J. Nonlinear Sci. Vol. 2: pp. 109–134 (1992)

Journal of Nonlinear Science

The Dressing Method and Nonlocal Riemann-Hilbert Problems

A. S. Fokas[1] and V. E. Zakharov[2]

[1] Department of Mathematics and Computer Science, Clarkson University, Potsdam, NY 13699-5815, USA

[2] Landau Institute of Theoretical Physics, Academy of Sciences, Moscow, Russia

Received May 22, 1991; accepted for publication October 17, 1991
Communicated by Miki Wadati

Summary. We consider equations in $2 + 1$ solvable in terms of a nonlocal Riemann-Hilbert problem and show that for such an equation there exists a unified dressing method which yields: (*i*) a Lax pair suitable for obtaining solutions that are perturbations of an arbitrary exact solution of the given equation; (*ii*) certain integrable generalizations of the given equation. Using this generalized dressing method large classes of solutions of these equations, including dromions and line dromions, can be obtained. The method is illustrated by using the N-wave interactions, the Davey-Stewartson I, and the Kadomtsev-Petviashvili I equations. We also show that a careful application of the usual dressing method yields a certain generalization of the N-wave interactions.

Key words. solitons in two spatial dimensions, dressing method
AMS/MOS classification numbers. 58F07, 35R58, 35Q51

1. Introduction

Many apparently disparate physical systems can be modeled in terms of integrable nonlinear equations. Such equations arise in ion-acoustic, electromagnetic, electrostatic, ionospheric, and water waves, in stimulated Raman scattering, in biology, in relativity, in quantum field theory, and so on (see, for example, [1]–[5]). This is a consequence of the fact that integrable equations express a certain physical coherence which is natural, at least asymptotically, to a variety of nonlinear phenomena. Indeed, Calogero and Eckhaus [6] have shown that large classes of nonlinear evolution PDEs, characterized by a dispersive linear part and by a largely arbitrary nonlinear part, after appropriate rescaling yield asymptotically equations (for the amplitude modulation) having a universal character. These "universal" equations are therefore likely to appear in many physical applications. Many integrable equations are precisely these universal models. Typical integrable equations in 1 + 1, i.e., in one spatial and one temporal dimension, and in 2 + 1, i.e., in two spatial and one temporal dimension, are the Korteweg-deVries equation, the nonlinear Schrödinger equation, and their 2 + 1 analogues, the Kadomtsev-Petviashvili (KP) and the Davey-Stewartson (DS) equations.

Integrable equations in 1 + 1 can support solitons, i.e., exponentially localized solutions with particle-like properties. Integrable equations in 2 + 1 can support line solitons. The line solitons, in contrast to one-dimensional solitons, do not decay in all directions; there exist certain "lines" on which these solutions are bounded but nondecaying. The typical integrable equations in 2 + 1 do not support exponentially localized solitons. However, it has been recently established [7], [8] that a certain modification of the DS equation, obtained by adding some new terms to the usual DS equation, can support exponentially localized solitons (see figure 1). These solutions, which possess several novel features not found in one-dimensional solitons, have been named dromions by the authors of [8].

For equations in 1 + 1, the initial value problem for decaying initial data can be solved by the so-called inverse scattering transform method. This method, which is based on the association of a given nonlinear integrable equation to a pair of linear equations known as the Lax pair, reduces the Cauchy problem to the solution of a local Riemann-Hilbert (RH) problem. A RH problem involves the determination of a function analytic in given sectors of the complex plane, from the knowledge of the

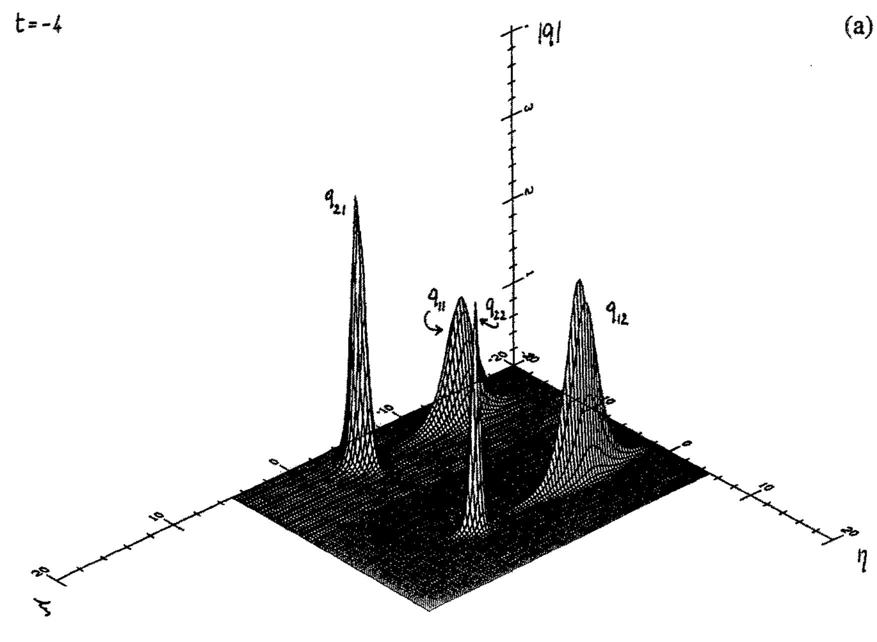
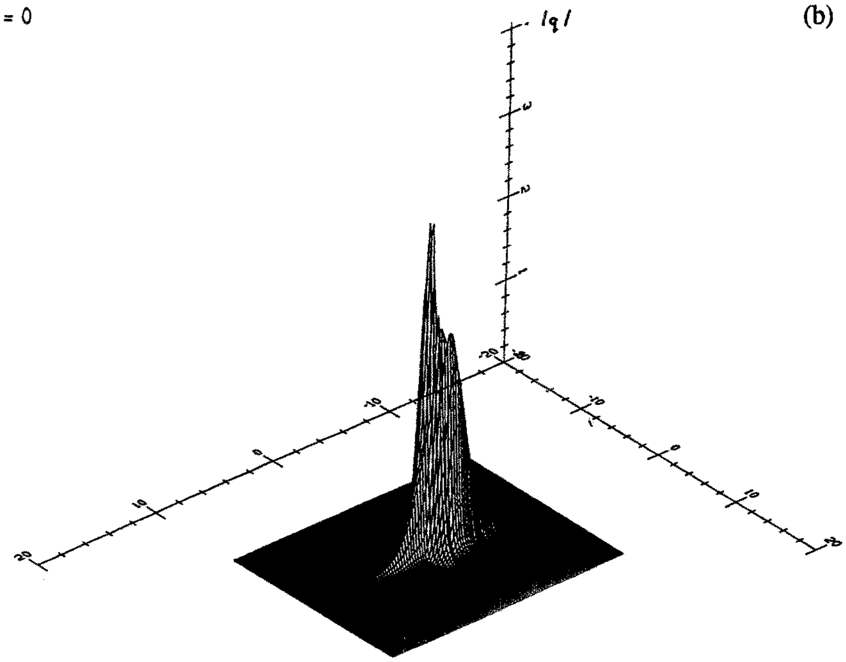


Fig. 1 describes the interaction of four localized exponentially decaying solutions of the Davey-Stewartson I equation. ξ, η are the spatial variables (characteristic coordinates) and $|q|$ is the amplitude of the wave. Shown are the waves before (a), during (b), and after (c) interaction. Note that after the interaction four localized waves appear but their amplitude has changed.

t = 0

(b)



t = 4

(c)

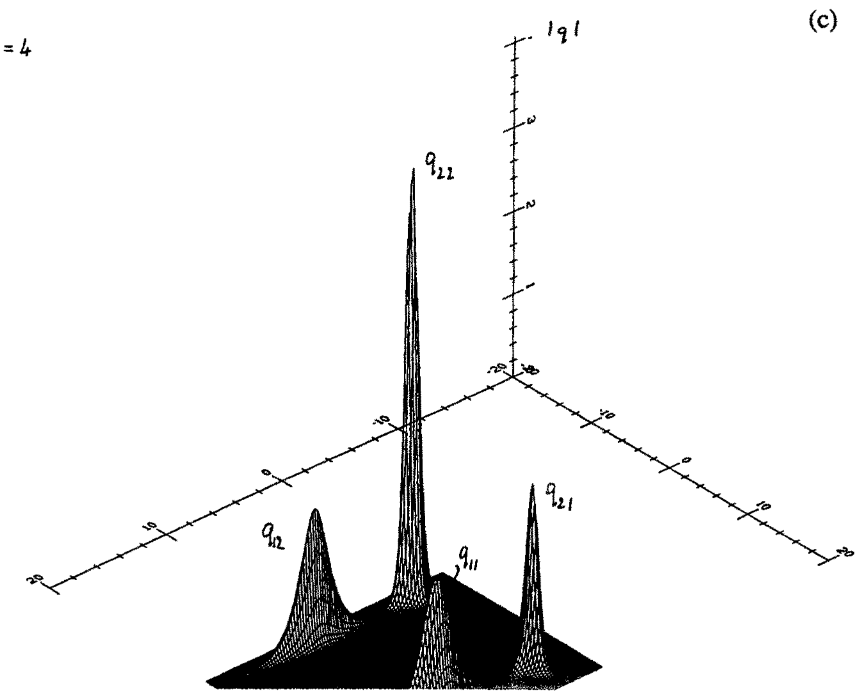


Figure 1 (*continued*)

jumps of this function across the boundaries of the given sectors. The extension of the above method for solving the initial value problem, for decaying initial data, and for equations in $2 + 1$ was achieved in the early 1980s [9]–[11]. The local RH problem is now replaced by either a nonlocal RH problem or by a $\bar{\partial}$ (DBAR) problem. In the case of a RH problem, a function "loses" its analyticity only on certain contours while, in the case of the $\bar{\partial}$ problem, the function loses its analyticity in a certain two-dimensional domain of the complex plane. The $\bar{\partial}$ (i.e., $\partial_{\bar{z}}$) derivative of a function measures its departure from analyticity. In both the RH and $\bar{\partial}$ cases knowledge of the $\bar{\partial}$ derivative of a function is enough to reconstruct this function. In spite of this progress, the question of solving an initial value problem for nondecaying initial data remains open. One very interesting such problem involves initial data that consist of a number of line solitons plus a decaying part. Another open question is the development of a method for systematically finding generalizations of integrable equations in $2 + 1$ (such as the modification of DS mentioned above) that support localized solutions.

In this paper we address both of these open problems by presenting an extension of the dressing method which: (*i*) provides an algorithm for finding certain generalizations of the well-known nonlinear equations as well as their associated Lax pairs. In the case of DS this generalization is precisely the usual DS with the additional terms studied in [7]–[8]. Thus, the extended dressing method is capable of yielding equations possessing dromion solutions. (Actually, our method yields equations that have several additional terms, but most of these terms can be transformed out via a gauge transformation.) The extension is also (*ii*) capable of capturing solutions that are perturbations of an exact solution of a given integrable nonlinear equation. In particular, it can characterize solutions that are perturbations of line solitons. As an example, we present the Lax pair that is appropriate for studying the initial value problem of the KP for initial data consisting of N-line solitons plus a decaying part. Our preliminary investigation in this case suggests a certain explicit formula (see equations (1.6) and (1.7)), but we have not carried out the inverse scattering transform in detail. The complete analytical treatment of this problem should be compared with the results of experiments under consideration [12] in water waves.

The dressing method introduced by one of the authors and Shabat [13] has been a powerful tool for obtaining new integrable nonlinear equations as well as characterizing large classes of solutions of these equations. This method is applicable to both equations in $1 + 1$, i.e., one spatial and one temporal dimension [13], [14], as well as to equations in $2+1$, i.e., two spatial and one temporal dimension [14], [15]. Regarding equations in $1 + 1$ there exist two formulations of the dressing method. One uses a local RH problem, and the other uses a Gel'fand-Levitan-Marchenko (GLM) type equation. The RH problem is equivalent to a linear integral equation whose Fourier transform yields the associated Gel'fand-Levitan-Marchenko equation. In this sense the two formulations of the dressing method are simply related. The usual dressing method for equations in $1 + 1$ yields solutions that can be thought of as perturbations of the zero solution [16]. The starting local RH problem is inadequate for capturing solutions that are perturbations of an arbitrary nonzero solution. For example, the problem of finding solutions of the Korteweg-deVries equation that are perturbations of an N-cnoidal solution yields to considering certain analytic structures on a Riemann surface of genus N [17].

With respect to the situation in $2 + 1$ we distinguish between two classes of integrable equations. One class, like KPI and DSI and the N-wave equation, can be solved via a nonlocal RH problem. The other class, like KPII and DSII, can be solved in terms of a $\bar{\partial}$ problem. Physical applications of DS include plasma physics, nonlinear optics, and water waves. In the context of water waves, it describes the amplitude of a surface wave packet interacting with a mean flow. One assumes small-amplitude, nearly monochromatic, nearly one-dimensional waves. The DS equation is the shallow water limit of the Benney-Roskes equation; DSI corresponds to the case of dominant surface tension. The KP equations play the role in $2 + 1$ that the KdV plays in $1 + 1$ [1]. Again, in the context of water waves KPI refers to the case of dominant surface tension. Regarding the dressing method of equations of the first class, again there exist two formulations; one uses a nonlocal RH problem and the other uses a GLM type equation. For a complete discussion of the relationship between these two formulations see [18]. Here we consider the dressing method for equations in $2 + 1$ of the first class, i.e., we consider equations solvable in terms of a nonlocal RH problem. Our starting point is to postulate the equation

$$\mu^+(x, y, t, k) = \int_R dl\, \mu^-(x, y, t, l)F(x, y, t, k, l),$$
$$\mu \sim I + \frac{\mu^{(1)}}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{1.1}$$

In (1.1), $+$ $(-)$ denotes holomorphicity in the upper (lower) half k-complex plane, $\int_R$ denotes integration over the real axis, and F denotes the underlying scattering data. We consider F given and regard (1.1) as a nonlocal RH problem which, given F, implies $\mu^{\pm}$. It is important to note that, although (1.1) appears as one equation for two unknowns (μ^+ and μ^-), it can be solved for μ^- in terms of F through a linear integral equation (this is a consequence of the fact that μ^+ and μ^- are not arbitrary functions but are analytic in the upper and lower complex k-plane). It is interesting that in $2 + 1$, in contrast to the case of $1 + 1$, one is able to use the same analytic structure associated with decaying solutions (namely, the nonlocal RH problem (1.1)) to capture bounded but nondecaying solutions. It seems that this is a consequence of the decay in one of the two dimensions.

In the usual dressing method one starts with (1.1), and then by using certain operators one finds nonlinear equations and Lax pairs that can be solved using this equation. The extension of the dressing method discussed here involves allowing certain additional terms in the operators of the usual dressing method. The two different applications mentioned above, for vector equations, correspond to the following. If these additional terms are diagonal matrices, one obtains suitable generalizations of the given integrable equation, while if they are off-diagonal matrices, one obtains solutions of a nonzero background. In more detail this paper is organized as follows.

In §2 we consider the KPI equation. Since this equation is scalar our method yields only a formulation of a scheme for obtaining solutions that are perturbations of a nonzero solution. In particular, let $q_0(x, y, t)$ be any solution of the KPI equation

$$q_t + \frac{1}{4}q_{xxx} + \frac{3}{2}qq_x - \frac{3}{4}\partial_x^{-1}q_{yy} = 0. \tag{1.2}$$

It is well known that (1.2) is integrable because it is associated with the linear equation $i\mu_y = \mu_{xx} + 2ik\mu_x + q\mu$. This equation is the x-part of the underlying Lax pair. Here we show that the x-part of the Lax pair associated with solutions of (1.2) that are perturbations of the solution q_0 is given by

$$i\mu_y = \mu_{xx} + 2ik\mu_x + q\mu - \frac{1}{2\pi}\int_{R^2} d\nu\, ds\, \mu(k-\nu)e^{-i\nu(x+s)}q_0(-s,y,t). \quad (1.3)$$

Furthermore,

$$q = q_0 - 2i\mu_x^{(1)}, \quad (1.4)$$

where $\mu^{(1)}$ can be obtained by solving the scalar RH problem (1.1). The scattering data F are given by $F(x,y,t,k,\ell) = f(y,t,k,\ell)\exp[i(\ell-k)x]$, and the Fourier transform of f, $\hat{f}(y,t,\xi,\eta) \doteq \int_R dk\, d\ell f(y,t,k,\ell)\exp[i(k\xi + \ell\eta)]$ satisfies

$$i\hat{f}_y = (\hat{f}_{\eta\eta} - \hat{f}_{\xi\xi}) + [q_0(\eta,y,t) - q_0(-\xi,y,t)]\hat{f}, \quad (1.5)$$

and a similar equation for $\hat{f}_t$.

The linear equation (1.5) can be solved by the method introduced in [8]. Having $\hat{f}$ and hence F, the RH problem (1.1) can be solved for μ; then the $O(1/k)$ term of μ yields $\mu^{(1)}$. Having $\mu^{(1)}$, (1.4) yields q. This procedure can also be used to solve a suitable Cauchy problem for KPI provided one analyzes (1.3) to obtain a map from $q(x,y,0)$ to $f(y,0,k,\ell)$. Here we consider only the particular case that q_0 is N-line solitons. It is known that the analytic expression for the N-line solitons is

$$q_0 = 2\partial_x^2 \ln\det(I + C), \qquad C_{nm} \doteq \frac{c_n\bar{c}_m}{p_n + \bar{p}_m}e^{-(p_n+\bar{p}_m)x}, \quad (1.6)$$

where $c_n = \gamma_n \exp[-ip_n^2 y + ip_n^3 t]$, γ_n, p_n are complex constants and $p_{n_R} > 0$. We shall show, bypassing the analysis of (1.3), that a special solution in this case is

$$q = q_0 - 2\partial_x^2 \ln\det[I + \rho(I + C^*)^{-1}], \quad (1.7)$$

where ρ is a constant matrix depending on the initial data. The additional part to q_0 appearing in (1.7) is a certain distortion of the N-line solitons. We call this part a line dromion. This is an analogy with the dromions: The external energy for the dromions is provided by the boundaries, while the "external" energy for the line dromions is provided by the line solitons. The line dromions like the line solitons do not decay on certain rays.

It should be noted that (1.7) could also be obtained via a Bäcklund transformation. This is also similar to the case of dromions. However, the approach via Bäcklund transformations cannot answer the important question of how generic these solutions are. Namely, suppose that q_0 and $q(x,y,0)$ are given; what will be the form of q as $t \to \infty$? Our speculation, in analogy with the situation in $1+1$, is that $q(x,y,0)$ will decompose into the N-line dromions given by (1.7) (the effect of initial data is only through ρ). To answer this question one has to perform an inverse spectral analysis of (1.3).

In §3 we consider DSI. We first indicate the approach to characterizing solutions that are perturbations of an exact solution of DSI. We then concentrate on finding integrable generalizations of the DSI equation. It is well known that DSI is a particular reduction ($q_2 = \pm\bar{q}_1$) of a more general system of equations for q_1, q_2,

$$iq_{1_t} + q_{1_{\xi\xi}} + q_{1_{\eta\eta}} - \frac{1}{2}q_1\left[\int_{-\infty}^{\eta} d\eta'(q_1q_2)_\xi + \int_{-\infty}^{\xi} d\xi'(q_1q_2)_\eta\right] = 0, \tag{1.8}$$

where q_2 satisfies a similar equation. We derive a generalization of this system, namely, (3.28), and the associated Lax pair. We show that a particular case of this system is *not* gauge equivalent to (1.8). This case is (1.8) with the additional terms $q_1(v_{2_\xi} + u_1 + u_2) + q_{1_\xi}v_2 + q_{1_\eta}v_1$, where $u_1(\eta, t)$, $u_2(\xi, t)$, $v_1(\eta, t)$, $v_2(\xi, t)$ are arbitrary functions of the arguments indicated. The special case $v_2 = v_1 = 0$ corresponds to the DS with the boundary terms discussed in [7]–[8]. The terms v_1, v_2 correspond to the first member of the DSI hierarchy, $iq_{1_t} + q_{1_\xi}v_2 + q_{1_\eta}v_1 = 0$. (It is well known that linear combinations of the members of an integrable hierarchy yield integrable equations.) It should be noted that even for the cases that are gauge equivalent to (1.8), it might still be useful to have a suitable dressing formulation, since gauge transformations usually map decaying to nondecaying solutions.

In §4 we consider the *N*-wave equations. We first present a missing equation! It is shown that a careful application of the usual dressing method actually implies an equation that is a generalization of the usual *N*-wave equation. This equation is given by

$$\varepsilon_{ikj}\left(I_i\frac{\partial Q}{\partial x_j}I_k - I_iQI_jQI_k\right) = 0, \tag{1.9}$$

where ε_{ikj} is the totally antisymmetric unit tensor and I^j, $j = 1, 2, 3$ are constant real $N \times N$ diagonal matrices with $I^j_\nu \neq I^j_\mu$, $\mu \neq \nu$. In the special case that $I^1 = J, I^2 = I, I^3 = C, Q$ reduces to an $N \times N$ off-diagonal matrix q, and one recovers the usual *N*-wave equation

$$q_{ij_t} = a_{ij}q_{ij_x} + (C_i - J_ia_{ij})q_{ij_y} + \sum_{k\neq j, k=1}^{N}(a_{ik} - a_{kj})q_{ik}q_{kj},$$
$$a_{ij} \doteq \frac{C_i - C_j}{J_i - J_j}, \quad i \neq j. \tag{1.10}$$

Equation (1.9) has been announced in [18].

We then apply our generalized dressing method to (1.9): (*i*) It is shown that the additional terms in (1.9) generated by our method can be transformed away by a gauge transformation. This is a peculiarity of the fact that all relevant operators of the dressing method are first order. Actually, it can be shown that if any of the operators is not of first order, then our method will yield equations that are *not* gauge equivalent to the starting nonlinear equation. (*ii*) We show how it is possible to characterize solutions that are perturbations of an exact solution q_0. For simplicity we consider (1.10) instead of (1.9). Let q_0 be a solution of (1.10). Then it is shown that the Lax pair associated with solutions of (1.10) that are perturbations of the solution q_0 is given by

$$\begin{aligned}\mu_x &= ik[J, \mu] + J\mu_y - \mu * \Gamma + q\mu,\\ \mu_t &= ik[C, \mu] + C\mu_y - \mu * \tilde{\Gamma} + A\mu,\end{aligned} \tag{1.11}$$

where the off-diagonal matrices a, A have components $a_{ij}, a_{ij}q_{ij}$ respectively, $\tilde{\Gamma}_{ij} \doteq \Gamma_{ij}a_{ij}$, and

$$(\mu * F)(k) \doteq \int_R d\nu\, \mu(k-\nu)F(\nu),$$
$$\Gamma(x,y,t,k) \doteq \frac{1}{2\pi}\int_R ds\, e^{ik(s-y)}q_0(x,s,t). \tag{1.12}$$

Furthermore,

$$q = q_0 - i[J, \mu^{(1)}], \tag{1.13}$$

where $\mu^{(1)}$ can be obtained by solving the RH problem (1.1). The scattering data F is given by $F = \exp[i(\ell - k)y]f(x,t,k,\ell)$, and the Fourier transform of $f, \hat{f}(x,t,\xi,\eta) \doteq 1/2\pi\int_{R^2} dk\, d\ell\, f(x,t,k,\ell)\exp[ik\xi + i\ell\eta]$ satisfies

$$\begin{aligned} \hat{f}_x &= J\hat{f}_\eta - \hat{f}_\xi J + q_0(x,\eta,t)\hat{f} - \hat{f}q_0(x,-\xi,t), \\ \hat{f}_y &= C\hat{f}_\eta - \hat{f}_\xi C + \tilde{q}_0(x,\eta,t)\hat{f} - \hat{f}\tilde{q}_0(x,-\xi,t)a, \end{aligned} \tag{1.14}$$

where $\tilde{q}_{0_{ij}} = q_{0_{ij}}a_{ij}$.

Given an arbitrary solution q_0 of (1.10), (1.14) yields $\hat{f}$ and then (1.1) and (1.13) yield q. Thus, large classes of solutions of (1.10) can be obtained. This procedure can also be used to solve a Cauchy problem for (1.10) provided that one is able to compute $f(x,0,k,\ell)$ from knowledge of $q(x,y,0)$. Finding the map from q to f requires an investigation of the "direct problem" associated with (1.11a). This is beyond the scope of this paper.

In the examples of N-waves and DSI we distinguished two important cases depending on whether the relevant matrices are diagonal or off-diagonal. It is clear that the most general case corresponds to considering full matrices. This will give rise to Lax pairs suitable for obtaining solutions of the generalized equations that are perturbations of any arbitrary exact solution.

It should be noted that throughout this paper q_0 is any exact solution of the associated equation such that the relevant integrals make sense. Appropriate estimates are beyond the scope of this paper. A rigorous investigation of (1.10) has been recently given in [19].

2. The KP Equation

Let $q_0(x,y,t)$ be any solution of the KPI equation

$$q_t + \frac{1}{4}q_{xxx} + \frac{3}{2}qq_x - \frac{3}{4}\partial_x^{-1}q_{yy} = 0. \tag{2.1}$$

Our aim is to develop an appropriate extension of the dressing method in order to capture solutions that are "perturbations" of the arbitrary solution q_0 instead of the zero solution. We first recall the usual dressing method.

2.1. The Usual Dressing Method

Let D_x, D_y, D_t be defined by

$$D_x = \partial_x + ik, \quad D_y = \partial_y + ik^2, \quad D_t = \partial_y + ik^3. \tag{2.2}$$

Since $iD_y\mu - D_x^2\mu \sim O(1)$, the unique solvability of the RH problem (1.1) implies

$$iD_y\mu = D_x^2\mu + q\mu \tag{2.3}$$

or

$$i\mu_y = \mu_{xx} + 2ik\mu_x + q\mu, \quad q = -2i\mu_x^{(1)}. \tag{2.4}$$

Similarly

$$D_t\mu = -D_x^3\mu + vD_x\mu + w\mu, \tag{2.5}$$

or

$$\mu_t = -(\mu_{xxx} + 3ik\mu_{xx} - 3k^2\mu_x) + v(\mu_x + ik\mu) + w\mu$$

or, using (2.4) to eliminate $3k^2\mu_x$,

$$\mu_t = -\mu_{xxx} + v\mu_x + w\mu + ik\left(v\mu + \frac{3}{2}\mu - \frac{3}{2}i\mu_y - \frac{3}{2}\mu_{xx}\right).$$

The $O(k)$ and $O(1)$ terms of this equation imply

$$v = 3i\mu_x^{(1)} = -\frac{3}{2}q, \tag{2.6}$$

$$w = \frac{3}{2}i\mu_{xx}^{(1)} - \frac{3}{2}\mu_y^{(1)} = -\frac{3}{4}q_x - \frac{3}{4}i\partial_x^{-1}q_y. \tag{2.7}$$

Using (2.4) to eliminate k we find

$$\mu_t = -\frac{1}{4}\mu_{xxx} + v\mu_x + w\mu + \frac{3}{4}(q\mu)_x + \frac{3}{4}\partial_x^{-1}\mu_{yy} + i\frac{3}{4}\partial_x^{-1}(q\mu)_y.$$

Expressing v, w, q in terms of $\mu^{(1)}$ (equations (2.4b), (2.6a), (2.7a)) and considering the $O(1/k)$ of the above equation we find

$$\mu_t^{(1)} = -\frac{1}{4}\mu_{xxx}^{(1)} + \frac{3}{2}i\mu_x^{(1)2} + \frac{3}{4}\partial_x^{-1}\mu_{yy}^{(1)}. \tag{2.8}$$

Equation (2.8) reduces to the KPI for q.

2.2. The Extended Dressing Method

We modify the operators D_x, D_y, D_t as follows

$$D_x = \partial_x + ik, \qquad D_y\mu = \mu_y + ik^2\mu + \mu * \Gamma,$$
$$D_t\mu = \mu_t + ik^3\mu + \mu * \Theta + ik * E, \tag{2.9}$$

where Γ, Θ, E are appropriate functions of k, x, y, t so that these operators commute with each other. To find the conditions for commutativity we consider the Fourier transforms of (2.9)

$$\hat{D}_x = \partial_x + \partial_s, \quad \hat{D}_y = \partial_y - i\partial_s^2 + \hat{\Gamma}, \quad \hat{D}_t = \partial_t - \partial_s^3 + (\hat{\Theta} + \hat{E}_s) + \hat{E}\partial_s, \tag{2.10}$$

where $\hat{}$ denotes the Fourier transform with respect to k, i.e.,

$$\hat{\Gamma}(s, x, y, t) = \int_R dk\, e^{iks}\Gamma(k, x, y, t), \tag{2.11}$$

and similarly for $\hat{\Theta}, \hat{E}$. Commutativity of $\hat{D}_x, \hat{D}_y, \hat{D}_t$ implies

$$\hat{\Gamma}_x + \hat{\Gamma}_s = 0, \quad \hat{\Theta}_x + \hat{\Theta}_s = 0, \quad (\partial_y - i\partial_{ss})(\hat{\Theta} + \hat{E}_s) = \hat{\Gamma}_t - \hat{\Gamma}_{sss} + \hat{E}\hat{\Gamma}_s, \tag{2.12}$$

$$\hat{E} = \frac{3}{2i}\hat{\Gamma}, \qquad \hat{\Theta}_s = -\frac{3}{4}\hat{\Gamma}_y + \frac{3i}{4}\hat{\Gamma}_{ss}. \tag{2.13}$$

Substituting (2.13) into (2.12) we find

$$\hat{\Gamma}_t - \frac{1}{4}\hat{\Gamma}_{sss} + \frac{3}{2i}\hat{\Gamma}_s\hat{\Gamma} + \partial_s^{-1}\frac{3}{4}\hat{\Gamma}_{yy} = 0, \tag{2.14}$$

while (2.12a), (2.13) yield

$$E = \frac{3}{2i}\Gamma, \quad \Theta = -\frac{3i}{4}\Gamma_x + \partial_x^{-1}\frac{3}{4}\Gamma_y, \quad \Gamma = e^{-ikx}\gamma(k, y, t). \tag{2.15}$$

Equation (2.14) implies that

$$i\Gamma(k, x, y, t) = \frac{1}{2\pi}\int_R d\xi\, e^{ik(\xi - x)}q_0(\xi, y, t). \tag{2.16}$$

To summarize, let q_0 be any solution of the KPI equation (2.1), and let

$$E = -\frac{3i}{2}\Gamma, \quad \Theta = -\frac{3i}{4}\Gamma_x + \frac{3}{4}\partial_x^{-1}\Gamma_y,$$

$$i\Gamma = \frac{1}{2\pi}\int_R d\xi\, e^{ik(\xi - x)}q_0(\xi, y, t). \tag{2.17}$$

Then the operators D_x, D_y, D_t defined by (2.9) commute.

We now derive the Lax pair associated with the operators (2.9). In the process we shall also motivate the specific modification of D_x, D_y, D_t considered above.

Equation (2.3) suggests that $D_y\mu = \mu_y + ik^2\mu + \mu * \Gamma$, or

$$i\mu_y + i\mu * \Gamma = \mu_{xx} + 2ik\mu_x + q\mu. \tag{2.18}$$

The $O(1)$ term of this equation implies

$$q = -2i\mu_x^{(1)} + i\tilde{\Gamma}, \qquad \tilde{\Gamma}(x, y, t) = \int_R dk\, \Gamma(k, x, y, t). \tag{2.19}$$

Equation (2.5) suggests $D_t\mu = \mu_t + ik^3\mu + \mu * \Theta + ik\mu * E$, or

$$\mu_t + \mu * \Theta = -\mu_{xxx} + v\mu_x + w\mu + ik\left(v\mu - \frac{3}{2}\mu_{xx} - \frac{3}{2}i\mu_y + \frac{3}{2}q\mu - \frac{3}{2}i\mu * \Gamma - \mu * E\right). \tag{2.20}$$

We choose Θ, Γ, E in such a way that v and w satisfy the same equations in terms of q as before. The $O(k)$ term of (2.20) yields

$$v + \frac{3}{2}q - \frac{3}{2}i\tilde{\Gamma} - \tilde{E} = 0;$$

thus, using (2.6b),

$$v = -\frac{3}{2}q, \qquad E = -\frac{3}{2}i\Gamma. \tag{2.21}$$

Similarly, the $O(1)$ term of (2.20) yields

$$w - \tilde{\Theta} = \frac{3}{2}i\mu^{(1)}_{xx} - \frac{3}{2}\mu^{(1)}_y, \tag{2.22}$$

or, using (2.7b),

$$w = -\frac{3}{4}q_x - \frac{3}{4}i\partial_x^{-1}q_y, \qquad \Theta = -\frac{3i}{4}\Gamma_x + \frac{3}{4}\partial_x^{-1}\Gamma_y. \tag{2.23}$$

Eliminating the k terms in (2.20) we find

$$\mu_t = -\frac{1}{4}\mu_{xxx} + v\mu_x + w\mu - \mu * \Theta + \frac{3}{4}(q\mu - i\mu * \Gamma)_x + \frac{3}{4}\partial_x^{-1}\mu_{yy} + \frac{3i}{4}\partial_x^{-1}(q\mu - i\mu * \Gamma)_y.$$

Considering the $O(1/k)$ term of this equation and using (2.19a), (2.22) to express $q - i\tilde{\Gamma}$ and $w - \tilde{\Theta}$ in terms of $\mu^{(1)}$, we find

$$\mu^{(1)}_t = -\frac{1}{4}\mu^{(1)}_{xxx} + \left(v - \frac{3i}{2}\mu^{(1)}_x\right)\mu^{(1)}_x + \frac{3}{4}\partial_x^{-1}\mu^{(1)}_{yy}.$$

Thus, since $v = -\frac{3}{2}q$ and $-2i\mu^{(1)}_x = q - i\tilde{\Gamma}$, the above equation implies that q satisfies the KPI equation provided that $i\tilde{\Gamma} = q_0$. Using this together with $\Gamma = e^{-ikx}\gamma(k, y, t)$ (which follows from the commutativity of D_x and D_y), we find that

$$i\int_R dk\, e^{-ikx}\gamma(k, y, t) = q_0(x, y, t). \tag{2.24}$$

Equations (2.21b), (2.23b), (2.24) are equations (2.15a), (2.15b), (2.16).

2.3 The Scattering Data

The equations for the scattering data $F(x, y, t, k, l)$ follow from the requirement that D_x, D_y, D_t commute with P_F.

$$F_x = i(l-k)F,$$
$$F_y = i(l^2-k^2)F + \int_R d\nu [\Gamma(\nu - l)F(k,\nu) - F(k-\nu, l)\Gamma(\nu)], \tag{2.25}$$
$$F_t = i(l^3-k^3)F + \int_R d\nu \{[\Theta(\nu - l) + i\nu E(\nu - l)]F(k,\nu) - F(k-\nu, l)[\Theta(\nu) + ikE(\nu)]\}. \tag{2.26}$$

Consider, for example, $[D_y, P_F] = 0$

$$D_y P_F \mu = D_y \int_R dl\, \mu(l) F(k,l)$$
$$= \int_R dl \left[\mu_y(l) + \mu(l)F_y + ik^2\mu(l)F + \int_R d\nu\, \mu(l)F(k-\nu, l)\Gamma(\nu)\right]$$
$$P_F D_y \mu = \int_R dl \left[\mu_y(l) + il^2\mu(l) + \int_R d\nu\, \mu(l-\nu)\Gamma(\nu)\right] F(k,l).$$

Thus, $[D_y, P_F] = 0$ implies (2.25b).

To solve (2.25) and (2.26) it is more convenient to work with the Fourier transform of F

$$F(x,y,t,k,l) = e^{i(l-k)x} f(y,t,k,l),\ \hat{f}(y,t,\xi,\eta) \doteqdot \frac{1}{2\pi}\int_{R^2} dk\, dl\, e^{i(k\xi + l\eta)} f(y,t,k,l). \tag{2.27}$$

Then $\hat{f}$ satisfies

$$i\hat{f}_y = \hat{f}_{\eta\eta} - \hat{f}_{\xi\xi} + [q_0(\eta, y, t) - q_0(-\xi, y, t)]\hat{f} \tag{2.28}$$

and a similar equation for $\hat{f}_t$. Since these equations are compatible, we concentrate on the solution of (2.28).

2.4. The N-Line Soliton Case

We now consider the particular case that q_0 is the N-line soliton of the KP, i.e., q_0 is given by (1.6). To analyze the equations satisfied by the scattering data we follow the method introduced in [8]. Using separation of variables it follows that (2.28) is intimately related with

$$i\varphi_y + \varphi_{xx} + q_0\varphi = 0. \tag{2.29}$$

In general, φ will contain two parts corresponding to the associated discrete and continuous spectrum. However, here we are only interested in the long-time behavior of solutions; thus, we consider only the associated discrete spectrum. Let $\varphi_j(x, y, t)$ satisfy

$$i\varphi_{j_y} + \varphi_{j_{xx}} + q_0\varphi_j = 0; \tag{2.30}$$

then (2.28) and the reality of q_0 yield

$$\hat{f} = \sum_{j,r=1}^{N} \rho_{jr}\varphi_j(-\xi)\bar{\varphi}_r(\eta), \tag{2.31}$$

where the matrix ρ depends on the initial data. The Fourier transform of (2.31) yields

$$f(y, t, k, l) = \sum_{j,r=1}^{N} \rho_{jr}\left[\frac{1}{\sqrt{2\pi}}\int_R d\xi\, e^{ik\xi}\varphi_j(\xi)\right]\left[\frac{1}{\sqrt{2\pi}}\int_R d\eta\, e^{-il\eta}\bar{\varphi}_r(\eta)\right].$$

Thus, if we represent f in the form

$$f(y, t, k, l) = \sum_{j=1}^{N}\left[\frac{1}{\sqrt{2\pi}}\int_R d\xi\, e^{ik\xi}f_j(\xi)\right]\left[\frac{1}{\sqrt{2\pi}}\int_R d\eta\, e^{-il\eta}\hat{g}_j(\eta)\right],$$

it follows that

$$\hat{f}_j(\xi) = \varphi_j(\xi), \qquad \hat{g}_j(\eta) = \sum_{r=1}^{N} \rho_{jr}\bar{\varphi}r(\eta). \tag{2.32}$$

Equations (2.32) show that the Fourier transform of f can be expressed in terms of solutions of (2.30). These equations can be solved by linear algebraic equations [8]

$$\varphi_n + \sum_{j=1}^{N} \frac{c_n\bar{c}_j}{p_n + \bar{p}_j}\exp[-(p_n + \bar{p}_j)x]\varphi_j = c_n e^{-p_n x},$$

$$q_0 = -2\partial_x \sum_{n=1}^{N} \bar{c}_n e^{-\bar{p}_n x}\varphi_n, \tag{2.33}$$

where $c_n = \gamma_n \exp(-i p_n^2 y + i p_n^3 t)$, γ_n, and p_n are complex constants and $p_{n_R} > 0$. Hence,

$$\varphi_n = \sum_{j=1}^{N}(I + C)_{nj}^{-1}c_j e^{-p_j x}, \qquad C_{nm} = \frac{c_n\bar{c}_m}{p_n + \bar{p}_m}\exp[-(p_n + \bar{p}_m)x]. \tag{2.34}$$

Equation (2.31) indicates that, for this case, the scattering data becomes degenerate. Thus, the RH problem (1.1) can be solved in closed form

$$\mu^+(k) - \mu^-(k) = \int_R d\,\ell\,\mu^-(\ell)\exp[i(\ell - k)x]\sum_{j=1}^{N} f_j(k)g_j(\ell). \tag{2.35}$$

Let

$$A_j \doteqdot \frac{1}{\sqrt{2\pi}}\int_R d\ell\,\mu^-(\ell)e^{i\ell x}g_j(\ell). \tag{2.36}$$

Taking the minus projection of (2.35), multiplying this equation by $(1/\sqrt{2\pi})e^{ikx}g_\nu(k)$, and integrating over k, we find

$$A_\nu = \hat{g}_\nu + \sum_{j=1}^{N} A_j a_{j\nu}, \qquad a_{j\nu} \doteq \int_R dk (f_j(k)e^{-ikx})^- g_\nu(k), \tag{2.37}$$

where the superscript $-$ denotes the application of the minus projection. But

$$\left(f_j(k)e^{-ikx}\right)^- = \frac{1}{2i\pi}\int_R dk' \frac{f_j(k')e^{-ik'x}}{k'-(k-i0)} = \frac{1}{2i\pi}\frac{1}{\sqrt{2\pi}}\int_{R^2} dk'\,dp \frac{e^{ik'(p-x)}\hat{f}_j(p)}{k'-(k-i0)}$$
$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} dp \hat{f}_j(p)e^{ik(p-x)}.$$

Thus,

$$a_{j\nu} = -\int_{-\infty}^{x} dp \hat{f}_j(p)\hat{g}_\nu(p).$$

Using (2.32) to express $\hat{f}_j, \hat{g}_j$ in terms of $\varphi_j, \ddot{\varphi}_j$, (2.37a) becomes

$$A_\nu = \sum_{r=1}^{N} \rho_{\nu r}\overline{\varphi}_r(x) - \sum_{j,r=1}^{N} \rho_{\nu r} A_j \int_{-\infty}^{x} dp \varphi_j(p)\overline{\varphi}_r(p). \tag{2.38}$$

It was shown in [8] that if φ_j solve (2.33), then

$$\varphi_j\bar{\varphi}_r = \partial_x(I+C)^{-1}_{jr}. \tag{2.39}$$

Hence, (2.38) yields

$$A_\nu + \sum_{j=1}^{N} M_{\nu j}A_j = \sum_{r=1}^{N} \rho_{\nu r}\bar{\varphi}_r(x), \qquad M \doteq \rho(I+C^*)^{-1}.$$

Thus, $A_j = \sum_{r=1}^{N}((I+M)^{-1}\rho)_{jr}\bar{\varphi}_r$, and since φ_n is given by (2.34), we find

$$A_j = \sum_{j=1}^{N}\left[(I+M)^{-1}\rho(I+C^*)^{-1}\right]_{j\ell}\bar{c}_\ell e^{-\bar{p}_\ell x}. \tag{2.40}$$

The large k asymptotics of (2.35) imply that

$$\mu^{(1)} = -i\sum_{j=1}^{N} A_j\bar{\varphi}_j.$$

Using (2.34) and (2.40) to express A_j, φ_j in terms of C, M it follows that

$$\mu^{(1)} = -i \operatorname{trace}(I+M)^{-1}\rho(I+C^*)^{-1}(\partial_x C^*)(I+C^*)^{-1},$$

where we have used that $\partial_x C_{\nu l} = c_\nu\bar{c}_l \exp[-(p_n+\overline{p}_l)x]$. But $M_x = \rho(I+C^*)^{-1}\times C_x^*(I+C^*)^{-1}$; thus

$$\mu^{(1)} = i \operatorname{trace}(I+M)^{-1}M_x = i\partial_x \operatorname{trace}\ln(I+M).$$

Rigorous results on KPI are given in [20]. The question of including line solitons to the usual inverse scattering of KPI is considered in [21].

3. The DSI Equation

Define the operators D_x, D_y, D_t by

$$\begin{aligned} D_x\mu &= \mu_x + ik\mu J + \mu * \Gamma, \\ D_y\mu &= \mu_y - ik\mu + \mu * \Theta, \\ D_t\mu &= \mu_t - ik^2\mu J + \mu * \Delta + ik\mu * E, \end{aligned} \tag{3.1}$$

where $\Gamma, \Theta, \Delta, E$ are functions of x, y, t, k. The Fourier transforms of these operators are defined by

$$\begin{aligned} \hat{D}_xM &= M_x + M_sJ + M\hat{\Gamma}, \\ \hat{D}_yM &= M_y - M_s + M\hat{\Theta}, \\ \hat{D}_tM &= M_t + iM_{ss}J + M\hat{Z} + M_s\hat{E}, \hat{Z} \doteqdot \hat{\Delta} + \hat{E}_s, \end{aligned} \tag{3.2}$$

where $\hat{\Gamma}, \hat{\Theta}, \hat{\Delta}, \hat{E}$ denote the Fourier transforms of $\Gamma, \Theta, \Delta, E$ respectively in the variable k, i.e., $\hat{\Gamma}(x, y, t, s) = \int_R dk\, e^{iks}\Gamma(x, y, t, k)$, etc. Commutativity of $\hat{D}_x, \hat{D}_y, \hat{D}_t$ among each other yields:

$$\begin{aligned} [J, \hat{\Theta}] = 0, \qquad & \hat{\Gamma}_y - \hat{\Gamma}_s + [\hat{\Gamma}, \hat{\Theta}] = \hat{\Theta}_x + \hat{\Theta}_sJ, \\ [J, \hat{E} - i\hat{\Gamma}] = 0, \qquad & \hat{E}_x + \hat{E}_sJ + [\hat{E}, \hat{\Gamma}] + [\hat{Z}, J] = 2i\hat{\Gamma}_sJ, \\ \hat{Z}_x + \hat{Z}_sJ + [\hat{Z}, \hat{\Gamma}] = \hat{\Gamma}_t + i\hat{\Gamma}_{ss}J + \hat{\Gamma}_s\hat{E}, & \\ \hat{E}_y - \hat{E}_s + [\hat{E}, \hat{\Theta}] = 2i\hat{\Theta}_sJ, \qquad & \hat{Z}_y - \hat{Z}_s + [\hat{Z}, \hat{\Theta}] = \hat{\Theta}_t + \hat{\Theta}_s\hat{E} + i\hat{\Theta}_{ss}J. \end{aligned} \tag{3.3}$$

There exist two important cases: (*i*) If matrices $\Gamma, \Delta, E, \Theta$ are off-diagonal, one can use (3.1) to characterize large classes of solutions that are perturbations of an arbitrary solution of the DSI equation. For economy of presentation we do not consider this case here; we simply note that the relevant methodology is similar to that used in §2. This methodology will be illustrated further in §3. (*ii*) If the matrices $\Gamma, \Theta, \Delta, E$ are diagonal, one obtains certain generalizations of the DSI equation. We first consider the special case that $\Theta = 0$. It will be shown later that the general case can be reduced to $\Theta = 0$ by a gauge transformation.

If $\Theta = 0$ equations (3.3) become

$$\begin{aligned} &\hat{\Gamma}_y - \hat{\Gamma}_s = 0, \quad \hat{E}_y - \hat{E}_s = 0, \quad \hat{\Delta}_y - \hat{\Delta}_s = 0, \\ &\hat{\Delta}_x + J\hat{\Delta}_s = \hat{\Gamma}_t + \hat{E}\hat{\Gamma}_s - iJ\hat{\Gamma}_{ss}, \qquad \hat{E}_x + J\hat{E}_s = 2iJ\hat{\Gamma}_s. \end{aligned} \tag{3.4}$$

3.1. The Lax Pair

To derive the Lax pair associated with the operators (3.1) we note that $D_x\mu \sim ikJ, D_y\mu \sim -ik$, and $D_t\mu \sim -ik^2J$ as $k \to \infty$. Thus,

$$D_x\mu + JD_y\mu + q\mu = 0, \tag{3.5}$$

or

$$\mu_x + J\mu_y + ik[\mu, J] + \mu * \Gamma + q\mu = 0. \tag{3.6}$$

The $O(1)$ term of this equation implies

$$q = i[J, \mu^{(1)}] - \tilde{\Gamma}, \qquad \tilde{\Gamma}(x, y, t) = \int_R dk\,\Gamma(k, x, y, t). \tag{3.7}$$

Similarly,

$$D_t\mu = iJD_y^2\mu + BD_y\mu + A\mu, \tag{3.8}$$

or

$$\begin{aligned}\mu_t = {} & iJ\mu_{yy} - k(\mu_x - J\mu_y + \mu * \Gamma + q\mu + i\mu * E + iB\mu) \\ & - \mu * \Delta + B\mu_y + A\mu.\end{aligned} \tag{3.9}$$

The $O(k)$ term of this equation implies

$$B = iq + i\tilde{\Lambda}, \quad \Lambda \doteqdot \Gamma + iE, \qquad \tilde{\Gamma} = \int_R dk\,\Gamma(k, x, y, t), \tag{3.10}$$

and $\tilde{E}, \tilde{\Delta}$ are defined in a similar way to $\tilde{\Gamma}$. Thus, (3.9) simplifies to

$$\mu_t = iJ\mu_{yy} - k(\mu_x - J\mu_y + \mu * \Lambda - \tilde{\Lambda}\mu) - \mu * \Delta + i(q + \tilde{\Lambda})\mu_y + A\mu. \tag{3.11}$$

Let

$$q = \begin{pmatrix} Q_1 & q_1 \\ q_2 & Q_2 \end{pmatrix}, \quad \mu^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mu^{(2)} = \begin{pmatrix} a^{(2)} & b^{(2)} \\ c^{(2)} & d^{(2)} \end{pmatrix}, \tag{3.12}$$

$$\xi = x + y, \quad \eta = x - y, \quad \tilde{\Gamma} = \mathrm{diag}(\tilde{\Gamma}_1, \tilde{\Gamma}_2), \quad J = \mathrm{diag}(1, -1). \tag{3.13}$$

Then (3.7a) becomes

$$Q_1 = -\tilde{\Gamma}_1, \quad Q_2 = -\tilde{\Gamma}_2, \quad b = \frac{q_1}{2i}, \quad c = -\frac{q_2}{2i}. \tag{3.14}$$

The $O(1/k)$ term of (3.6) yields

$$\begin{aligned} a &= \frac{1}{4i}\int_{-\infty}^{\xi} d\xi' q_1 q_2, & b^{(2)} &= -\frac{q_{1_\xi}}{2} + \frac{\tilde{\Gamma}_1 - \tilde{\Gamma}_2}{4}q_1 + \frac{q_1}{8}\int_{-\infty}^{\eta} d\eta' q_1 q_2, \\ d &= -\frac{1}{4i}\int_{-\infty}^{\eta} d\eta' q_1 q_2, & c^{(2)} &= -\frac{q_{2_\eta}}{2} + \frac{\tilde{\Gamma}_2 - \tilde{\Gamma}_1}{4}q_2 + \frac{q_2}{8}\int_{-\infty}^{\xi} d\xi' q_1 q_2. \end{aligned} \tag{3.15}$$

The $O(1)$ terms of (3.11) yield

$$A_{11} = \frac{1}{2i}\int_{-\infty}^{\xi} d\xi'(q_1q_2)\eta + \tilde{\Delta}_1, \qquad A_{22} = -\frac{1}{2i}\int_{-\infty}^{\eta} d\eta'(q_1q_2)_\xi + \tilde{\Delta}_2,$$

$$A_{12} = -iq_{1_\eta} + \frac{1}{2i}(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)q_1, \qquad A_{21} = iq_{2\xi} + \frac{1}{2i}(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)q_2. \tag{3.16}$$

To derive the associated nonlinear equations we consider the $O(1/k)$ term of (3.11). In particular, the "12" component yields

$$\begin{aligned} b_t &= ib_{yy} + b_y^{(2)} - b_x^{(2)} + (\tilde{\Lambda}_1 - \tilde{\Lambda}_2)b^{(2)} + (A_{11} - \tilde{\Delta}_2 - \tilde{\tilde{\Lambda}}_2)b \\ &\quad + i(\tilde{\Lambda}_1 + Q_1)b_y + iq_1dy + A_{12}d, \end{aligned} \tag{3.17}$$

where

$$\tilde{\tilde{\Lambda}}_2 = \int_R dk\,k\Lambda_2(k,x,y,t). \tag{3.18}$$

Using (3.14)–(3.16), (3.17) reduces to

$$\begin{aligned} &iq_{1_t} + q_{1_{\xi\xi}} + q_{1_{\eta\eta}} - \frac{1}{2}q_1\left[\int_{-\infty}^{\eta} d\eta'(q_1q_2)_\xi + \int_{-\infty}^{\xi} d\xi'(q_1q_2)_\eta\right] \\ &\quad + q_1\left[\frac{1}{2}(\tilde{\Lambda}_1 - \tilde{\Lambda}_2)(\tilde{\Gamma}_1 - \tilde{\Gamma}_2) + (\tilde{\Gamma}_2 - \tilde{\Gamma}_1)_\eta + \tilde{\Lambda}_\xi - \tilde{\Lambda}_\eta + i(\tilde{\Delta}_2 - \tilde{\Delta}_1)\right] \\ &\quad + q_{1_\xi}(\tilde{\Lambda}_2 - \tilde{\Gamma}_1) + q_{1_\eta}(\tilde{\Gamma}_2 - \tilde{\Lambda}_1) = 0. \end{aligned} \tag{3.19}$$

The "21" component of (3.11) yields a similar equation for q_2.

We now determine the relationships among the functions $\tilde{\Gamma}, \tilde{\Delta}, \tilde{E}$ and express Γ, Δ, E in terms of them. Equations (3.4) yield

$$\Gamma = e^{iky}\gamma(x,t,k), \quad \Delta = e^{iky}\delta(x,t,k), \quad E = e^{iky}\varepsilon(x,t,k), \tag{3.20}$$

and, also,

$$\tilde{\Gamma} = \int_R dk\,\Gamma(x,y,t,k) = \int_R dk\,e^{iky}\gamma(x,t,k). \tag{3.21}$$

Thus,

$$\begin{aligned} \Gamma(x,y,t,k) &= \frac{1}{2\pi}\int_R dz\,\tilde{\Gamma}(x,t,z)\exp[ik(y-z)], \\ \hat{\Gamma}(x,y,t,s) &= \tilde{\Gamma}(x,t,y+s), \end{aligned}$$

and similarly for Δ, E. Then, (3.4) imply that $\tilde{\Gamma}(x,t,y), \tilde{\Delta}(x,t,y)$, and $\tilde{E}(x,t,y)$ solve

$$\tilde{E}_x + J\tilde{E}_y = 2iJ\tilde{\Gamma}_y, \qquad \tilde{\Delta}_x + J\tilde{\Delta}_y = \tilde{\Gamma}_t + \tilde{E}\tilde{\Gamma}_y - iJ\tilde{\Gamma}_{yy}. \tag{3.22}$$

Also,

$$\tilde{\tilde{\Lambda}}_2 = -i\tilde{\Lambda}_y.$$

In Summary: Let $\tilde{\Gamma}(x,t,y)$ be an arbitrary diagonal matrix, and let $\tilde{E}, \tilde{\Delta}$ be defined by (3.22). Let Γ, Δ, E be given by

$$\Gamma(x,y,t,k) = \frac{1}{2\pi}\int_R dz\,\tilde{\Gamma}(x,t,z)\exp[ik(y-z)],$$
$$\Delta = \frac{1}{2\pi}\int_R dz\,\tilde{\Delta}\exp[ik(y-z)],$$
$$E = \frac{1}{2\pi}\int_R dz\,\tilde{E}\exp[ik(y-z)], \tag{3.23}$$

and let $\Lambda = \Gamma + iE$. Then the nonlinear equation (3.19) (where q_2 satisfies a similar equation) admits the Lax pair given by (3.6) and (3.11).

We now solve (3.22). It is convenient to define $\tilde{g}(x,t,y)$ in terms of $\tilde{\Gamma}$ as follows; $\tilde{\Gamma} = \tilde{g}_x + J\tilde{g}_y$ (the motivation for this substitution is given in §3.3). Then it is easily verified that $\tilde{E}_p = 2iJ\tilde{g}_y$ and $\tilde{\Delta}_p = \tilde{g}_t + iJ\tilde{g}_y^2 - iJ\tilde{g}_{yy}$ are particular solutions of (3.22). Hence,

$$\tilde{\Gamma} = \tilde{g}_x + J\tilde{g}_y, \quad \tilde{E} = \tilde{E}_0 + \tilde{E}_p, \quad \tilde{\Delta} = \tilde{\Delta}_0 + \tilde{\Delta}_p, \tag{3.24}$$

where

$$\tilde{E}_{0_x} + J\tilde{E}_{0_y} = 0, \qquad \tilde{\Delta}^0_x + J\tilde{\Delta}^0_y = \tilde{E}_0(\tilde{g}_x + J\tilde{g}_y)_y, \tag{3.25}$$

are the solutions of (3.22). Equations (3.25b) can be simplified by the substitution $\tilde{\Delta}^0 = \tilde{\Delta}_0 + \tilde{g}_y\tilde{E}_0$ to give a homogeneous equation for $\tilde{\Delta}_0$.

Having expressed $\tilde{\Gamma}, \tilde{E}, \tilde{\Delta}$ in terms of $\tilde{g}, \tilde{E}_0$, and $\tilde{\Delta}_0$, the terms appearing in (3.19) can also be expressed in terms of $\tilde{g}, \tilde{E}_0$, and $\tilde{\Delta}_0$.

In Summary: Let

$$\tilde{\Gamma} = \tilde{g}_x + J\tilde{g}_y, \quad \tilde{E} = \tilde{E}_0 + 2iJ\tilde{g}_y, \quad \tilde{\Delta} = \tilde{\Delta}_0 + \tilde{g}_y\tilde{E}_0 + \tilde{g}_t + iJ\tilde{g}_y^2 - iJ\tilde{g}_{yy}, \tag{3.26}$$

where $\tilde{g}$ is an arbitrary function of x, t, y and

$$\tilde{E}_{0_1} = iv_1(\eta,t), \quad \tilde{E}_{0_2} = -iv_2(\xi,t), \quad \tilde{\Delta}_{0_1} = iu_1(\eta,t), \quad \tilde{\Delta}_{0_2} = -iu_2(\xi,t). \tag{3.27}$$

Let Γ, Δ, E be defined by (3.23). Then the nonlinear equation (3.19)

$$\begin{aligned} &iq_{1_t} + q_{1_{\xi\xi}} + q_{1_{\eta\eta}} - \frac{1}{2}q_1\left[\int_{-\infty}^{\eta} d\eta'(q_1q_2)_\xi + \int_{-\infty}^{\xi} d\xi'(q_1q_2)_\eta\right] + q_{1_\xi}(2\nu_\xi + \nu_2) \\ &\quad + q_{1_\eta}(2\nu_\eta + \nu_1)q_1(i\nu_t + \nu_{\xi\xi} + \nu_{\eta\eta} + \nu_\xi^2 + \nu_\eta^2 + \nu_{2_\xi} + \nu_2\nu_\xi\nu_1\nu_\eta + u_1 + u_2) \\ &\quad = 0, \end{aligned} \tag{3.28}$$

where $\nu = \tilde{g}_2 - \tilde{g}_1$ (and q_2 satisfies a similar equation), admits the Lax pair given by (3.6) and (3.11). In (3.6) q is given by (3.12a), and in (3.11) A is given by (3.16), while $\Lambda = \Gamma + iE$.

The astute reader will notice that the transformation $q_1 = Q_1 e^{-\nu}, q_2 = Q_2 e^{\nu}$ maps (3.28) to an equation with $\nu = 0$. This will be further discussed in §3.3.

3.2. The Scattering Data

The equations for the scattering data $F(x, y, t, k, l)$ follow from the requirement that D_x, D_y, D_t commute with P_F. This implies

$$F_x = i\ell JF - ikFJ + \int_R d\nu\,[\Gamma(\nu - \ell)F(k, \nu) - F(k - \nu, \ell)\Gamma(\nu)] \tag{3.29}$$

$$F_y = i(k - \ell)F \tag{3.30}$$

$$\begin{aligned} F_t &= -i\ell^2 JF + ik^2 FJ \\ &\quad + \int_R d\nu\{[\Delta(\nu - \ell) + i\nu E(\nu - l)]\,F(k, \nu) - F(k - \nu, l)\,[\Delta(\nu) + ikE(\nu)]\}. \end{aligned} \tag{3.31}$$

Equation (3.30) implies that the dependence on y is exponential. To solve (3.29), (3.31) it is convenient to introduce the Fourier transform of F

$$\begin{aligned} F(x, y, t, k, l) &= \exp[i(k - l)y]\,f(x, t, k, l), \\ \hat{f}(x, t, \xi, \eta) &= \frac{1}{2\pi}\int_{R^2} dk\,dl\,f(x, t, k, l)\exp[i(k\xi + l\eta)]. \end{aligned} \tag{3.32}$$

Then (3.29) and (3.31) reduce to

$$\hat{f}_x = iJ\hat{f}_\eta - i\hat{f}_\xi J + \tilde{\Gamma}(x, -\eta, t)\hat{f} - \hat{f}\tilde{\Gamma}(x, \xi, t), \tag{3.33}$$

$$\begin{aligned} \hat{f}_t &= iJ\hat{f}_{\eta\eta} - i\hat{f}_{\xi\xi}J + \tilde{\Delta}(x, -\eta, t)\hat{f} - \hat{f}\tilde{\Delta}(x, \xi, t) \\ &\quad + \tilde{E}(x, -\eta, t)\hat{f}_\eta - [\hat{f}\tilde{E}(x, \xi, t)]_\xi. \end{aligned} \tag{3.34}$$

Indeed, one of the convolution terms in (3.29) becomes

$$\begin{aligned} &\frac{1}{2\pi}\int_{R^3} d\nu\,dk\,dl\,\Gamma(\nu - l)f(k, \nu)\exp(ik\xi + il\eta) \\ &= \frac{1}{2\pi}\int_{R^3} d\nu\,dk\,dl\,\Gamma(\nu - l)\exp[i(l - \nu)\eta]\,f(k, \nu)\exp(ik\xi + i\nu\eta) \\ &= \tilde{\Gamma}(x, -\eta, t)\hat{f}(x, t, \xi, \eta), \end{aligned}$$

where we used (3.21).

In Summary: Let the scattering data F satisfy (3.32)–(3.34); then the RH problem (1.1) yields solutions of the nonlinear equation (3.28) via $q_1 = 2i(\mu^{(1)})_{12}$. Equations (3.33) and (3.34) can be solved by the method introduced in [8].

3.3. A Gauge Transformation

In this section we investigate the question of gauge equivalence of the nonlinear equation (3.28) to the equation obtained by the usual dressing method.

Recall that (3.28) is associated with the operators

$$\begin{aligned} \hat{D}_x M &= M_x + M_s J + M\hat{\Gamma}, \\ \hat{D}_y M &= M_y - M_s, \\ \hat{D}_t M &= M_t + iM_{ss}J + M(\hat{\Delta} + \hat{E}_s) + M_s\hat{E}. \end{aligned} \tag{3.35}$$

The question of gauge equivalence reduces to finding a transformation $\Psi = Me^{\hat{g}}$, $\hat{g}$ a diagonal matrix, such that (3.35) reduce to

$$\hat{D}_x^0\Psi = \Psi_x + \Psi_s J, \quad \hat{D}_y^0\Psi = \Psi_y - \Psi_s, \quad \hat{D}_t^0\Psi = \Psi_t + i\Psi_{ss}J. \tag{3.36}$$

This is the case if and only if

$$\hat{g}_y - \hat{g}_s = 0, \tag{3.37}$$

and

$$\hat{\Gamma} = \hat{g}_x + J\hat{g}_y, \quad \hat{E} = 2iJ\hat{g}_y, \quad \hat{\Delta} = \hat{g}_t + iJ\hat{g}_y^2 - iJ\hat{g}_{yy}. \tag{3.38}$$

On the other hand, we know that $\hat{\Gamma}, \hat{\Delta}, \hat{E}$ satisfy (3.4). Therefore, there exist cases in (3.35) that are *not* gauge equivalent to (3.36), if and only if there exist solutions $\hat{\Gamma}, \hat{\Delta}, \hat{E}$ of (3.4) that *cannot* be written in the form (3.38). It is easily verified that the particular solutions

$$\hat{\Gamma} = \hat{g}_x + J\hat{g}_y, \quad \hat{E}_p = 2iJ\hat{g}_y, \quad \hat{\Delta}_p = \hat{g}_t + iJ\hat{g}_y^2 - iJ\hat{g}_{yy}$$

satisfy (3.4). However, the general solution of (3.4) is given by (3.26), and it involves the arbitrary functions $v_1(\eta, t)$, $v_2(\xi, t)$, $u_1(\eta, t)$, $u_2(\xi, t)$.

In Summary: The particular case of the operators (3.35) given by

$$\begin{aligned} \hat{D}_x M &= M_x + M_s J, \quad \hat{D}_y M = M_y - M_s, \\ \hat{D}_t M &= M_t + iM_{ss}J + M(\hat{\Delta}_0 + \hat{E}_{0_s}) + M_s\hat{E}_0, \end{aligned}$$

where $\hat{\Delta}_0, \hat{E}_0$ are given by (3.27), is *not* gauge equivalent to (3.36). This corresponds to the equation

$$\begin{aligned} iq_{1_t} + q_{1_{\xi\xi}} + q_{1_{\eta\eta}} + q_1\left[-\frac{1}{2}\int_{-\infty}^{\eta} d\eta'(q_1q_2)_\xi - \frac{1}{2}\int_{-\infty}^{\xi} d\xi'(q_1q_2)_\eta + v_{2_\xi} + u_1 + u_2\right] \\ + q_{1_\xi}v_2 + q_{1_\eta}v_1 = 0, \end{aligned} \tag{3.39}$$

and a similar equation for q_2. The system of equations of q_1, q_2 admits the reduction $q_2 = \pm\bar{q}_1$ provided that $v_2 = v_1 = v(t)$ and $u_1, u_2, v \in \mathbb{R}$. The terms v_1, v_2 correspond to the first member of the DS hierarchy. Indeed, commutativity of the operators

$$L \doteqdot \partial_y + J\partial_x + [J, q], \quad M \doteqdot \partial_t + H\partial_x + [H, q], \quad H \doteq -i\ \mathrm{diag}(v_1, v_2),$$

implies

$$iq_{1_t} + q_{1_\xi}v_2 + q_{1_\eta}v_1 = 0.$$

We now examine the case $\Theta \neq 0$. The operators (3.2) reduce to (3.36) if and only if

$$\hat{\Theta} = \hat{g}_y - \hat{g}_s, \quad \hat{\Gamma} = \hat{g}_x + J\hat{g}_s, \quad \hat{E} = 2iJ\hat{g}_s, \quad \hat{\Delta} = \hat{g}_t + iJ(-\hat{g}_{ss} + \hat{g}_s).$$

However, the general solution of (3.3) (with $\hat{\Gamma}, \hat{\Delta}, \hat{E}, \hat{\Theta}$ diagonal) is given by

$$\hat{\Theta} = \hat{g}_y - \hat{g}_s, \quad \hat{\Gamma} = \hat{\Gamma}_0 + \hat{g}_x + J\hat{g}_s, \quad \hat{E} = \hat{E}_0 + 2iJ\hat{g}_s,$$
$$\hat{\Delta} = \hat{\Delta}_0 + \hat{E}_0\hat{g}_s + \hat{g}_t + iJ(-\hat{g}_{ss} + \hat{g}_s^2),$$

where $\hat{\Gamma}_0, \hat{E}_0, \hat{\Delta}_0$ satisfy (3.4). Hence, the general case of $\Theta \neq 0$ is reduced via a gauge transformation to the case $\Theta = 0$.

3.4. The DSI with Nonzero Boundary Conditions

Let $\Gamma = E = 0, q_2 = \varepsilon\bar{q}_1, \varepsilon = \pm 1$. Then

$$iq_t + q_{\xi\xi} + q_{\eta\eta} + q\left[-\frac{\varepsilon}{2}\int_{-\infty}^{\eta} d\eta' |q|_\xi^2 - \frac{\varepsilon}{2}\int_{-\infty}^{\xi} d\xi' |q|_\eta^2 + u_1(\eta, t) + u_2(\xi, t)\right] = 0. \tag{3.40}$$

Assuming that the scattering data F is off-diagonal with entries F_{12}, F_{21}, it follows that $\hat{f}_{12}$ solves

$$\hat{f}_{12_x} = i(\hat{f}_{12_\eta} + \hat{f}_{12_\xi}),$$
$$i\hat{f}_{12_t} + \hat{f}_{12_{\xi\xi}} + \hat{f}_{12_{\eta\eta}} + [u_1(x+\eta, t) + u_2(x+\xi, t)]\hat{f}_{12} = 0.$$

Thus,

$$\begin{aligned} F_{12} &= g_{12}(t, k, l)\exp[il(x-y) + ik(x+y)], \\ F_{21} &= g_{21}(t, k, l)\exp[-i\ell(x-y) - ik(x+y)], \end{aligned} \tag{3.41}$$

where

$$\hat{g}_{12}(t, \xi, \eta)\frac{1}{2\pi}\int_{R^2} dk\, dl\, g_{12}(t, k, l)\exp[i(k\xi + l\eta)],$$

solves

$$i\hat{g}_{12_t} + \hat{g}_{12_{\xi\xi}} + \hat{g}_{12_{\eta\eta}} + [u_1(\eta, t) + u_2(\xi, t)]\hat{g}_{12} = 0. \tag{3.42}$$

It can be shown that $g_{21}(t, k, l) = \bar{g}_{12}(t, l, k)$.

In Summary: Equation (3.40) is associated with the RH problem (1.1) where the off-diagonal matrix F is given by (3.41) and the Fourier transform of g_{12} solves (3.42), while $g_{21}(t, k, l) = \bar{g}_{21}(t, l, k)$.

Equation (3.40) has been investigated in [7,8]. Here we note that the functions u_1, u_2 can be thought of as nontrivial boundary conditions for the velocity potential U, where

$$U_{1_\eta} = -\frac{\varepsilon}{2}|q|_\xi^2, \qquad U_{2_\xi} = -\frac{\varepsilon}{2}|q|_\eta^2.$$

4. The N-Wave Equation

We first derive (1.9). Consider the operators D_{x_j} defined by

$$D_{x_j}\mu = \mu_{x_j} + ik\mu I^j, \quad j = 1, 2, 3.$$

Since $D_{x_j}\mu$ is asymptotic to ikI^j it follows that

$$I^i D_{x_j}\mu - I^j D_{x_i}\mu = U_{ij}\mu.$$

Using the $O(1/k)$ of the above equation to compute U_{ij} it follows that

$$I^i(\mu_{x_j} + ik\mu I^j) - I^j(\mu_{x_i} + ik\mu I^i) = i(I^i\mu^{(1)}I^j - I^j\mu^{(1)}I^i)\mu.$$

Multiplying this equation by I^k from the right, using cyclic permutation, and adding the resulting equations we find

$$\varepsilon_{ijk}\left[I^i\mu_{x_j}I^k - iI^i\mu^{(1)}I^j\mu I^k\right] = 0.$$

The $O(1/k)$ of this equation implies (1.9), where $\mu^{(1)} = -iQ$.

We now consider the extended dressing method. Define the operators $D_{x_j}, j = 1, 2, 3$ by

$$D_{x_j}\mu = \frac{\partial\mu}{\partial x_j} + ik\mu I^j + \mu * \Gamma^j, \tag{4.1}$$

where I^j are constant diagonal matrices, $x_1 = x, x_2 = y, x_3 = t$, and the $*$ operator is defined in (1.12). Let $\hat{}$ denote Fourier transform in k. Then

$$\hat{D}_{x_j}M = \frac{\partial M}{\partial x_j} + \frac{\partial M}{\partial s}I^j + M\hat{\Gamma}^j, \tag{4.2}$$

Demanding that $[\hat{D}_{x_j}, \hat{D}_{x_i}] = 0, i, j = 1, 2, 3$ we find,

$$\hat{\Gamma}^j I^i + I^j\hat{\Gamma}^i = \hat{\Gamma}^i I^j + I^i\hat{\Gamma}^j, \tag{4.3}$$

$$\hat{\Gamma}^j_{x_i} + \hat{\Gamma}^j_s I^i + \hat{\Gamma}^j\hat{\Gamma}^i = \hat{\Gamma}^i_{x_j} + \hat{\Gamma}^i_s I^j + \hat{\Gamma}^i\hat{\Gamma}^j, \quad i \neq j, \quad i, j = 1, 2, 3. \tag{4.4}$$

We again distinguish two important cases: (*i*) $\hat{\Gamma}^j$ are diagonal matrices; (*ii*) $\hat{\Gamma}^j$ are off-diagonal matrices.

In the first case it is a straightforward matter following the approach of §3, to find the Lax pair of a nonlinear equation that is a generalization of the N-wave equation.

This equation involves certain functions that are generated from the solution of the following system of diagonal matrices,

$$\hat{\Gamma}^j_{x_i} + I^i \hat{\Gamma}^j_s = \hat{\Gamma}^i_{x_j} + I^j \hat{\Gamma}^i_s, \quad i \neq j, \quad i, j = 1, 2, 3. \tag{4.5}$$

However, these functions can be transformed out by a gauge transformation. Let us find the appropriate gauge in a step-by-step procedure. The general solution of (4.5) can be written as

$$\hat{\Gamma}^1 = \hat{g}_{x_1} + I^1 \hat{g}_s, \qquad \hat{\Gamma}^j = \hat{\Gamma}^j_0 + \hat{g}_{x_j} + I^j \hat{g}_s, \quad j = 2, 3, \tag{4.6}$$

where $\hat{g}$ is an arbitrary diagonal matrix and $\hat{\Gamma}^j_0$ solve

$$\hat{\Gamma}^j_{0_{x_1}} + I^1 \hat{\Gamma}^j_{0_s} = 0, \quad j = 2, 3, \qquad \hat{\Gamma}^2_{0_{x_3}} + I^3 \hat{\Gamma}^2_{0_s} = \hat{\Gamma}^3_{0_{x_2}} + I^2 \hat{\Gamma}^3_{0_s}. \tag{4.7}$$

Thus, the gauge transformation $M = \Psi e^{\hat{g}}$ maps the operators (4.2) to

$$\hat{D}_{x_1} \Psi = \Psi_{x_1} + \Psi_s I^1, \qquad \hat{D}_{x_j} \Psi = \Psi_{x_j} + \Psi_s I^j + \Psi \hat{\Gamma}^j_0, \quad j = 2, 3, \tag{4.8}$$

where Γ^j_0 solve (4.7). However, the general solution of (4.7) is given by

$$\hat{\Gamma}^2_0 = \hat{G}_{x_2} + I^2 \hat{G}_s, \qquad \hat{\Gamma}^3_0 = \hat{f} + \hat{G}_{x_3} + I^3 \hat{G}_s, \tag{4.9}$$

where

$$\hat{G} = \hat{G}(x_1 I^1 - s, x_2, x_3), \qquad \hat{f} = \hat{f}(x_1 I^1 - s, x_2 I^2 - s, x_3), \tag{4.10}$$

and, for any diagonal matrices G, J, we use the notation $(\hat{G}(xJ))_{jj} = \hat{G}_{jj}(xJ_j)$. Hence, the gauge transformation $\Psi = \Phi e^{\hat{G}}$ reduces the operators (4.8) to

$$\hat{D}_{x_j} \Phi = \Phi_{x_j} + \Phi_s I^j, \quad j = 1, 2 \qquad \hat{D}_{x_3} \Phi = \Phi_{x_3} + \Phi_s I^3 + \Phi \hat{f}, \tag{4.11}$$

where $\hat{f}$ is given by (4.10b). Finally, the gauge transformation $\Phi = m e^{\hat{h}}$, where

$$\hat{f} = \hat{h}_{x_3} + I^3 \hat{h}_s, \qquad \hat{h} = \hat{h}(x_1 I^1 - s, x_2 I^2 - s, x_3), \tag{4.12}$$

maps the operators (4.11) to

$$\hat{D}_{x_j} m = m_{x_j} + m_s I^j, \quad j = 1, 2, 3. \tag{4.13}$$

4.1 Dressing Off Nonzero Background

We now consider the case that Γ^j are off-diagonal. For the sake of simplicity we consider the usual N-wave equation, i.e., we consider the particular case of (1.9) where

$$I^1 = J, \quad I^2 = I, \quad I^3 = C. \tag{4.14}$$

Then (4.3) and (4.4) yield

$$[\hat{\Gamma}^2, J] = [\hat{\Gamma}, C] = 0, \qquad [\hat{\Gamma}^3, J] = [\hat{\Gamma}^1, C], \tag{4.15}$$

$$
\begin{aligned}
\hat\Gamma^2_x + \hat\Gamma^2_s J + \hat\Gamma^2\hat\Gamma^1 &= \hat\Gamma^1_y + \hat\Gamma^1_s + \hat\Gamma^1\hat\Gamma^2,\\
\hat\Gamma^3_x + \hat\Gamma^3_s J + \hat\Gamma^3\hat\Gamma^1 &= \hat\Gamma^1_t + \hat\Gamma^1_s C + \hat\Gamma^1\hat\Gamma^3,\\
\hat\Gamma^2_t + \hat\Gamma^2_s C + \hat\Gamma^2\hat\Gamma^3 &= \hat\Gamma^3_y + \hat\Gamma^3_s + \hat\Gamma^3\hat\Gamma^2.
\end{aligned} \tag{4.16}
$$

Letting $\hat\Gamma^1 = \hat\Gamma$, (4.15) and (4.16) imply

$$
\hat\Gamma^2 = 0, \qquad \hat\Gamma^3_{ij} = \hat\Gamma_{ij}a_{ij}, \quad a_{ij} = \frac{C_i - C_j}{J_i - J_j}, \quad i \neq j, \tag{4.17}
$$

while $\hat\Gamma$ solves

$$
\begin{aligned}
&\hat\Gamma_y + \hat\Gamma_s = 0,\\
&\hat\Gamma_{ij_t} = \hat\Gamma_{ij_x}a_{ij} + \hat\Gamma_{ij_y}(C_j - J_j a_{ij}) + \sum_{\nu\neq i,\nu=1}^{N} \hat\Gamma_{i\nu}\hat\Gamma_{\nu j}(a_{i\nu} - a_{\nu j}).
\end{aligned} \tag{4.18}
$$

Equations (4.18) imply $\Gamma = e^{-iky}f(x,t,k)$, where $\hat f(x,t,s)$ solves the N-wave equation (1.10). Hence,

$$
\Gamma^2 = 0, \quad \Gamma^3_{ij} = \Gamma_{ij}a_{ij}, \quad \Gamma = \frac{1}{2\pi}\int_R ds \exp[ik(s-y)]\, q_0(x,s,t), \tag{4.19}
$$

where q_0 is a solution of the N-waves. Using (4.19) into (4.1), we find

$$
\begin{aligned}
D_x\mu &= \mu_x + ik\mu J + \mu * \Gamma, \qquad & D_y\mu &= \mu_y + ik\mu,\\
D_t\mu &= \mu_t + ik\mu C + \mu * \tilde\Gamma, \qquad & \tilde\Gamma_{ij} &= \Gamma_{ij}a_{ij}.
\end{aligned} \tag{4.20}
$$

We next require that the operators D_{x_j} commute with the RH problem (1.1). Commutativity of D_{x_j} yields

$$
\begin{aligned}
F_x &= i\ell JF - ikFJ + \int_R d\nu\, [\Gamma(\nu-\ell)F(k,\nu) - F(k-\nu,\ell)\Gamma(\nu)],\\
F_y &= i(\ell - k)F,\\
F_t &= i\ell CF - ikFC + \int_R d\nu \left[\tilde\Gamma(\nu-\ell)F(k,\nu) - F(k-\nu,\ell)\tilde\Gamma(\nu)\right].
\end{aligned}
$$

Thus,

$$
F(k,\ell) = \exp[i(\ell-k)y]\, f(x,t,k,\ell)
$$

and if

$$
\hat f(x,t,\xi,\eta) \doteq \frac{1}{2\pi}\int_{R^2} dk\, d\ell\, f(x,t,k,\ell)\exp[i(k\xi + \ell\eta)], \tag{4.21}
$$

$\hat f$ solves (1.14).

Next we proceed with the usual dressing method based on the operators D_{x_j}, $j = 1, 2, 3$. Let $\mu(k)$ be the unique solution of (1.1) where F satisfies (4.21), (1.14).

We seek linear combinations of $D_{x_j}\mu$ that also solve (1.1). Since $D_x\mu, D_y\mu$ are asymptotic to ikJ and ik as $k \to \infty$, it follows that

$$D_x\mu = JD_y\mu + q\mu, \quad \text{or} \quad \mu_x = ik[J, \mu] + J\mu_y - \mu * \Gamma + q\mu. \tag{4.22}$$

Similarly,

$$D_t\mu = CD_y\mu + A\mu, \quad \text{or} \quad \mu_t = ik[C, \mu] + C\mu_y - \mu * \tilde{\Gamma} + A\mu. \tag{4.23}$$

The $O(1)$ of these equations imply

$$q = q_0 - i[J, \mu^{(1)}], \qquad A_{ij} = a_{ij}q_{ij}. \tag{4.24}$$

Compatibility of (4.22), (4.23), or equivalently the $O(1/k)$ terms of (4.23) imply that q satisfies the usual N-wave equation.

Acknowledgments

One of the authors (ASF) has been partially supported by the Office of Naval Research under Grant Number N00014-88K-0447, National Science Foundation under Grant Number DMS-8803471, and Air Force Office of Scientific Research under Grant Number 87-0310.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Stud. Appl. Math., Philadelphia (1981).
[2] V. G. Makhankov, *Soliton Phenomenology*, Kluwer Academic Publishers, Mathematics and Its Applications, Holland (1990).
[3] A. S. Davydov, *Solitons in Molecular Systems*, D. Reidel Publishing Co., Mathematics and Its Applications, Holland (1985).
[4] D. J. Kaup, Physica **6D,** 143–54 (1983).
[5] O. Legrand and C. Montes, J.E. Phys. **50**, 3–147 (1989).
[6] F. Calogero, in *What is Integrability?*, ed. by V. E. Zakharov, Springer-Verlag, New York (1990).
[7] M. Boiti, J. Leon, L. Martina, and F. Pempinelli, Phys. Lett. A, **132**, 432 (1988).
[8] A. S. Fokas and P. M. Santini, Phys Rev. Lett. **63**, 1329 (1989); *Dromions and a Boundary Value Problem for the Davey-Stewartson I Equation*, Physica D **44**, 99 (1990).
[9] S. V. Manakov, Physica D **3,** 420 (1981).
[10] A. S. Fokas and M. J. Ablowitz, Stud. Appl. Math. **69,** 211 (1983).
M. J. Ablowitz, D. BarYaacov, and A. S. Fokas, Stud. Appl. Math. **69**, 135 (1983).
A. S. Fokas, Phys. Rev. Lett. **51**, 3 (1983).
A. S. Fokas and M. J. Ablowitz, J. Math. Phys. **25**, No. 8, 2494–2505 (1984).
[11] R. Beals and R. R. Coifman, Proc. Symp. Pure Math. **43**, Amer. Math. Soc. Providence, 45 (1985); *The Spectral Problem for the Davey-Stewartson and Ishimori Hierarchies*, Proc. Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods, Como, University of Manchester (1988).
[12] J. Bona, Private communication.
[13] V. E. Zakharov and P. B. Shabat, Funct. Anal. Appl. **8**, 226 (1974).
[14] V. E. Zakharov and P. B. Shabat, Funct. Anal. Appl. **13**, 166 (1979).

[15] V. E. Zakharov and S. V. Manakov, Funct. Anal. App. **19**, No. 2, 11 (1985).
[16] P. M. Santini, M. J. Ablowitz, and A. S. Fokas, J. Math. Phys. **25**, 2619 (1984).
[17] E. A. Kuznetsov and A. V. Mikhailov, Zh. Eksp. Teor. Fiz. **67**, 1718 (1974).
[18] V. E. Zakharov, *On the Dressing Method*, to appear (1990).
[19] L.-Y. Sung and A. S. Fokas, Inverse Problems of $N \times N$ Hyperbolic Systems on the Plane and the N wave Interactions, Comm. Pure and Appl. Math. **XLIV**, 535–571 (1991).
[20] X. Zhou, Inverse Scattering Transform for the Time Dependent Schrödinger Equation and KPI, to appear in Comm. in Math. Phys.
[21] M. Boiti, F. Pempinelli, A. K. Pogrebkov, and M. C. Polivanov, Inverse Problems **7**, 43 (1991).