SHOCK WAVES PROPAGATING ON SOLITONS ON THE SURFACE OF A FLUID

V. E. Zakharov

UDC 532.59+530.182

Using an inversion method, we construct an exact solution of the Kadomtsev—Petviashvili equation, which describes weakly nonlinear waves on the surface of a shallow fluid. The solution describes a "rectifying wave" which propagates as a distorted soliton. We have simultaneously obtained classes of exact solutions for the one-dimensional equation of thermal conduction with a source.

1. In 1974 it was established [1] that the method of the inverse scattering problem (MISP) was applicable to certain nonlinear Hamiltonian wave systems in two-dimensional space. The best known of these systems is the Kadomtsev—Petviashvili (KP) equation:

$$\frac{\partial}{\partial x}(u_t + 6\,uu_x + u_{xxx}) = 3\alpha^2\, u_{yy}, \tag{1.1}$$

which describes weakly nonlinear waves in media where the dispersion is small. Two cases can be distinguished, depending on the sign of the dispersion: $\alpha^2 = -1$ (KP-1 equation), and $\alpha^2 = +1$ (KP-2 equation). In the latter case, Eq. (1.1) describes gravity waves on the surface of a shallow fluid. In both cases, Eq. (1.1) has an exact solution in the form of a simple soliton:

$$u = \frac{2\varkappa^2}{\mathrm{ch}^2\,\varkappa\,(x - 4\varkappa^2 t)}\,. \tag{1.2}$$

Even in 1970 it had been established [2] that, in the framework of the KP-1 equation, the soliton is unstable, and has a tendency to bend spontaneously. On the other hand, in the framework of the KP-2 equation, the soliton is stable. Oscillations of acoustic type may propagate along the soliton. These oscillations undergo a certain amount of damping, due to radiation of acoustic waves into space behind the propagating soliton. In [3], we used MISP to calculate precisely the growth rate of soliton instability (for the KP-1 case) and the acoustic dispersion law, including damping (for the KP-2 case). (In this regard, see also the work in [4].)

In the present paper, we present exact solutions of the KP-2 equation which describe the nonlinear propagation of sound in the form of a soliton. Corresponding solutions have already appeared in [1] and [3], but they were not interpreted properly. Consideration shows that these solutions describe the propagation on a soliton of "rectifying waves" similar to those which may propagate as a predistorted extended thread in a viscous medium. In the course of "rectification," soliton energy is lost in the form of sound propagating behind. These waves can therefore be considered as rarefaction shock waves, which are accompanied by bending of the soliton. It is probable that the solutions which we have derived may turn out to be useful in the study of the propagation of solitons in real situations. For example, they may be applied to describe the behavior of a tsunami wave traveling in the inhomogeneous zone on the bottom of the ocean.

2. We consider a situation which occurs when one attempts to apply the exact methods which can be amalgamated under the general label of "the method of inverse problems" to nonlinear equations, including the KP-2 equation. In carrying out this application, it is convenient to proceed in two stages. In the first stage, it is advisable to present a method of constructing a rather broad class of exact solutions of the equation under consideration. This is relatively easy to do, if it is possible to do it in general, i.e. if the equation belongs hypothetically to the set of equations which can be integrated by means of MISP. (The KP equation belongs to this set.) It is further necessary to select out of the set of solutions obtained in this way those solutions which are actually of interest to us, i.e. the

Institute of Theoretical Physics, Academy of Sciences of the USSR. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 29, No. 9, pp. 1073-1079, September, 1986.

0033-8443/86/2909-0813$12.50

solutions of the equation for the initial or boundary value problem which was previously posed. This may turn out to be a difficult job for even the simplest problem. Thus, the Cauchy problem for the KP-2 equation over the entire xy-plane with initial conditions which fall off rapidly as $|x^2 + y^2| \to \infty$ was solved only in 1983 in [5]. We will show below that the results of [5] can be reproduced very simply. It is very important to apply the Cauchy problem in the xy-plane with initial data which falls off as $|x| \to \infty$ but is finite as $|y| \to \infty$: up to the present time, this has not been solved, and it is not known how one should go about solving it. This applies all the more to the boundary value problem. The reason for the difficulty consists in the fact that the exact solutions which have been derived have more or less pronounced singularities in the xy plane, and they do not make much physical sense. It is therefore a reasonable problem to pick out those exact solutions which are certainly free of singularities, even if they are possibly not solutions of any initial or boundary value problem. Solutions of such a kind make sense physically, and they may be used in applications. This class of solutions also contains the solutions which we have constructed of the type "rectifying wave solitons."

3. We consider the method of constructing exact solutions of Eq. (1.2) which were described in [1]. Suppose we are given the Gel'fand—Levitan—Marchenko equation:

$$K(x,z) + F(x,z) + \int_x^\infty K(x,s)\, F(s,z)\, ds = 0. \tag{3.1}$$

Here, F(x, z) is a known function, but K(x, z) is unknown. Both of these functions also depend on the additional variables y and t. Suppose the function F obeys the two equations

$$\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0; \tag{3.2}$$

$$\frac{\partial F}{\partial t} + 4\left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3}\right) = 0. \tag{3.3}$$

Then the function u(x, y, t) = 2dK(x, x, y, t)/dx satisfies Eq. (1.1).

Instead of Eq. (3.1), we can use the equation

$$K(x,z) + F(x,z) + \int_{-\infty}^{x} K(x,s)\, F(s,z)\, ds = 0. \tag{3.4}$$

The solutions obtained from this will differ from the preceding solutions by substituting x for —x and t for —t: this is permitted by Eq. (1.1).

Before we turn to analyzing individual solutions of Eq. (1.1) (which we may obtain by using the procedure we have just described), we should mention that this procedure may be substantially generalized [6, 7]. Let us write the functions F and K in the form

$$F = \iint\limits_{\substack{\mathrm{Re}\,\lambda_1>0\\ \mathrm{Re}\,\lambda_2>0}} T(\lambda_1, \bar\lambda_1, \lambda_2, \bar\lambda_2) \exp\left[-(\lambda_1 x + \lambda_2 z) - (\lambda_1^2 - \lambda_2^2)\, y + 4(\lambda_1^3 - \lambda_2^3)\, t\right] d\lambda_1\, d\bar\lambda_1\, d\lambda_2\, d\bar\lambda_2; \tag{3.5}$$

$$K = \int\limits_{\mathrm{Re}\,\lambda>0} K(\lambda, \bar\lambda, x, y, t) \exp(-\lambda z + \lambda^2 y + 4\lambda^3 t)\, d\lambda\, d\bar\lambda. \tag{3.6}$$

Obviously, F satisfies Eqs. (3.2) and (3.3). Substituting (3.5) and (3.6) into (3.1) we obtain an integral equation:

$$K(\lambda, \bar\lambda, x, y, t) + \int T(\lambda_1, \bar\lambda_1, \lambda, \bar\lambda) \exp(-\lambda_1 x - \lambda_1^2 y + 4\lambda_1^3 t)\, d\lambda_1\, d\bar\lambda_1 \tag{3.7}$$

$$+ \iint \left(\frac{K(\lambda_1, \bar\lambda_1, x, y, t)\, T(\lambda_2, \bar\lambda_2, \lambda, \bar\lambda)}{\lambda_1 + \lambda_2} \exp\left[-(\lambda_1 + \lambda_2)\, x + (\lambda_1^2 + \lambda_2^2)\, y + 4(\lambda_1^3 + \lambda_2^3)\, t\right] \right) d\lambda_1\, d\bar\lambda_1\, d\lambda_2\, d\bar\lambda_2 = 0.$$

In Eq. (3.7) we deliberately do not indicate the regions over which the integrations are performed. According to the scheme in [1], the regions of integration should be those where Re $\lambda_1 > 0$ and Re $\lambda_2 > 0$. In such a case, the denominator in (3.7) does not vanish. However, it has been shown in [6] and [7] that Eq. (3.7) can be used by assuming that the integration extends over the entire complex λ_1, λ_2 plane. In that case, the integral in (3.7) is to be understood in the sense of its principal value:

$$\frac{1}{\lambda_1+\lambda_2}=\lim_{\varepsilon\to 0}\frac{\bar{\lambda}_1+\bar{\lambda}_2}{|\lambda_1+\lambda_2|^2+\varepsilon^2}\ . \tag{3.8}$$

For the solution u(x, y, t) we now have

$$u(x,y,t)=2\frac{d}{dx}\int K(\lambda,\bar{\lambda},x,y,t)\exp(-\lambda x+\lambda^2 y+4\lambda^3 t)\,d\lambda\,d\bar{\lambda}. \tag{3.9}$$

Eq. (3.9) allows us to guess what form we should look for in the kernel $T(\lambda_1, \bar{\lambda}_1, \lambda, \bar{\lambda})$ such that we can obtain solutions which fall off in the plane. Suppose the solution we are seeking is quite small. Then we obtain approximately

$$K(\lambda,\bar{\lambda},x,y,t)=-\int T(\lambda_1,\bar{\lambda}_1,\lambda,\bar{\lambda})\exp(-\lambda_1 x-\lambda_1^2 y+4\lambda_1^3 t)\,d\lambda_1\,d\bar{\lambda}_1,$$

$$u=-2\frac{d}{dx}\int T(\lambda_1,\bar{\lambda}_1,\lambda,\bar{\lambda})\exp[-(\lambda+\lambda_1)x+(\lambda^2-\lambda_1^2)y+4(\lambda_1^3-\lambda^3)t]\,d\lambda_1\,d\bar{\lambda}_1. \tag{3.10}$$

In order that u decline to zero as $x^2 + y^2 \to 0$, it is necessary that

$$\mathrm{Re}(\lambda+\lambda_1)=0,\quad \mathrm{Re}(\lambda_2-\lambda_1^2)=0, \tag{3.11}$$

This yields $\lambda_1 = -\bar{\lambda}$, i.e.,

$$T(\lambda_1,\bar{\lambda}_1,\lambda,\bar{\lambda})=R(\lambda,\bar{\lambda})\,\delta(\lambda_1+\bar{\lambda})\,\delta(\bar{\lambda}_1+\lambda).$$

Eq. (3.7) now takes on the form

$$K(\lambda,\bar{\lambda},x,y,t)+R(\lambda,\bar{\lambda})\exp(\bar{\lambda}x-\bar{\lambda}^2 y-4\bar{\lambda}^3 t)+\iint\frac{K(\lambda_1,\bar{\lambda}_1,x,y,t)\,R(\lambda,\bar{\lambda})}{\lambda_1-\bar{\lambda}} \tag{3.12}$$

$$\times\exp[-(\lambda_1-\bar{\lambda})x+(\lambda_1^2-\bar{\lambda}^2)y]\,d\lambda_1\,d\bar{\lambda}_1=0$$

This becomes equivalent to the integral equation which was derived in [5] for the rapidly declining case by a completely different method. Note that for arbitrary $R(\lambda, \bar{\lambda})$ we obtain solutions of Eq. (1.1) which are in general complex. In order to obtain real solutions, we need to impose the requirement

$$\bar{R}(\bar{\lambda},\lambda)=R(\bar{\lambda},\lambda). \tag{3.13}$$

By considering iterations in (3.12), it can be proved easily by induction that the solution falls off rapidly in the xy plane for finite T. When $R(\lambda, \bar{\lambda})$ is sufficiently small, a series of iterations will converge. However, the precise conditions for convergence are unknown.

4. We now turn to the construction of exact solutions of another kind, which do not reduce to solutions with rapid fall-off. We start with the Marchenko equation (3.1) and we seek a solution in the form

$$F(x,z,y,t)=\varphi(x,y,t)\,\psi(z,y,t). \tag{4.1}$$

The functions φ and ψ satisfy the equations

$$\frac{\partial\varphi}{\partial y}+\frac{\partial^2\varphi}{\partial x^2}=0,\quad \frac{\partial\psi}{\partial y}-\frac{\partial^2\psi}{\partial z^2}=0, \tag{4.2}$$

as well as the equations

$$\frac{\partial\varphi}{\partial t}+4\frac{\partial^3\varphi}{\partial x^3}=0,\quad \frac{\partial\psi}{\partial t}+4\frac{\partial^3\psi}{\partial z^3}=0. \tag{4.3}$$

Substituting (4.1) in Eq. (3.1) we have

$$u=2\frac{d^2}{dx^2}\ln\Delta,\quad \Delta=1+\int_x^\infty\varphi(x)\,\psi(x)\,dx. \tag{4.4}$$

The question of choosing functions φ and ψ is far from trivial. The function ψ satisfies the equation of thermal conduction, uncorrected as $y \to -\infty$, while the function φ satisfies the inverse equation of thermal conduction, uncorrected as $y \to \infty$. If the function $\Delta(x, y)$ vanishes on some curve $x = x_0(y)$, singularities of the type

$$u = 2\frac{1}{(x - x_0(y))^2}$$

will occur on this curve. If singularities are to be avoided, we must satisfy the strict condition

$$\Delta > 0. \tag{4.5}$$

This condition can be satisfied if we assume

$$\varphi = \int_0^\infty f(\lambda)\exp(-\lambda x - \lambda^2 y + 4\lambda^3 t)\, d\lambda; \tag{4.6}$$

$$\psi = \int_0^\infty g(\lambda)\exp(-\lambda x + \lambda^2 y + 4\lambda^3 t)\, d\lambda, \tag{4.7}$$

where $f(\lambda) \geqslant 0$, $g(\lambda) \geqslant 0$ are real nonnegative functions. In the event that

$$f(\lambda) = g(\lambda) \tag{4.8}$$

solution (4.4) is symmetric with respect to the substitution $y \to -y$. The soliton (1.2) belongs to solutions of the type (4.4). In this case

$$f(\lambda) = g(\lambda) = \sqrt{2\varkappa}\,\delta(\lambda - \varkappa). \tag{4.9}$$

Let us consider solutions of the type (4.4) which are similar to the soliton (1.2) in the sense that the functions $f(\lambda)$ and $g(\lambda)$ are concentrated into narrow regions near the point $\varkappa$: $(\varkappa - a) < \lambda < (\varkappa + a)$, where $a \ll \varkappa$. In this case, the expression for Δ

$$\Delta = 1 + \int_0^\infty\int_0^\infty \frac{f(\lambda_1)\, g(\lambda_2)}{\lambda_1 + \lambda_2}\exp[-(\lambda_1 + \lambda_2)x - (\lambda_1^2 - \lambda_2^2)y + 4(\lambda_1^3 + \lambda_2^3)t]\, d\lambda_1\, d\lambda_2 \tag{4.10}$$

simplifies to the form

$$\Delta = 1 + e^{-2\varkappa x'}\Phi(x' + 2\varkappa y - 8\varkappa^2 t)\,\Psi\,(x' - 2\varkappa y - 8\varkappa^2 t) \tag{4.11}$$

Here, $x' = x - 4\varkappa^2 t$ is a coordinate in the soliton frame of reference, and

$$\Phi(s) = \sqrt{2\varkappa}\int_{\varkappa - a}^{\varkappa + a} f(\lambda)\, e^{-\lambda s}\, d\lambda; \tag{4.12}$$

$$\Psi(s) = \sqrt{2\varkappa}\int_{\varkappa - a}^{\varkappa + a} g(\lambda)\, e^{-\lambda s}\, d\lambda. \tag{4.13}$$

Suppose $f(\lambda) = g(\lambda) = A = \text{const}$, such that the solution is symmetric with respect to replacing y with $-y$.

Let us consider expression (4.11) in the asymptotic region $2\varkappa y \gg 8\varkappa^2 t$. Here, the asymptote is independent of time:

$$\Delta = 1 + \frac{A^2}{y^2}\exp[-2\varkappa(x + 2\varkappa a y)]. \tag{4.14}$$

This asymptote corresponds to a slightly curved soliton whose peak is located on the lines

$$x = -2\varkappa a y - \frac{1}{2\varkappa}\ln\frac{A^2}{y^2}. \tag{4.15}$$

Along these lines, the amplitude of the soliton is a constant, and is equal to $\varkappa^2$.

Suppose now that $y \ll 4\varkappa t$. Here, the y-dependence can be neglected, and we have

$$\Delta \simeq 1 + \frac{A^2}{16\varkappa^4 t^2}\exp\{-2[(\varkappa - a)x' + 8\varkappa^2 a t]\}, \quad x' = x - 4\varkappa^2 t. \tag{4.16}$$

Expression (4.16) corresponds to a "rectifying soliton" of reduced amplitude $2(\varkappa - a)^2$, propagating backward with velocity

$$V = \frac{8\varkappa^2 a}{\varkappa - a} + \frac{1}{2\varkappa t}.$$

The reduction in the amplitude of the soliton can be explained by the backward radiation of small amplitude waves into the zone where the coordinate x' is negative. Thus, a rectifying wave can be treated as a running down shock wave.

For more complicated choices of $f(\lambda)$ and $g(\lambda)$, a rather rich set of exact solutions occurs for the KP-2 equation. If these functions consist of singular components in the form of δ-functions in λ, then solitons of smaller amplitude will be present in the radiation background $x' \to -\infty$. These solitons break away as a result of the rectification of the main soliton.

5. Application of the inverse problem method to Eq. (1.1) is based on the fact that this equation is the condition of compatibility for the redefined linear system:

$$\alpha \frac{\partial \Phi}{\partial y} + \frac{\partial^2 \Phi}{\partial x^2} + u\Phi = 0; \tag{5.1}$$

$$\Phi_t + 4\Phi_{xxx} + 6u\Phi_x + 3(u_x + \alpha w)\Phi = 0, \quad w_x = uy. \tag{5.2}$$

If we construct exact solutions of the KP equation, this will enable us simultaneously to construct exact solutions of this linear system. They are given by the formula

$$\Phi(x, y, t) = \Phi_0(x, y, t) + \int_x^\infty K(x, s, y, t)\,\Phi_0(s, y, t)\,ds. \tag{5.3}$$

Here, Φ_0 is an arbitrary solution of the unperturbed system

$$\alpha \frac{\partial \Phi_0}{\partial y} + \frac{\partial^2 \Phi_0}{\partial x^2} = 0, \quad \Phi_{0t} + 4\Phi_{0xxx} = 0, \tag{5.4}$$

and $K(x, z, y, t)$ is a solution of Marchenko's equation. In this particular case, K has the form

$$K(x, z, y, t) = -\frac{\varphi(x, y, t)\,\psi(z, y, t)}{1 + \int_x^\infty \psi(s, y, t)\,\psi(s, y, t)\,ds}. \tag{5.5}$$

When $\alpha = \pm 1$, Eq. (5.1) is the heat-conduction equation with sources, and there is great applied value in constructing exact solutions of this equation for geophysical problems, particularly for problems in electroprospecting. It is therefore a matter of very real importance to expand the class of functions $u(x, y)$ for which solutions of this kind are possible. (Simultaneously, this will also enlarge the class of exact solutions of the KP equation). This might be achieved by generalizing the procedure which was used above, by choosing a kernel T in (3.7) in the form of a decomposition:

$$T(\lambda_1, \overline{\lambda_1}, \lambda, \overline{\lambda}) = \sum_{n=1}^{N} f_n(\lambda_1, \overline{\lambda_1})\, g'_n(\lambda, \overline{\lambda}). \tag{5.6}$$

In this case, Eq. (3.7) transforms into a finite system of linear algebraic equations which can be solved easily. However, the following question is still unsolved: how much restriction should we place on the functions f_n and g_n such that the solutions will have no singularities? The solution of this question is a task for the near future.

LITERATURE CITED

1. V. E. Zakharov and A. B. Shabat, Funkts. Anal. Prilozhen., 6, No. 3, 43 (1974).
2. B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR, 192, No. 4, 753 (1970).
3. V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz., 22, 364 (1975).
4. S. P. Burtsev, Zh. Eksp. Teor. Fiz., 88, No. 2, 461 (1985).
5. M. J. Ablowitz, D. Bar Yaacov, and A. S. Fokas, Studies Appl. Math., 69, 135 (1983).
6. V. E. Zakharov and S. V. Manakov, Zap. Nauchn. Sem. LOMI, 153, 190 (1984).
7. V. E. Zakharov and S. V. Manakov, Funkts. Anal. Prilozhen., 19, No. 2, 11 (1985).