Then, besides a slow density variation under the action of Langmuir oscillations there arises a quickly oscillating density variation

$$\delta\tilde{n} \simeq \frac{(k_1 - k_2)^2 E_1 E_2^*}{(\omega_{k_1} - \omega_{k_2})^2 - c_s^2(k_1 - k_2)^2} . \tag{3.4.11}$$

Substituting (3.4.11) into (3.4.9), we see that in the equations a non-linear frequency shift appears which does not result in a slow density variation. Within a static limit, when in the second equation of

(3.4.9) the term $\partial^2 n/\partial t^2$ can be neglected, the density variation is explicitly expressed by the plasmon number. From (3.4.9) it follows that

$$\tilde{\omega}_k = \omega_k + \tilde{T} \int n_k \,\mathrm{d}\boldsymbol{k}, \qquad \tilde{T} = T_{0,0}. \tag{3.4.12}$$

It follows from eq. (3.4.12) with the correct answer $\tilde{\omega}_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} + 2\int T_{\boldsymbol{kk}'} n_{\boldsymbol{k}'}\,\mathrm{d}\boldsymbol{k}'$ that besides the error in a factor two, the Vedenov–Rudakov equations substitute $T_{\boldsymbol{kk}'}$ by its static value $T_{0,0}$. Such a substitution is impossible even for qualitative estimates, since $T_{\boldsymbol{kk}'}$ is a sign-variable function.

The modulational monochromatic wave instability was first discovered just in such a way (Vedenov and Rudakov, 1964 [24]). However, in so doing it is possible to find only the limit of the instability growth rate as $p \to 0$. It is explained by the fact that when transforming from the non-linear Schrödinger equation to the Vedenov–Rudakov equations the dispersion and diffraction effects limiting the instability are lost. Thus, the applicability criterion of the Vedenov–Rudakov equations is the Langmuir wave spectrum narrowness:

$$(\delta k/k)^2 < W/nT.$$

The collisionless kinetic equations are applicable, on the contrary, for wide packets in $\boldsymbol{k}$-space, $(\delta k/k)^2 > W/nT$. In the intermediate case $(\delta k/k)^2 \sim W/nT$ giving an averaged description of weakly inhomogeneous turbulence, simple equations were not obtained.

3. As an example of an application of the derived equations, let us consider the effect of the finite packet width on the evolution of the modulational instability. Any uniform distribution of oscillations satisfies the equation (3.4.5).

The modulational instability is the space turbulence modulation appearance. For perturbations $\sim\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{pr})$ we have a dispersion equation similar to that for plasma oscillations:

$$1 + 2\tilde{T} \int \frac{(\boldsymbol{p}\,\partial n/\partial \boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}}{\omega - \boldsymbol{p\kappa}} = 0. \tag{3.4.13}$$

When integrating over $\boldsymbol{\kappa}$, the pole should be rounded along the lower semicircle. It should be noted that only the plasmon distribution function averaged over $\kappa_\perp p$ enters into (3.4.13). If the distribution width Δ in the direction $\boldsymbol{p}$ is sufficiently small* $\Delta^2 < \tilde{T}N_0$ ($\int n_{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{k} = N_0$) the pole contribution can be ignored, and eq. (3.4.13) gives

$$\omega = |\boldsymbol{p}|\,(2\tilde{T}N_0)^{1/2}; \tag{3.4.14}$$

when $\tilde{T} < 0$, the modulational instability takes place. If $\omega'' < 0$, the instability criterion is of the form

$$\left(\omega'' \cos^2\Theta + \frac{v_{\mathrm{gr}}}{k_0}\sin^2\Theta\right)\tilde{T} < 0; \qquad \cos\Theta = \frac{(\boldsymbol{pk})}{pk}$$

which coincides with the monochromatic wave stability criterion.

* Phase randomness is provided by a large spectral width in the transverse direction.

The growth rate (3.4.14) infinitely increases with wavevector $\boldsymbol{p}$. This is connected with ignoring diffraction effects. The maximum instability growth rate is achieved on the quasi-classic approximation applicability boundary $p \sim \sqrt{2\tilde{T}N_0}$, and as to order of magnitude it is equal to the non-linear frequency shift.

If a packet is considered which is narrow in all directions $\Delta^2 < \tilde{T}N_0$, the numerical coefficient of the growth rate (3.4.14) differs from the exact expression obtained with the help of exact dynamical equations mentioned in chapter 2. It is connected with the fact that $\Delta^2 < \tilde{T}N_0$ is a condition inverse to the applicability criterion (2.3.4) of the self-consistent field equations. The equation (3.4.14) is applicable only for describing packets which are wide in the direction normal to $\boldsymbol{p}$ ($\Delta^2 > \tilde{T}N_0$). Due to this fact the individual wave phase randomness is provided.

To investigate a finite width packet effect to fix the ideas it is assumed that the packet has a Lorentz configuration

$$\int n_\kappa \, \mathrm{d}\boldsymbol{\kappa}_\perp = \frac{1}{\pi}\frac{N_0\Delta}{\kappa_\parallel^2 + \Delta^2} .$$

Then after integration of (3.4.13) we obtain

$$\omega = |\boldsymbol{p}|\{(2\tilde{T}N_0)^{1/2} - \mathrm{i}\Delta\} .$$

It is evident that the finite packet width, as was firstly mentioned in [24], stabilizes the modulational instability; for an isotropic distribution the stability criterion coincides with the wave phase randomness condition.

To explain the stabilization mechanism it is appropriate to remember that the equations (3.4.5) are similar to the kinetic equation describing a gas of attractive particles; in these terms the finite packet width is equivalent to a thermal spread. Therefore the instability is stabilized similarly to the process of the gravitating gas instability stabilization by a finite temperature.

4. Now let us consider the effect of a wide turbulent background on the monochromatic wave stability. Let us consider $a_{\boldsymbol{k}}$ as a sum of coherent and stochastic parts

$$a_{\boldsymbol{k}} = A_{\boldsymbol{k}} + \tilde{a}_{\boldsymbol{k}} .$$

Writing out the equation for the coherent part and transforming to the $\boldsymbol{r}$-representation, due to the turbulent background we obtain a parabolic equation for the complex envelope A of the coherent packet

$$\mathrm{i}\left(\frac{\partial A}{\partial t} + v_{\mathrm{gr}}\frac{\partial A}{\partial r}\right) + \frac{\omega''}{2}\frac{\partial^2 A}{\partial r^2} + \frac{v_{\mathrm{gr}}}{2k_0}\nabla_\perp^2 A = \left(\tilde{T}|A|^2 + 2\int T_{\boldsymbol{k}_0\boldsymbol{k}} n_{\boldsymbol{k}} \, \mathrm{d}\boldsymbol{k}\right) A .$$

For the stochastic part we obtain

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} + \frac{\partial \tilde{\omega}_{\boldsymbol{k}}}{\partial \boldsymbol{k}}\frac{\partial n_{\boldsymbol{k}}}{\partial \boldsymbol{r}} - \frac{\partial \tilde{\omega}_{\boldsymbol{k}}}{\partial \boldsymbol{r}}\frac{\partial n_{\boldsymbol{k}}}{\partial \boldsymbol{k}} = 0 .$$

Here $\tilde{\omega}_{\boldsymbol{k}}$ is the frequency renormalized due to the turbulent background and the coherent wave

$$\tilde{\omega}_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} + 2\int T_{\boldsymbol{k}\boldsymbol{k}'} n_{\boldsymbol{k}'}\, \mathrm{d}\boldsymbol{k}' + T_{\boldsymbol{k}\boldsymbol{k}_0}|A|^2 .$$

If the turbulent background is also a narrow packet positioned in the vicinity of $\boldsymbol{k}_0$, the system of equations may be simplified, and after transforming to dimensionless variables in the reference system moving with the group velocity the following expression is obtained

$$\begin{aligned} &\mathrm{i}\, \partial A/\partial t + \tfrac{1}{2}\Delta A = \tilde{T}(|A|^2 + 2N_0)A \\ &\frac{\partial n_{\boldsymbol{\kappa}}}{\partial t} + \boldsymbol{\kappa}\frac{\partial n_{\boldsymbol{\kappa}}}{\partial \boldsymbol{r}} - 2\tilde{T}\frac{\partial}{\partial \boldsymbol{r}}(N_0 + |A|^2)\frac{\partial n_{\boldsymbol{\kappa}}}{\partial \boldsymbol{\kappa}} = 0 . \end{aligned} \tag{3.4.15}$$

The obtained system of equations has solutions in the form of a monochromatic wave on a homogeneous turbulent background. Linearizing (3.4.15) and ignoring diffraction effects in the parabolic equation, the following equation is obtained:

$$\omega^2 = p^2 \tilde{T} A_0^2 (1 - 4\tilde{T}N_0/\Delta^2) . \tag{3.4.16}$$

It is evident that the turbulent background weakly affects the monochromatic wave instability.

3.5. Quasi-dynamic description of singular spectra

The collisionless wave kinetic equations derived in the previous section do not describe some of the important effects arising out of the framework of a quasi-classical description. Thus, for example, using these equations it is impossible to obtain a correct structure of the modulational instability growth rate, and, consequently, to describe adequately its non-linear stage. Thus, these equations are not suitable for describing packets narrow in any direction. In the meantime, for plasma turbulence the case with singular spectra when excited oscillations are concentrated on lines or surfaces within $\boldsymbol{k}$-space, is typical. The vicinity of wavevectors to this surface or line allows us to obtain simplified equations, on the one hand, using the interaction weakness and the phase randomness appearing due to the wavepacket prolongation in one (or two) directions, and, on the other hand, changing to wavepacket envelopes in the third direction.

For simplicity, let us consider the case when the spatial inhomogeneity is one-dimensional [25] (z-axis in the inhomogeneity direction). Then $n_{\boldsymbol{k}\boldsymbol{k}'}$ is a δ-function in transverse directions

$$n_{\boldsymbol{k}\boldsymbol{k}'} = n_{\boldsymbol{k}_\perp}(k_z, k_z')\, \delta(\boldsymbol{k}_\perp - \boldsymbol{k}_\perp') .$$

Taking into account that for fixed $\boldsymbol{k}_\perp$ the packet $n_{\boldsymbol{k}_\perp}(k_z, k_z')$ is concentrated within a narrow layer $\Delta k_z \ll k_z$, in eq. (3.4.1) it is possible to expand $\omega_{\boldsymbol{k}}$ in a series in $k_z - k_z^\circ$ (k_z° is the coordinate of the packet centre, $k_z^\circ = f(\boldsymbol{k}_\perp)$) and to ignore the $\tilde{T}$ dependence on $k_z - k_z^\circ$.

The derived equations are essentially simplified by passing to the $\boldsymbol{r}$-representation along the

z-coordinate

$$\left\{\frac{\partial}{\partial t}+\mathrm{i}(\tilde{\omega}_{\boldsymbol{k}}(z)-\tilde{\omega}_{\boldsymbol{k}}(z'))+v_{\mathrm{gr}}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z'}\right)+\mathrm{i}\frac{\omega''}{2}\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial z'^2}\right)\right\}n_{\boldsymbol{k}}(z,z')=0\,. \tag{3.5.1}$$

Here

$$v_{\mathrm{gr}}=\partial\omega_{\boldsymbol{k}}/\partial k_z\,;\qquad \omega''=\partial^2\omega_{\boldsymbol{k}}/\partial k_z^2$$

$$\tilde{\omega}_{\boldsymbol{k}}(z)=\omega_{\boldsymbol{k}}+2\int T_{\boldsymbol{k}\boldsymbol{k}'}n_{\boldsymbol{k}'}(z,z')\,\mathrm{d}\boldsymbol{k}_\perp \tag{3.5.2}$$

$$n_{\boldsymbol{k}}(z,z')=\frac{1}{(2\pi)^3}\int n_{\boldsymbol{k}_\perp}(k_z,k_z')\exp(\mathrm{i}[k_zz-k_z'z'-k_z^\circ(z-z')])\,\mathrm{d}k_z\,\mathrm{d}k_z'\,.$$

Like the equations derived above, (3.5.1) is an equation with a self-consistent field. The neglect of the collision term is also significant. In a non-conservative medium there appear in (3.5.1) terms describing linear and non-linear dampings (3.3.5). In a homogeneous medium $n_{\boldsymbol{k}}(z,z')$ depends only on the difference $(z-z')$. If the space inhomogeneity is slow, i.e. $n_{\boldsymbol{k}}(z,z')$ depends on $z+z'$ more weakly than on $z-z'$, in eq. (3.5.1) it is possible to expand

$$\tilde{\omega}_{\boldsymbol{k}}(z)-\tilde{\omega}_{\boldsymbol{k}}(z')\simeq\frac{\partial\omega_k}{\partial z}(z-z')$$

and change to the z-coordinate in the $\boldsymbol{k}$-representation. Then we can again operate with the collisionless kinetic equations.

However, there exists another method of simplification of (3.5.1) which is not connected with the assumption on a quasi-classical medium parameter change [25]. It is evident that it has a partial solution:

$$n_{\boldsymbol{k}}(z,z')=A(z)\,A(z')\,. \tag{3.5.3}$$

In this case $A_{\boldsymbol{k}_\perp,k_z^\circ}(z)$ satisfies the equation

$$\left\{\mathrm{i}\left(\frac{\partial}{\partial t}+v_{\mathrm{gr}}\frac{\partial}{\partial z}\right)+\frac{\omega''}{2}\frac{\partial^2}{\partial z^2}+\tilde{\omega}_k(z)\right\}A=0 \tag{3.5.4}$$

where

$$\tilde{\omega}_{\boldsymbol{k}}=\omega_{\boldsymbol{k}}+2\int T_{\boldsymbol{k}\boldsymbol{k}'}|A_{\boldsymbol{k}'}|^2\,\mathrm{d}\boldsymbol{k}_\perp\,.$$

Evidently it represents the generalized equation (1.5.8) for envelopes of a monochromatic wave for the case of an extended packet. $A_{\boldsymbol{k}_\perp\boldsymbol{k}_0}$ is a function of $\boldsymbol{k}_\perp$. If $A_{\boldsymbol{k}_\perp k_z^\circ}\sim\delta^{1/2}(\boldsymbol{k}_\perp-\boldsymbol{k}_\perp^\circ)$ is substituted into

(3.5.4), for $A_{k_0}(z)$ an equation is obtained differing from (1.5.8) only by the coefficient 2 before $T_{k_0k_0}$. The origin of the factor 2, as in (3.5.4), is explained by the phase randomness.

It is worth noting that for the validity of (3.5.4) it is also necessary that the relation (3.5.3) would be satisfied on the boundary of the medium, which is, generally speaking, not necessary. For example, it is not fulfilled for the cases when the turbulence is excited within a layer on the boundaries on which the oscillation dissipation takes place.

However, for the most interesting physical problems of the modulational instability of turbulent spectra of localized turbulent bunches evolution eq. (3.5.4) gives an adequate description.

It is not difficult to make some simplifications of (3.5.1). If the non-homogeneity is not assumed one-dimensional, eqs. (3.5.1) keep their form; however, the expression for the renormalized frequency does not have a local structure (3.5.2). But if the wave spectrum is almost one-dimensional (an arbitrary line), and the non-homogeneity is two-dimensional, for the quantity

$$n_{k_\perp, k_z}(r, r') = \frac{1}{(2\pi)^2} \int \mathrm{d}k_\perp \, \mathrm{d}k'_\perp \, n_{k_z, k'_\perp, k_\perp} \exp\{\mathrm{i}(k_\perp r_\perp - k'_\perp r'_\perp - k^\circ_\perp (r - r'))\}$$

a simple equation can be derived:

$$\left\{\mathrm{i}\left(\frac{\partial}{\partial t} + v_{\mathrm{gr}}\left(\frac{\partial}{\partial r_\perp} - \frac{\partial}{\partial r'_\perp}\right)\right) + \mathrm{i}(\tilde{\omega}_k(r_\perp) - \tilde{\omega}_k(r'_\perp))\right\} n_k(r_\perp, r'_\perp) = 0$$
$$\tilde{\omega}_k(r) = \omega_k + 2\int T_{kk'} n_{k'}(r_\perp, r'_\perp)\, \mathrm{d}k' . \qquad (3.5.5)$$

For the sake of conciseness here the diffraction term $\propto \omega''_{k^*}$ is not written down. The expression for $\tilde{\omega}_k$ can be obtained for an arbitrary inhomogeneity if $T_{kk'} = \text{constant}$, and also in some other particular cases. In the most interesting case in the study of the transverse modulation of Langmuir oscillations propagating in one direction the analogue of (3.5.4) takes the form

$$\mathrm{i}\psi_k(r_\perp) + \tfrac{1}{2} r_{\mathrm{D}}^2 \nabla^2 \psi_k + \int \tilde{T}_{kk'} |\psi_{k'}|^2 \, \mathrm{d}k' \, \psi_k + \mathrm{i}\int T_{kk'} |\psi_{k'}|^2 \, \mathrm{d}k' \, \psi_k = 0 . \qquad (3.5.6)$$

If a medium is slowly inhomogeneous even when oscillations are absent, it can be taken into account for only a quadratic Hamiltonian

$$\mathcal{H} = \int \omega_{kk'} a_k^* a_{k'} \, \mathrm{d}k \, \mathrm{d}k' .$$

Because $\omega_{kk'}$ in a quasi-classical case depends on the difference of argument more strongly than on the sum, repeating all the above-mentioned considerations, we again come to the equations (3.5.4), where the ω_k now depend on r,

$$\omega_k \to \omega_k(r) = \frac{1}{\sqrt{2\pi}} \int \omega_{k_\perp \kappa} \exp(\mathrm{i}\kappa r)\, \mathrm{d}\kappa .$$

Table 3.1
Induced scattering on ions matrix elements $\omega_1(\boldsymbol{k}) \to \omega_2(\boldsymbol{k}') + |\boldsymbol{k}-\boldsymbol{k}'|v_{T_i}$. Induced scattering on ions matrix elements have the form

$$T_{\boldsymbol{k}\boldsymbol{k}'} = f(\boldsymbol{k}, \boldsymbol{k}')\, \mathrm{Im}\, G\left(\frac{\omega_1(\boldsymbol{k}) - \omega_2(\boldsymbol{k}')}{|\boldsymbol{k}-\boldsymbol{k}'|v_{T_i}}\right)$$

Process	$f_2(\boldsymbol{k}, \boldsymbol{k}')$
$\omega^{\ell}_{\boldsymbol{k}} \to \omega^{\ell}_{\boldsymbol{k}'} + \lvert\boldsymbol{k}-\boldsymbol{k}'\rvert v_{T_i}$	$\dfrac{\omega_{\mathrm{p}}^2}{2n_0T_{\mathrm{e}}}\dfrac{(\boldsymbol{k}\boldsymbol{k}')^2}{k^2k'^2}$
$\omega^{\pm}_{\boldsymbol{k}} \to \omega^{\pm}_{\boldsymbol{k}'} + \lvert\boldsymbol{k}-\boldsymbol{k}'\rvert v_{T_i}$	$\dfrac{\omega_{\mathrm{p}}^4}{2n_0T_{\mathrm{e}}}\dfrac{\omega^{\pm}_{\boldsymbol{k}}\omega^{\pm}_{\boldsymbol{k}'}\lvert(\boldsymbol{k}\boldsymbol{k}')^2 - \omega_{\mathrm{H}}^2 k_z k'_z/\omega^{\pm}_{\boldsymbol{k}}\omega^{\pm}_{\boldsymbol{k}'} + \mathrm{i}(\omega_{\mathrm{H}}[\boldsymbol{k}\boldsymbol{k}']_z/(\omega^{\pm}_{\boldsymbol{k}}\omega^{\pm}_{\boldsymbol{k}'})^{1/2})\rvert^2}{(\omega^{+2}_{\boldsymbol{k}} - \omega^{-2}_{\boldsymbol{k}})(\omega^{+2}_{\boldsymbol{k}'} - \omega^{-2}_{\boldsymbol{k}'})}$
$\omega^{\mathrm{t}}_{\boldsymbol{k}} \to \omega^{\ell}_{\boldsymbol{k}'} + \lvert\boldsymbol{k}-\boldsymbol{k}'\rvert v_{T_i}$	$\dfrac{\omega_{\mathrm{p}}^2}{2n_0T_{\mathrm{e}}}\dfrac{(\boldsymbol{k}' S^{\lambda}_{\boldsymbol{k}})^2}{k'^2}$

In particular, ignoring the interaction we obtain dynamic equations (see chapter 2) which describe the wave propagation through a non-homogeneous medium.

Now let us use the obtained equations in order to investigate the above-considered problem of the singular spectrum stability. Considering a narrow packet and assuming $T_{\boldsymbol{k}\boldsymbol{k}'} = \tilde{T}$ for perturbations $\sim\exp(-\mathrm{i}\Omega t + \mathrm{i}\boldsymbol{\kappa}\boldsymbol{r})$ from (3.5.4) we obtain the dispersion relation

$$\Omega = (\boldsymbol{\kappa}\boldsymbol{v}_{\mathrm{gr}}) \pm \sqrt{\frac{\omega''\kappa^2}{2}\left(\frac{\omega''\kappa^2}{2} + 4\tilde{T}\int |A^0_{\boldsymbol{k}}|^2\, \mathrm{d}\boldsymbol{k}\right)}\,.$$

When $\kappa\omega'' \ll 4\tilde{T}\int |A^0_{\boldsymbol{k}}|\,\mathrm{d}\boldsymbol{k}$ it transforms to the relation (3.4.6). But, by virtue of the fact that the initial equations contain diffraction effects, it gives a full structure of the growth rate which is qualitatively similar to the growth rate behaviour of the modulational instability (chapter 2).

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