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THE ALGEBRA OF INTEGRALS OF MOTION OF TWO-DIMENSIONAL HYDRODYNAMICS IN CLEBSCH VARIABLES

V. E. Zakharov

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1. We will consider the equations of two-dimensional hydrodynamics of an incompressible fluid

$$\frac{\partial\Omega}{\partial t}+(V\nabla)\Omega=0,\quad \mathbf{V}=(V_x,V_y),\quad \Omega=\frac{\partial V_x}{\partial y}-\frac{\partial V_y}{\partial x},\quad \operatorname{div}\mathbf{V}=0 \tag{1.1}$$

on a torus Γ, i.e., in a square $-L < x < L$, $-L < y < L$ with periodic conditions on the velocity field. Moreover, the mean vorticity equals zero ($\langle\Omega\rangle = \int_\Gamma \Omega dxdy = 0$). We also must equate to zero the mean flow of the fluid

$$\langle V\rangle=\int_\Gamma V\,dx\,dy=0. \tag{1.2}$$

Then one can introduce a periodic function of the current $\psi(V_x = -(\partial\psi/\partial y)$, $V_y = (\partial\psi/\partial x)$ and rewrite (1) in the form

$$\frac{\partial\Omega}{\partial t}+\{\psi,\Omega\}=0,\quad \{A,B\}=AxBy-AyBx. \tag{1.3}$$

Let us remark that $\psi = -(\delta H/\delta\Omega)$, where $H = -1/2\int_\Gamma \psi\Omega dxdy$ is the kinetic energy of the fluid.

Equation (1.3) is a Hamiltonian system, the phase space of which is the space U of smooth periodic functions $\Omega(x, y)$ with zero mean, the Hamiltonian is the energy H, and the Poisson bracket between the functionals α and β of Ω is determined by the formula

$$[\alpha,\beta]=\int_\Gamma \Omega\left\{\frac{\delta\alpha}{\delta\Omega},\frac{\delta\beta}{\delta\Omega}\right\}dx\,dy. \tag{1.4}$$

Equation (1.3), in which $\psi(x, y, t)$ is an arbitrary given function, has an infinite set of integrals of motion of the form

$$I=\int_\Gamma F(\Omega)\,dx\,dy. \tag{1.5}$$

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Among them, this is true for the hydrodynamic equations (1.1) and for a general form of Hamiltonian systems with the bracket (1.4), i.e., for the equations

$$\frac{\partial\Omega}{\partial t}+\left\{\Omega,\frac{\delta H}{\delta\Omega}\right\}=0. \tag{1.6}$$

Here, H is an arbitrary functional of Ω.

The existence of the integrals (1.5) has a geometric interpretation (see [1]) and is simply a consequence of the ability to change in the hydrodynamic equations [and in more general equations of form (1.3) with an arbitrary function ψ] to Lagrangian variables. Let us denote by G the group $S_0\,\mathrm{Diff}\,T^2$ of diffeomorphisms of Γ, preserving area and the position of the center of gravity. It consists of substitutions of coordinates of the form

$$\begin{gathered}X^1=x'(x,y),\; y^1=y^1(x,y),\; \{x',y'\}=1,\\ X^1(L,y)-x'(-L,y)=2L,\\ J'(x,L)-y'(x,-L)=2L,\\ \langle x'\rangle=\langle y'\rangle=0.\end{gathered} \tag{1.7}$$

The corresponding Lie algebra g consists of vector fields of the form $-\varphi_x(\partial/\partial y)+\varphi_y(\partial/\partial x)$ with a periodic function φ. The commutation in g of the elements determined by the functions φ_1, φ_2 leads to an element determined by the function $\{\varphi_1,\ \varphi_2\}$.

The phase space U is broken under the action of G into orbits — manifolds of functions, translating one to another by the transformations (1.7). The functionals (1.5) are invariant with respect to the action of G and are determined by the choice of orbit. On the other hand, (1.3) describes for arbitrary ψ motion along the orbit and, due to this, preserves the integrals (1.5).

2. The bracket (1.4) is degenerate — the integrals (1.5) commute with any functionals of the vorticity Ω, including themselves. This greatly complicates the diagonalization of the bracket (1.4) — the introduction into two-dimensional hydrodynamics of canonical variables. The degeneracy of the bracket indicates that (1.6) must be considered as a collection of independent Hamiltonian systems, defined on the orbits of G. The canonical variables must be introduced on each orbit separately, which requires as a minimum the effective description of the orbits. Meanwhile, there was proposed yet in the last century, a way to avoid these difficulties — passage to Clebsch variables (see [2]). In the two-dimensional case, it can be generalized to systems of type (1.6) and consists in considering two equations

$$\frac{\partial\lambda}{\partial t}+\left\{\lambda,\frac{\delta H}{\delta\Omega}\right\}=0,\quad \frac{\partial\mu}{\partial t}+\left\{\mu,\frac{\delta H}{\delta\Omega}\right\}=0, \tag{2.1}$$

$$\Omega=\{\lambda,\mu\}. \tag{2.2}$$

It is easy to see that Eq. (2.1) can be rewritten in the form

$$\frac{\partial\lambda}{\partial t}=\frac{\delta H}{\delta\mu},\quad \frac{\partial\mu}{\partial t}=-\frac{\delta H}{\delta\lambda}. \tag{2.3}$$

For $\langle\Omega\rangle=0$ the functions λ, μ can be chosen as periodic. Moreover, the Eqs. (2.3) are Hamiltonian on the phase space $U\times U$. The Hamiltonian, as previously, is H, the Poisson bracket is diagonal, and the variables $\lambda$and μ are canonical. They can be defined with a large degree of nonuniqueness — with a precision up to the transformation

$$\lambda'=\lambda'(\lambda,\mu),\quad \mu'=\mu'(\lambda,\mu),\quad \langle\lambda,\mu\rangle=1. \tag{2.4}$$

Here and in what follows for $F = F(\lambda, \mu)$, $G = G(\lambda, \mu)$

$$\langle F,G\rangle=F_\lambda G_\mu-F_\mu G_\lambda. \tag{2.5}$$

Further, two very simple lemmas are required, presented without proofs.

LEMMA 1. Let the functions A_i (i = 1, 2) from U be subject to the equation

$$\frac{\partial A_i}{\partial t}+\{\psi,A_i\}=0. \tag{2.6}$$

Then, their Jacobian $\{A_1, A_2\}$ is subject to the same equation.

LEMMA 2. Let the functions A_i ($i = 1, \ldots, N$) from U be subject to (2.6). Then the functional

$$J = \int_\Gamma F(A_1, \ldots, A_N)\, dx\, dy \tag{2.7}$$

is an integral of motion of system (2.6).

Lemma 1 explains the connection of systems (2.1) and (1.6) – formula (2.2) implements a projection of any solution of (2.1) to a solution of (1.6). Let us remark that these systems are by no means equivalent, if only because they have phase spaces of different dimensions. Furthermore, it follows from Lemma 1 that the infinite set of equations of the form (2.6) follows along with (1.6) from (2.1). For example,

$$\frac{\partial}{\partial t}\{\lambda, \Omega\} + \left\{\{\lambda, \Omega\}, \frac{\delta H}{\delta \Omega}\right\} = 0. \tag{2.8}$$

Lemma 2 shows that Eqs. (2.1) have a vast set of integrals of motion not depending on the choice of the integral H. Actually, there is conserved any expression of the form

$$J = \int_\Gamma F(\lambda, \mu, \Omega, \{\lambda, \Omega\}, \{\mu, \Omega\} \ldots)\, dx\, dy. \tag{2.9}$$

All the integrals (2.9) have a purely geometric origin. They are conserved under the transformations (1.7), i.e., they are invariant under motion along the orbits of the natural action of the group in U × U. The integrals (1.5) are the simplest frequent case of the integrals (2.9).

Let us state a question concerning the completeness of the system of integrals (2.9): with what arbitrariness can one determine one of the functions; for example $\mu(x, y)$, having fixed the integrals (2.9) and the second function $\lambda(x, y)$: (The motivation for asking this question will be explained below.) The solution of this question is equivalent to determining a subgroup G_0 of G conserving $\lambda(x, y)$. It is clear that the Lie algebra to this subgroup consists of the vector fields vanishing under the action on λ, i.e., having the form

$$Dg_0 = -\frac{\partial}{\partial y} f(\lambda) \frac{\partial}{\partial x} + \frac{\partial}{\partial x} f(\lambda) \frac{\partial}{\partial y}, \quad Dg_0 \lambda = 0. \tag{2.10}$$

Here, $f(\lambda)$ is an arbitrary function.

The subgroup G_0 is a flow along the level lines of $\lambda(x, y)$ and is commutative. Thus, the system of integrals (2.9) is incomplete, and to determine $\mu(x, y)$, an additional set of integrals is needed, determined by a function of one variable. One such integral is the Hamiltonian H.

3. On the integrals J, there is defined the Poisson bracket

$$[J_1, J_2] = \int_\Gamma \left(\frac{\delta J_1}{\delta \lambda(r)} \frac{\delta J_2}{\delta \mu(r)} - \frac{\delta J_1}{\delta \mu(r)} \frac{\delta J_2}{\delta \lambda(r)} \right) dx\, dy, \tag{3.1}$$

immersing them in some Lie algebra. The structure of this algebra is quite complicated. It has two important subalgebras. Let us consider integrals of the form

$$J^0 = \int_\Gamma F(\lambda, \mu)\, dx\, dy. \tag{3.2}$$

Evidently,

$$[J_1^0, J_2^0] = \int_\Gamma \langle F_1, F_2 \rangle\, dx\, dy. \tag{3.3}$$

The integrals J^0 form a subalgebra ℓ_0 of ℓ, isomorphic to the algebra of Hamiltonian vector fields on the plane λ, μ.

Let us consider integrals of the form

$$J = \int_\Gamma F(\lambda, \mu, \Omega)\, dx\, dy \tag{3.4}$$

and compute the variational derivative $\delta J/\delta\lambda$. We have after simple computations

$$\frac{\delta J}{\delta\lambda(r)} = \frac{\partial}{\partial\lambda}\left(F - \Omega\frac{\partial F}{\partial\Omega}\right) + \{\mu, \Omega\}\frac{\partial^2 F}{\partial\Omega^2}. \tag{3.5}$$

Let us restrict ourselves to considering the case when F is analytic according to Ω in a neighborhood of zero, and let us consider integrals of the form

$$J^n = \int_\Gamma F(\lambda, \mu)\, \Omega^m\, dx\, dy. \tag{3.6}$$

Using formula (3.5), it is easy to find

$$[J_1^n, J_2^m] = -\frac{(n-1)(m-1)}{n+m-1}\int_\Gamma \langle F_1, F_2\rangle\, \Omega^{n+m}\, dx\, dy. \tag{3.7}$$

Thereby, is actually proved

<u>THEOREM.</u> Integrals of the form (3.4) form a closed algebra isomorphic to the algebra of currents of the area-conserving diffeomorphism group of the plane.

Let us remark that for periodic λ, μ

$$J^1 = \int_\Gamma F(\lambda, \mu)\, \Omega\, dx\, dy \equiv 0,$$

that agrees with (3.5), (3.7).

Let us consider integrals of the form

$$I_m^n = \int_\Gamma \lambda^m \Omega^n\, dx\, dy. \tag{3.8}$$

Due to (3.7), the integrals (3.8) commute with each other. This is a two-parameter set of integrals, and it can be used to integrate the system (1.6), applying the Liouville theorem. For this, it is necessary on the first stage, fixing $I_m{}^n$ and $\lambda(x, y)$, to express the function $\mu(x, y)$. It follows from the results of Sec. 2 that it is impossible to do this, even using instead of the integrals (3.8) the entire set of integrals (2.9).

Thus, the set of commuting integrals $I_m{}^n$ is insufficient for integrating (1.6). Nevertheless, the question is of interest whether one can imbed the set of integrals $I_m{}^n$ in a wider (but automatically also incomplete) commutative subalgebra of the complete algebra ℓ of integrals (2.9).

4. All the results obtained above can be transferred to the hydrodynamics of a compressible barotropic fluid. Instead of Eq. (1.1), we now have

$$\frac{\partial\Omega}{\partial t} + \operatorname{div}\Omega V = 0. \tag{4.1}$$

Here

$$V_x = -\frac{\partial\psi}{\partial y} + \frac{\partial\Phi}{\partial x}, \qquad V_y = \frac{\partial\psi}{\partial x} + \frac{\partial\Phi}{\partial y},$$

Φ is the velocity potential. Equation (4.1) is supplemented by the system

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho V, \quad \frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + W(\rho) = 0, \quad W = \frac{\partial R}{\partial\rho}, \tag{4.2}$$

$\varepsilon(\rho)$ is the density of the internal energy of the fluid.

System (4.1), (4.2) has a Poisson structure, connected with the diffeomorphism group of the torus Diff T^2 (see, for example, [3]) and the Hamiltonian

$$H = \frac{1}{2}\int_{\Gamma} \rho V^2\, dx\, dy + \int_{\Gamma} \varepsilon(\rho)\, dx\, dy, \tag{4.3}$$

coinciding with the energy of the fluid. "Geometric" integrals also occur, generalizing the integrals (1.5)

$$I = \int_{\Gamma} \rho F\left(\frac{\Omega}{\rho}\right) dx\, dy. \tag{4.4}$$

In passing to Clebsch variables, instead of (4.1), there arises a pair of equations

$$\frac{\partial \lambda}{\partial t} + \operatorname{div} \lambda V = 0, \quad \frac{\partial \mu}{\partial t} + (V\nabla)\mu = 0. \tag{4.5}$$

The projection of the system (4.2), (4.5) to the initial system is realized with the help of the formulas

$$V = \nabla\Phi + \frac{\lambda}{\rho}\nabla\mu, \quad \Omega = \left\{\frac{\lambda}{\rho}, \mu\right\}. \tag{4.6}$$

The system (4.5), (4.6), considered in an extended phase space ρ, Φ, λ, μ, is Hamiltonian, where the variables are broken into the canonically conjugate pairs

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \Phi}, \quad \frac{\partial \Phi}{\partial t} = -\frac{\delta H}{\delta \rho}; \quad \frac{\partial \lambda}{\partial t} = \frac{\delta H}{\delta \mu}, \quad \frac{\partial \mu}{\partial t} = \frac{\delta H}{\delta \lambda}. \tag{4.7}$$

System (4.7) has the same set of geometric integrals, that two-dimensional incompressible hydrodynamics does. In order to be convinced of this, let us remark that $\lambda = \lambda/\rho$ satisfies

$$\frac{\partial \tilde{\lambda}}{\partial t} + (V\nabla)\tilde{\lambda} = 0. \tag{4.8}$$

The following are simply verifiable.

LEMMA 3. If the functions A_i ($i = 1, 2$) satisfy

$$\frac{\partial A_i}{\partial t} + (V\nabla) A_i = 0, \tag{4.9}$$

then $1/\rho\{A_1, A_2\}$ satisfies this same equation.

LEMMA 4. If the set of functions A_i ($i = 1, \ldots, N$) satisfies (4.9), then the integral

$$J = \int_{\Gamma} \rho F(A_1, \ldots, A_N)\, dx\, dy \tag{4.10}$$

is conserved.

Thus, the geometric integrals of (4.7) have the form

$$J = \int_{\Gamma} \rho F\left(\mu, \frac{\lambda}{\rho}, \frac{\Omega}{\rho}, \frac{1}{\rho}\left\{\mu, \frac{\Omega}{\rho}\right\} \ldots\right) dx\, dy. \tag{4.11}$$

5. Among the systems (1.6), there are systems to which the method of the inverse-scattering problem is applicable (more precisely the "vesture method," see [4]). Let us consider the linear redefined system of equations with respect to the complex-valued function φ $(x, y, t, \lambda, \bar{\lambda})$

$$\lambda D_1 \varphi + \{\Omega, \varphi\} = 0, \tag{5.1}$$

$$\frac{\partial\varphi}{\partial t} + \lambda D_2\varphi + \{S, \varphi\} = 0. \tag{5.2}$$

Here

$$D_1 = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}, \quad D_2 = \gamma\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y}$$

are constant vector fields on the plane, Ω and S are unknown real functions of x, y, t.

The consistency condition of (5.1), (5.2) has the form

$$D_1 S = D_1\Omega, \tag{5.3}$$

$$\Omega_t + \{\Omega, S\} = 0. \tag{5.4}$$

Equations (5.3), (5.4) are a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\int_\Gamma \Omega S\,dx\,dy = \frac{1}{2}\int_\Gamma \Omega D_2 D_1^{-1}\Omega\,dx\,dy. \tag{5.5}$$

Except for the Hamiltonian, system (5.3), (5.4) conserves an arbitrary quadratic functional of the form

$$\tilde{H} = \frac{1}{2}\int_\Gamma \Omega D_3 D_1^{-1}\Omega\,dx\,dy, \tag{5.6}$$

where $D_3 = S_1(\partial/\partial x) + S_2(\partial/\partial y)$ is any constant vector field on the plane. For $D_3 = D_1$, we have one of the geometric integrals of the form (1.5).

The system (5.3), (5.4) has infinite sets of integrals of any integral positive degree of uniformity according to Ω. Among them are contained all the geometric integrals (1.5).

Physical applications of (5.3), (5.4) are yet unknown.

Let us consider a general system of form (1.6) with a quadratic translation-invariant Hamiltonian. It is convenient to write it, performing a Fourier transform in the coordinates and letting $L \to \infty$. Then

$$H = \frac{1}{2}\int f_k |\Omega_k|^2. \tag{5.7}$$

For incompressible hydrodynamics $f_k = 1/k^2$, and in the integrable case (5.3), (5.4) $f_k = (\alpha k_x + \beta k_y)/(\gamma k_x + \delta k_y)$. Introducing the complex variable $a_k = 1/\sqrt{2}\,(\lambda_k + i\mu_k)$, let us rewrite (2.3) for this case in the form of one equation

$$\frac{\partial a_k}{\partial t} = i\frac{\delta H}{\delta a_k^*}. \tag{5.8}$$

Here

$$H = \int W_{kk_1,\,k_2k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta k + k_1 - k_2 - k_3\,dk\,dk_1\,dk_2\,dk_3, \tag{5.9}$$

where

$$W_{kk_1,\,k_2k_3} = \frac{1}{4}\{k_1, k_3\}\{k_2, k_4\}[f(k_1 - k_3) + f(k_1 - k_4)] + \frac{1}{4}\{k_1, k_4\}\{k_2, k_3\}[f(k_1 - k_4) + f(k_2 - k_3)]. \tag{5.10}$$

All the Hamiltonians of a system of type (5.8), (5.9), the Hamiltonian of which is given by (5.10), have infinite sets of geometric integrals, including commuting ones. Among the functionals of form (5.9), determined by giving W_{kk_1,k_2k_3} for an arbitrary choice of f_k, except for H, a functional is conserved for which w = const, and also a functional for which W_{kk_1,k_2k_3} is given by (5.10), where f_k = const. In the integrable case (5.3), (5.4) all the functionals of form (5.9), (5.10) are conserved for which f_k is an arbitrary rational function of the ratio k_x/k_y. In this case, the geometric integrals of type (2.9) are immersed in a much wider Lie algebra, concerning the construction of which almost nothing is known.

It would be of significant interest to study systems of type (5.8)-(5.10), for which f_k is close to a rational function of k_x/k_y, and to analyze the possibility of the existence for them of additional integrals of motion.

Systems of type (5.8)-(5.10), including also two-dimensional incompressible hydrodynamics, can also become an object of numerical experimentation in the spirit of the classical Fermi—Ulam—Pasta experiments. The object of these experiments is to clarify to what degree the existence of an infinite number of geometric integrals presents an obstacle to the stochastization of a dynamical system and the arising of turbulence in it.

In general, the existence of additional geometric integrals for system (2.1) compels one to look anew at the problem of two-dimensional hydrodynamic turbulence. Disregarding the "hidden" nature of these integrals, their existence must be considered in constructing phenomenological models of turbulence and computing indices of Kolmogorov spectra of various types.

6. The construction described above has a direct finite-dimensional analog. Let R be a semisimple finite-dimensional Lie group, r be its Lie algebra, realized by matrices of finite order, and S be an element from r. The Lie—Berezin—Kirillov—Konstant bracket gives on the numerical functions α(S) the structure of a Lie algebra

$$\langle\alpha_1,\alpha_2\rangle_s = \operatorname{Sp} S\left[\frac{\partial\alpha_1}{\partial S},\frac{\partial\alpha_2}{\partial S}\right], \tag{6.1}$$

and the equation

$$\frac{\partial S}{\partial t} = \left[S,\frac{\partial H}{\partial S}\right] \tag{6.2}$$

is a Hamiltonian system with Hamiltonian H and Poisson bracket (6.1), which is nondegenerate on the orbits of action on r of the group R.

The bracket (6.1) is nondegenerate on the orbits of R fibering the space r, and it is necessary to generate the diagonalization of this bracket on each orbit separately.

Let the pair (p, q) be an element of the space r × r, being mapped to r by the formula

$$S = [p, q]. \tag{6.3}$$

Then any solution of

$$\frac{\partial p}{\partial t} = \left[p,\frac{\partial H}{\partial S}\right],\quad \frac{\partial q}{\partial t} = \left[q,\frac{\partial H}{\partial S}\right] \tag{6.4}$$

is carried with the help of (6.3) to a solution of (6.2). It is easy to verify that we have

$$\frac{\partial p}{\partial t} = \frac{\partial H}{\partial q},\quad \frac{\partial q}{\partial t} = -\frac{\partial H}{\partial p}. \tag{6.5}$$

Thus, the system (6.4) is Hamiltonian on p, q with respect to the canonical Poisson bracket. It has a rich set of integrals not depending on the form of H. There are conserved all the variables of the form

$$I = S_t F(p, q), \tag{6.6}$$

where F(p, q) is the product of any number of noncommuting matrices p, q, taken in arbitrary order. The space r × r is exfoliated under the action of R into orbits; the integrals (6.6) are invariant with respect to the group action and, thereby, are purely geometric, and the question of the construction of the algebra of these integrals remains open. If (6.2) is integrable, then the extended system (6.5) is also integrable.

Thus, the construction shown allows one, starting from integrate systems of type (6.2), to construct integrable systems in the phase space of doubled dimension. Thus, from the integrability of the simplest Landau—Lifshits equation for a one-dimensional ferromagnet

$$\frac{\partial S}{\partial t} = [S, S_{xx}] \tag{6.3}$$

follows the integrability of the Hamiltonian system of equations

$$\frac{\partial p}{\partial t} = [p,[pq]_{xx}];\quad \frac{\partial q}{\partial t} = [q,[p,q]_{xx}]. \tag{6.4}$$

Physical applications of (6.4) are unknown to the author; however, it is impossible to exclude that the described way of doubling the dimension of an integrable system will prove to be useful from an applied point of view.

In conclusion, the author conveys thanks to V. I. Arnol'd for a useful discussion.

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A SPINOR REPRESENTATION OF AN INFINITE-DIMENSIONAL ORTHOGONAL SEMIGROUP AND THE VIRASORO ALGEBRA

Yu. A. Neretin

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Most infinite-dimensional representations of Lie groups can be easily realized by means of operators which are products of the change in variables and the multiplication by a function. In the case of infinite-diminsional groups, two very special classes of operators, acting in the boson and fermion Fock space are almost as important; this means that representations of infinite-dimensional groups have a habit of "passing through" the Weyl representation and the spinor representation (see, e.g., [3, 8, 9, 14]).

A spinor representation of the automorphism group of the canonical anticommutation relations (CAR) has been constructed by Berezin in [1]. The aim of our paper is to extend this representation onto as large a domain as possible; this domain is a semigroup (which is not surprising, cf. [11]), containing some linear transformations of CAT, in general unbounded (there are many more bounded transformation CAR than had been usually assumed, see Sec. 2.3). Speaking of unbounded operators, it is natural to use the language of their graphs, in other words, our semigroup consists of linear relations between CAR. Notice that even in the finite-dimensional case our construction does not coincide with the standard sources on spinor representations [4, 1, 15, 2].

The considered construction (a part of it has been announced in [8]) implies a number of corollaries for the theory of representations of infinite-dimensional groups. In Sec. 3, we show that any irreducible representation of the Virasoro algebra with the highest-order weight, no necessarily unitary, can be integrated to a projective representation of the group Diff of diffeomorphisms of the circle which, in turn, extends onto the complex extension of the group Diff constructed in [10]. Further, we consider a problem arising in conformal quantum field theory concerning the construction of an operator with respect to an arbitrary Riemannian surface in such a way that the operators should multiply by each other when the Riemannian surfaces are patched together (notice that recently there appeared a number of articles in which the patching of Riemannian surfaces and the Virasoro algebra are considered, cf. [5-7, 10, 16]). Some other applications of the construction (in which only the group part of our subgroup has been used) have been considered in [8] and [9, Sec. 9].

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Moscow Institute of Electronic Machinery Construction.

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