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Integrable (1+1)-dimensional systems and the Riemann problem with a shift

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Abstract. We study (1+1)-dimensional integrable systems considering them as special cases of the more general (2+1)-dimensional systems. Using the non-local $\bar\partial$-problem approach in (2+1) dimensions, we show that the $\bar\partial$-problem with a shift and (for the decreasing solutions) the Riemann problem with a shift arise naturally in (1+1) dimensions. The Boussinesq equation and the first-order relativistically-invariant systems are investigated. The approach developed also allows us to investigate the structure of the continuous spectrum and the inverse scattering problem for an arbitrary-order ordinary differential operator on the infinite line.

1. Introduction

The starting point of our work is the dressing method based on the non-local $\bar\partial$-problem [1–5]. This is a powerful method of constructing (2+1)-dimensional integrable equations together with a broad class of their solutions. Removing the dependence on one variable, we go to the one-dimensional case. In terms of the $\bar\partial$-problem this leads us to the problem with a special kind of non-locality—the $\bar\partial$-problem with a shift and the Riemann problem with a shift. It appears that these *scalar non-local* problems are a general and natural technical tool in the (1+1)-dimensional case.

The main objects of our investigation are the KP equation and the system of equations of N-wave type [4]. Slightly modifying the KP equation and descending to (1+1) dimensions, one obtains the Boussinesq equation. The approach developed enables us to construct the small non-singular decreasing solutions of this equation and to investigate the continuous spectrum of its L-operator. The geometry of this spectrum is rather interesting; the spectral data are localized on a hyperbola in the complex plane and on a segment of the real axis and the decreasing solutions are given by the Riemann problem with a shift on this curve (see another approach in [6]).

Using the N-wave type equations in (2+1) dimensions, we obtain the relativistically-invariant systems in (1+1) dimensions. These systems were first integrated by Zakharov and Mikhailov [7], using the *matrix local* Riemann problem. In our work we use the *scalar non-local* problem and obtain some new spectral information about these equations.

Finally, we use the developed technique to study the inverse scattering problem for the differential operators of arbitrary order on the line. A special case of this problem was considered in detail in [8] . Our investigation is not so detailed. We use the ideology of the dressing method and do not treat the direct scattering problem at all. The dressing method yields the structure of the continuous spectrum, which is quite non-trivial. The spectral data are localized on certain algebraic curves in the complex plane, and the inverse problem reduces to Riemann problem with a shift on these curves.

We would like to emphasize that the calculations in this work are based on the $\bar\partial$-dressing technique and they are mainly formal. A rigorous investigation of the underlying analysis is beyond the scope of this paper. The integral operators are supposed to be 'small' in some sense, so that the integral equations are uniquely solvable. But we hope that this work produces a link between the formal technique and the spectral theory.

2. The $\bar\partial$-problem and its special cases

The main idea of this paper is the following. We consider (1+1)-dimensional systems as special cases of the more general (2+1)-dimensional systems. For instance, the Boussinesq equation is a special case of the more general Kadomtsev–Petviashvili (KP) equation. We will show that, in this case, the Riemann problem with a shift naturally arises.

In some sense the technique of constructing (2+1)-dimensional integrable systems is not in the least more complicated than the (1+1)-dimensional case [1–5]. We briefly outline it in this paper, restricting ourselves to the scalar case as the simplest. We would like to emphasize that the contents of this section are not original and are in the main described in the papers mentioned above. The dressing method uses the non-local $\bar\partial$-problem with the special dependence of the kernel on additional (space and time) variables

$$\bar\partial(\psi(x,\lambda)-\eta(x,\lambda))=\hat R\psi(x,\lambda) \tag{1}$$

$$\hat R\psi=\iint\psi(\lambda)R(\lambda,\mu)\exp(\phi_i x_i)\,\mathrm{d}\mu\wedge\mathrm{d}\bar\mu \qquad \phi_i=K_i(\mu)-K_i(\lambda),\ 1\leqslant i\leqslant 3 \tag{2}$$

where $\lambda\in\mathbb{C}$, $\bar\partial=\partial/\partial\bar\lambda$ and $\eta(x,\lambda)$ is a rational function of λ (normalization), $K_i(\lambda)$ are rational functions, the choice of which determines the equation that can be solved using (1). We suppose that the kernel $R(\lambda,\mu)$ equals zero in a neighbourhood of λ and μ, which appear in the divisor of the poles of the functions $K_i(\lambda)$, tends to zero as $\lambda,\mu\to\infty$ and that for the chosen kernel $R(\lambda,\mu)$ the problem (1) is uniquely solvable (at least for sufficiently small x). The solution of (1) normalized by η is the function

$$\psi(x,\lambda)=\eta(x,\lambda)+\varphi(x,\lambda)$$

where $\eta(x,\lambda)$ is a rational function of λ (normalization), $\varphi(x,\lambda)$ decreases as $\lambda\to\infty$ and is *analytic* in a neighbourhood of the poles of $K_i(\lambda)$.

The problem (1) reduces to the integral equation for the function φ

$$\varphi(x,\lambda)=\bar\partial^{-1}\hat R(\varphi(x,\lambda)+\eta(x,\lambda)) \tag{3}$$

here

$$(\bar\partial^{-1}\varphi)(\lambda)=(2\pi\mathrm{i})^{-1}\iint\frac{\varphi(\lambda')}{(\lambda'-\lambda)}\,\mathrm{d}\lambda'\wedge\mathrm{d}\bar\lambda'$$

$$=(2\pi\mathrm{i})^{-1}\lim_{\epsilon\to0}\iint\frac{\varphi(\lambda')(\bar\lambda'-\bar\lambda)}{(|\lambda'-\lambda|^2+\epsilon)}\,\mathrm{d}\lambda'\wedge\mathrm{d}\bar\lambda'$$

which is supposed to be uniquely solvable for the given R. Solvability is guaranteed if the operator $\bar{\partial}^{-1}R$ is 'small enough' (i.e. the norm of this operator is less than one for some properly chosen space of functions).

Let us introduce $\rho(\lambda, \bar{\lambda}) = \bar{\partial}\varphi$. Now

$$\psi(x, \lambda) = \eta + (2\pi\mathrm{i})^{-1} \iint \frac{\rho(\lambda')}{(\lambda' - \lambda)} \,\mathrm{d}\lambda' \wedge \mathrm{d}\bar{\lambda}'. \tag{4}$$

Substituting (4) into (3), we can get another form of the basic integral equation, resolving the non-local $\bar{\partial}$-problem

$$\rho(x, \lambda) = \hat{R}\big(\eta + \bar{\partial}^{-1}\rho\big). \tag{5}$$

In the most important cases the kernel $R(\lambda, \mu)$ is a singular function localized on some manifold in $\mathbb{C}^2$. This means that the kernel contains the δ-function localized on the corresponding manifold, or, in other words, that the measure of integration in the operator $\bar{\partial}^{-1}\hat{R}$ is localized on this manifold. The operator $\bar{\partial}^{-1}\hat{R}$ in this case is still well defined. In a typical situation this manifold is a covering of the complex λ-plane, defined by the equation

$$f(\lambda, \bar{\lambda}, \mu, \bar{\mu}) = 0 \tag{6}$$

where f is some function in $\mathbb{C}^2$. Equation (6) defines a multi-valued shift function $\mu = \mu_i(\lambda, \bar{\lambda})$. The kernel of the problem (1) in this case reads

$$R = \sum_i R_i(\lambda, \bar{\lambda})\delta(\mu - \mu_i(\lambda, \bar{\lambda})).$$

We will call this case a *$\bar{\partial}$-problem with a shift.*

Another special case of the problem (1) is a non-local Riemann problem. Let $\gamma = \lambda(\xi)$, $\xi \in \mathbb{R}$ be an oriented curve in a complex plane (which need not be connected), and the kernel of the problem (1) be concentrated on the product of two of these curves in the λ and μ planes. In other words

$$R(\lambda, \mu) = \delta_\gamma(\lambda)R_\gamma(\lambda, \mu)\delta_\gamma(\mu) \tag{7}$$

where $\delta_\gamma(\lambda)$ is a δ-function picking out points on γ. The solution ψ of the problem (1) with the kernel (7) is rational outside γ and has boundary values ψ^+, ψ^- on γ. After regularizing δ_γ we obtain from the problem (1) with the kernel (7) the non-local Riemann problem

$$\psi^+ - \psi^- = \tfrac{1}{2}\int_\gamma (\psi^+ + \psi^-)R_\gamma(\lambda, \mu)\,\mathrm{d}\mu \tag{8}$$

the integration in (8) goes along the curve γ.

A combination of these two special cases leads to the Riemann problem with a shift (or Carleman's problem). The shift function $\mu = \mu_i(\lambda)$ is now defined on the curve γ ($\lambda, \mu \in \gamma$). In this case

$$R_\gamma(\lambda, \mu) = \sum_i R^i_\gamma(\lambda)\delta_\gamma(\mu - \mu_i(\lambda))$$

and

$$\psi^+ - \psi^- = \tfrac{1}{2}\sum_i (\psi^+(\mu_i(\lambda)) + \psi^-(\mu_i(\lambda)))R^i_\gamma(\lambda) \tag{9}$$

where $\mu_i(\lambda)$ is a multi-valued shift function on the curve $\lambda(\xi)$. We will write the problem (9) symbolically in the form

$$\Delta(\psi(\lambda(\xi))) = R_\gamma(\lambda, \mu(\lambda))\psi(\mu(\lambda(\xi))) \tag{10}$$

where $\gamma = \lambda(\xi)$ ($\xi \in \mathbb{R}$) is a curve in the complex plane, Δ is a jump of the function across the curve, the value of the function on the curve is the half-sum of the boundary values and $\mu(\lambda)$ is the shift function (which may be multi-valued).

In all these three cases the problem is equivalent to a certain integral equation which can be obtained by a proper reduction of equations (3), (5). Let us do that for a Riemann problem with a shift. Introducing

$$\rho_\gamma(\lambda) = \psi^+ - \psi^-|_{\lambda\in\gamma}$$

we can restore the function ψ in a form

$$\psi = \eta + \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{\rho_\gamma(\lambda')}{(\lambda-\lambda')}\,\mathrm{d}\lambda'.$$

Hence

$$\frac{1}{2}(\psi^+ + \psi^-)|_{\lambda\in\gamma} = \eta(\lambda) + \frac{1}{2\pi \mathrm{i}}\mathrm{v.p.}\int \frac{\rho_\gamma(\lambda')}{(\lambda-\lambda')}\,\mathrm{d}\lambda'$$

and from equation (9) one gets

$$\rho_\gamma(\lambda) = \sum_i \left(\eta(\mu_i(\lambda)) + \frac{1}{2\pi \mathrm{i}}\mathrm{v.p.}\int \frac{\rho_\gamma(\lambda)}{(\mu_i(\lambda)-\lambda')}\,\mathrm{d}\lambda'\right)R^i_\gamma(\lambda) \qquad \lambda\in\gamma. \tag{11}$$

Let the curve γ consist of n connected branches $\gamma_i = \lambda_i(\xi)$, $\xi \in \mathbb{R}$, and $\rho_i(\xi)$ be the jump of the function ψ across the corresponding branch. Then the expression for the function ψ takes the form

$$\psi = \eta + \frac{1}{2\pi \mathrm{i}}\sum_{i=1}^{n}\int \frac{\rho_i(\xi')}{(\lambda-\lambda_i(\xi'))}\frac{\mathrm{d}\lambda_i}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \tag{12}$$

and the integral equation (11) reads

$$\rho_k(\xi) = \sum_i \left(\eta(\mu_i(\lambda_k(\xi))) + \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{n}\mathrm{v.p.}\int \frac{\rho_j(\xi')}{(\mu_i(\lambda_k(\xi))-\lambda_j(\xi'))}\frac{\mathrm{d}\lambda_j}{\mathrm{d}\xi'}\,\mathrm{d}\xi'\right)R^{ik}(\xi). \tag{13}$$

Thus we have obtained a system of n singular integral equations.

3. The KP equation and the N-wave system of equations

The non-local $\bar{\partial}$-problem and its special cases ($\bar{\partial}$-problem with a shift, non-local Riemann problem, Riemann problem with a shift) are powerful tools for constructing integrable nonlinear wave equations and their solutions (see [2–4, 9]).

The algebraic scheme of constructing equations is based on the following property of the problem (1): if $\psi(x, \lambda)$ is a solution of the problem (1), then the functions

$$u(x)\psi \qquad D_i\psi = (\partial/\partial x_i + K_i)\psi \tag{14}$$

are also solutions. Combining this property with the unique solvability of the problem (1), one obtains the differential relations between the coefficients of expansions of the functions $\psi(x, \lambda)$ into powers of $(\lambda - \lambda_p)$ at the poles of $K_i(\lambda)$. Let us outline the basic steps of this scheme for the equations which will be used in this work, i.e. for the KP equation and for the N-wave type system of equations [4].

For the KP equation $D_1 = \partial/\partial x + \mathrm{i}\lambda$, $D_2 = \partial/\partial y + \alpha^{-1}\lambda^2$, $(\alpha = 1, \mathrm{i})$, $D_3 = \partial/\partial t + \mathrm{i}\lambda^3$. Let us introduce the solution of the problem (1) normalized by 1 $(\eta = 1)$

$$\psi(\lambda, x, y, t)_{\lambda\to\infty} \to 1 + \psi_0(x, y, t)\lambda^{-1} + \cdots.$$

It follows from the unique solvability of the problem (1) that this solution satisfies the relations

$$(\alpha D_2 + D_1^2 + 2v(x, y, t))\psi = 0$$

$$(D_3 + D_1^3 + g(x, y, t)D_1 + h(x, y, t))\psi = 0.$$

The successive use of the coefficients of expansion of these relations as $\lambda \to \infty$ allows us to define the functions v, g, h

$$v = -\mathrm{i}\frac{\partial}{\partial x}\psi_0 \qquad g = 3v \qquad h_x = \frac{3}{2}(v_{xx} - \alpha v_y)$$

and to derive the KP equation for the first coefficient of expansion of the function ψ as $\lambda \to \infty$

$$\frac{\partial}{\partial x}\left(v_t + \frac{1}{4}v_{xxx} + 3v_x v\right) = -\frac{3}{4}\alpha^2 v_{yy}. \tag{15}$$

To construct the N-wave type system of equations, we use the functions $K_i(\lambda)$ with an arbitrary number of simple and distinct poles

$$K_i(\lambda) = \sum_{\alpha=1}^{n_i} \frac{a_i^\alpha}{\lambda - \lambda_i^\alpha} \tag{16}$$

where $a_i^\alpha, \lambda_i^\alpha \in \mathbb{C}$, $1 \leqslant \alpha \leqslant n_i$, $\lambda_i^\alpha \neq \lambda_j^\beta$. Let us introduce the solutions $\psi_i^\alpha(x, \lambda)$ of the problem (1), normalized by $(\lambda - \lambda_i^\alpha)^{-1}$. These solutions satisfy the relations

$$\begin{aligned} &D_i\psi_j^\beta - K_i(\lambda_j^\beta)\psi_j^\beta - \psi_{ji}^{\beta\alpha}a_i^\alpha\psi_i^\alpha = 0 \\ &\psi_{ji}^{\beta\alpha}(x) = \psi_j^\beta(x, \lambda_i^\alpha) \end{aligned} \tag{17}$$

where $i \neq j$ and summation over β is understood. The leading order of expansion of the relation (17) as $\lambda \to \lambda_k^\gamma$, $i \neq j \neq k$ yields the equation

$$\frac{\partial}{\partial x_i}\psi_{jk}^{\beta\gamma} + (K_i(\lambda_k^\gamma) - K_i(\lambda_j^\beta))\psi_{jk}^{\beta\gamma} - \psi_{ji}^{\beta\alpha} a_i^\alpha \psi_{ik}^{\alpha\gamma} = 0 \tag{18}$$

summation over β is understood. If the different permutations ijk and substitutions of the indices β, γ are taken into account, (18) is a closed set of equations for the functions $\psi_{ji}^{\beta\alpha}(\boldsymbol{x})$. The system of equations (18) is formally Lagrangian with the action density

$$\mathcal{L}(\boldsymbol{x}) = \epsilon_{ijk}(\tfrac{1}{2}\psi_{ij}^{\alpha\beta} a_j^\beta \partial_k \psi_{ji}^{\beta\alpha} a_i^\alpha + K_k(\lambda_i^\alpha)\psi_{ij}^{\alpha\beta} a_j^\beta \psi_{ji}^{\beta\alpha} a_i^\alpha + \tfrac{1}{3} a_i^\alpha \psi_{ik}^{\alpha\gamma} a_k^\gamma \psi_{kj}^{\gamma\beta} a_j^\beta \psi_{ji}^{\beta\alpha}) \tag{19}$$

summation is over α, β, γ as well as over i, j, k.

4. The decreasing solutions

A solution given by the problem (1) in a general case is defined only locally in a vicinity of the point $\boldsymbol{x} = 0$, where the $\bar{\partial}$-problem is uniquely solvable. Solvability may be lost on some manifold in a space (x_1, x_2, x_3), where the solution has a singularity. To get 'good enough' solutions having no singularities and bounded (decreasing) as $|\boldsymbol{x}| \to \infty$ one should put some restrictions on the kernel $R(\lambda, \mu)$. These restrictions were discussed in our article [5]; the main result of this article can be formulated as follows. Let us choose a unit vector n_i ($\sum n_i^2 = 1$) defining a direction in the $\boldsymbol{x}$-space. The solution given by the problem (1) is regular in a neighbourhood of the straight line $x_i = n_i\xi$ and decreases along this line as $\xi \to \pm\infty$, if the condition

$$\operatorname{Re}\sum_{i=1}^{3} n_i(K_i(\lambda) - K_i(\mu)) = 0 \tag{20}$$

is satisfied (this condition is in fact the condition for the kernel $R(\lambda, \mu)$, it means that we should use the kernel localized on the manifold (20)), and the kernel $R(\lambda, \mu)$ is 'small enough'.

To get a solution which is 'good enough' in a neighbourhood of some plane, defined by two vectors n_i, m_i, one has to satisfy two conditions

$$\operatorname{Re}\sum_{i=1}^{3} n_i(K_i(\lambda) - K_i(\mu)) = 0$$

$$\operatorname{Re}\sum_{i=1}^{3} m_i(K_i(\lambda) - K_i(\mu)) = 0.$$

In a generic case a pair of conditions (20) define some manifold with real dimension two in the space $\mathbb{C}^2$ of complex variables λ, μ.

Let us illustrate this result for the simple example of the KP equation. To obtain the small non-singular solution decreasing in the plane (x, y), it is sufficient to use the problem (1) with the kernel localized on the manifold defined by the system of conditions (20)

$$\operatorname{Im}(\lambda - \mu) = 0 \tag{21}$$

$$\operatorname{Re}\alpha^{-1}(\lambda^2 - \mu^2) = 0. \tag{22}$$

If $\alpha = \mathrm{i}$, the system (21), (22) has a solution $\lambda, \mu \in \mathbb{R}$, which defines a non-local Riemann problem on the real axis. So the small decreasing solutions of the KP1 equation are given by the non-local Riemann problem

$$\psi^+ - \psi^- = \int_\gamma (\psi^+ + \psi^-) R_\gamma(\lambda, \mu) \exp(\phi_i x_i) \,\mathrm{d}\mu \tag{23}$$

which was originally used by Manakov [10] to integrate the KP1 equation.

If $\alpha = 1$, the solution of the system (22) is $\mu = -\bar{\lambda}$. Thus the small decreasing solutions of the KP2 equation are given by the $\bar{\partial}$-problem with a conjugation

$$\bar{\partial}\psi(x, y, t, \lambda) = R(\lambda, -\bar{\lambda}) \exp(\phi_i x_i) \psi(x, y, t, -\bar{\lambda}) \tag{24}$$

and we reproduce the result of Ablowitz *et al* [11].

A technique used in the $\bar{\partial}$-problem is flexible enough to construct solutions with a different given type of asymptotic behaviour. For instance, to get a solution, periodic in x and decreasing in y, one can use the kernel localized on the countable system of manifolds

$$\lambda - \mu = 2\pi n/X \qquad -\infty < n < \infty \tag{25}$$

$$\operatorname{Re} \alpha^{-1}(\lambda^2 - \mu^2) = 0. \tag{26}$$

If $\alpha = \mathrm{i}$, the system (26) has a solution $\lambda, \mu \in \mathbb{R}$, $\lambda - \mu = \pm 2\pi n/X$, and the solutions of the KP1 equation, periodic in x and decreasing in y, are given by the Riemann problem with a shift on the real axis with the shift function (25).

If $\alpha = 1$, the system (26) has a solution $\operatorname{Re}\lambda, \mu = \pm\pi n/X$, $\lambda - \mu = \pm 2\pi n/X$. Thus the solutions of the KP2 equation, periodic in x and decreasing in y, are given by the Riemann problem with a shift on the system of lines $\operatorname{Re}\lambda, \mu = \pm\pi n/X$ with the shift function $\mu = -\bar{\lambda}$. This interesting problem is quite complicated, but we will not treat it here in detail.

5. The (1+1)-dimensional case

The solutions independent of the variable x_j can be obtained from the problem (1) with the kernel localized on the manifold

$$K_j(\lambda) - K_j(\mu) = 0. \tag{27}$$

This observation allows us to use the (2+1)-dimensional dressing method for (1+1)-dimensional equations, and leads us naturally to the $\bar{\partial}$-problem with a shift and, for decreasing solutions, to the Riemann problem with a shift. Let us consider this observation in more detail.

If we have a (2+1)-dimensional integrable equation, defined by the functions $K_i(\lambda)$, we can descend to the (1+1)-dimensional case, using the condition (27) for some coordinate x_i in the original or rotated coordinate system. For example, the y-independent KP equation gives the KdV equation

$$(v_t + \tfrac{1}{4} v_{xxx} + 3 v_x v) = 0.$$

The condition (27) in this case reads

$$\lambda^2 - \mu^2 = 0$$

and the solutions of the KdV equation are given by the $\bar{\partial}$-problem with a shift [3]

$$\bar{\partial}\psi(\lambda) = R(\lambda, -\lambda)\exp(\phi_i x_i)\psi(-\lambda) \tag{28}$$

the shift function for this case is quite simple ($\mu = -\lambda$), and it is easy to transform the problem (28) to the local matrix (2×2) Riemann problem.

We may also consider the case of the t-independent KP equation, which corresponds to the simplified Boussinesq equation

$$\tfrac{3}{4}\alpha^2 v_{yy} = -(\tfrac{1}{4}v_{xx} + \tfrac{3}{2}v^2)_{xx}. \tag{29}$$

The condition (27) in this case reads

$$\lambda^3 - \mu^3 = 0$$

and the solutions of the simplified Boussinesq equation (29) are given by the $\bar{\partial}$-problem

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^{3} R_i \psi(e_i\lambda)$$

where $e_i^3 = 1$. This simplified variant of the Boussinesq equation was considered in [12]. Let us show that for decreasing solutions our approach leads us to the Riemann problem with a shift for the functions analytic in sectors (such a geometry for the local matrix Riemann problem arose in [12] from the analytic properties of the direct scattering problem). Combining the condition (27) with the condition (20)

$$\mathrm{Im}(\lambda - \mu) = 0$$

we obtain

$$\lambda - e_i\mu = 0$$

$$\lambda - \mu = \xi \qquad \xi \in \mathbb{R}.$$

The solution of this system is

$$\lambda = \xi(1 - e_i)^{-1}$$

$$\mu = -\xi(1 - e_i^{-1})^{-1}$$

which defines a Riemann problem with a shift on the pair of straight lines with the vectors $\exp(\mathrm{i}\pi/6)$, $\exp(-\mathrm{i}\pi/6)$, the shift function is $\mu = -\bar{\lambda}$. So we arrive at the problem for the function analytic in corresponding sectors.

For the arbitrary rational function $K_i(\lambda)$ the condition (27) defines a multi-valued shift function $\mu_i(\lambda)$, and the corresponding $\bar{\partial}$-problem reads

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^{n} R_i \psi(\lambda_i(\mu)). \tag{30}$$

Let us introduce a modification of the KP equation

$$\frac{\partial}{\partial x}\left((v_t - \beta v_x) + \frac{1}{4}v_{xxx} + 3v_x v\right) = -\frac{3}{4}\alpha^2 v_{yy} \qquad \beta^2 = 1. \tag{31}$$

The solutions of this equation are given by the problem (1) with the dependence of the kernel on the variables x, y, t defined by the expressions (compare (2), (14))

$$\begin{aligned} D_1 &= \partial/\partial x + \mathrm{i}\lambda \\ D_2 &= \partial/\partial y + \alpha^{-1}\lambda^2 \qquad (\alpha = 1, \mathrm{i}) \\ D_3 &= \partial/\partial t + \mathrm{i}\lambda^3 + \mathrm{i}\beta\lambda \end{aligned} \tag{32}$$

the derivation of this statement for the KP equation is given in section 2, in this case it is completely analogous (or otherwise one may treat equation (31) as the KP equation with transformed variables x, y, t).

The time-independent solutions of equation (31) satisfy the Boussinesq equation

$$(\tfrac{3}{4}\alpha^2 v_{yy} - \beta v_{xx}) = -(\tfrac{1}{4}v_{xx} + \tfrac{3}{2}v^2)_{xx}. \tag{33}$$

Such solutions are given by the problem (1) ($v = -\mathrm{i}(\partial/\partial x)\psi_0$), if the support of the kernel $R(\lambda, \mu)$ belongs to the manifold defined by the condition (27)

$$(\lambda^3 + \beta\lambda - \mu^3 - \beta\mu) = 0 \qquad \lambda \neq \mu \tag{34}$$

or

$$\lambda^2 + \lambda\mu + \mu^2 + \beta = 0.$$

This relation defines a $\bar\partial$-problem with a shift

$$\bar\partial\psi(\lambda, x, y) = R(\lambda, \mu(\lambda))\exp(\phi_i x_i)\psi(\mu(\lambda), x, y) \qquad \mu = \tfrac{1}{2}(-\lambda \pm (4\beta - 3\lambda^2)^{1/2}). \tag{35}$$

The solutions of the Boussinesq equation, given by the problem (35) ($v = -\mathrm{i}(\partial/\partial x)\psi_0$), are defined locally in the neighbourhood of the point $x = 0$, $y = 0$. We consider the Boussinesq equation as a dynamic equation with respect to the variable y. To obtain solutions decreasing as $|x| \to \infty$, we should investigate the intersection of the manifold (27) with the manifold defined by the condition (20)

$$\mathrm{Im}(\lambda - \mu) = 0. \tag{36}$$

The conditions (34), (36) define the Riemann problem with a shift (Carleman's problem) which is a proper tool for solving Boussinesq's equation. Introducing $\xi = \frac{1}{2}(\lambda - \mu)$, $\nu = -\mathrm{i}\frac{1}{2}(\lambda + \mu)$, $\xi \in \mathbb{R}$, one can obtain

$$\beta + \xi^2 - 3\nu^2 = 0. \tag{37}$$

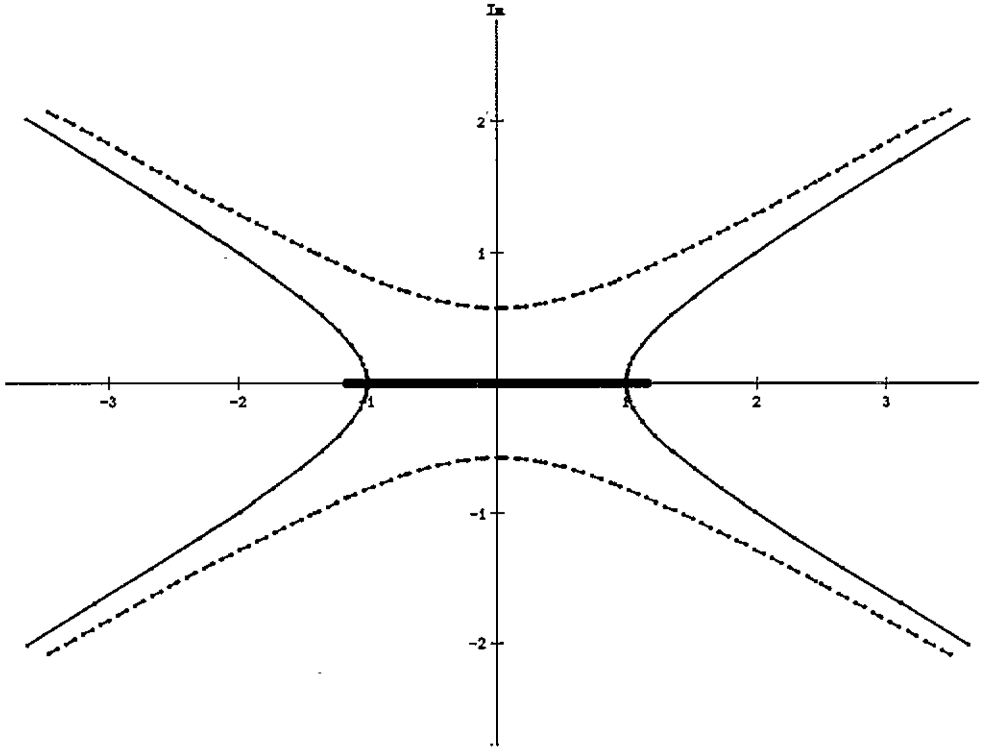

Figure 1. The localization of the continuous spectrum for the 'plus' (broken curve) and 'minus' (full curve) Boussinesq equations.

5.1. The 'plus' Boussinesq equation

One can see that the properties of the Boussinesq equation depend essentially on the sign of β. Let $\beta = 1$, then the corresponding equation (the plus Boussinesq equation) has a form

$$\tfrac{3}{4}\alpha^2 v_{yy} - v_{xx} + \tfrac{1}{4}v_{xxxx} + (\tfrac{3}{2}v^2)_{xx} = 0. \tag{38}$$

In the case $\alpha^2 = 1$ this is a nonlinear wave equation, having an approximate linear monochromatic solution

$$v \simeq e^{i(\omega y + kx)} \qquad \omega^2 = \tfrac{4}{3}(k^2 + \tfrac{1}{4}k^4).$$

In the case $\alpha^2 = -1$ it is a nonlinear elliptic equation. In both cases equation (38) can be solved by the following shifted Riemann problem

$$\begin{aligned} &-3\nu^2 + \xi^2 + 1 = 0 \\ &\lambda = -\bar{\mu} \\ &\lambda = \xi + i\nu \qquad \mu = -\xi + i\nu. \end{aligned} \tag{39}$$

Equation (39) defines a hyperbola with branches belonging to the upper and the lower half-planes, respectively (figure 1, broken curve). The shift is the change of the sign of the real part of λ. Let us introduce

$$\rho_\pm(\xi) = \Delta\psi\mid_\pm$$

to denote the jumps of the function $\psi(\lambda)$ across the upper and lower branches of the hyperbola. The function ψ can be represented in the form

$$\psi = 1 + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'} \mathrm{d}\xi' + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'} \mathrm{d}\xi'$$

where

$$\lambda_\pm(\xi) = \xi \pm \mathrm{i}\sqrt{\frac{1+\xi^2}{3}}.$$

The Riemann problem with a shift (39) is equivalent to the system of two integral equations (13)

$$\begin{aligned}
\rho_+(\xi) = \Bigg(1 &+ \frac{1}{2\pi \mathrm{i}} \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_+(-\xi) - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'} \mathrm{d}\xi' \\
&+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_+(-\xi) - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'} \mathrm{d}\xi' \Bigg) R^+(\xi) \\
&\times \exp\left(\frac{4\mathrm{i}}{\sqrt{3}\alpha} \xi \sqrt{1+\xi^2} y + 2\mathrm{i}\xi x\right) \\
\rho_-(\xi) = \Bigg(1 &+ \frac{1}{2\pi \mathrm{i}} \mathrm{v.p.} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_-(-\xi) - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'} \mathrm{d}\xi' \\
&+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_-(-\xi) - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'} \mathrm{d}\xi \Bigg) R^-(\xi) \\
&\times \exp\left(\frac{-4\mathrm{i}}{\sqrt{3}\alpha} \xi \sqrt{1+\xi^2} y - 2\mathrm{i}\xi x\right).
\end{aligned}$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\rho_+(\xi) \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi} + \rho_-(\xi) \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi}\right) \mathrm{d}\xi.$$

5.2. The 'minus' Boussinesq equation

This equation

$$\tfrac{3}{4}\alpha^2 v_{yy} + v_{xx} + \tfrac{1}{4} v_{xxxx} + (\tfrac{3}{2} v^2)_{xx} = 0$$

arises after putting $\beta = -1$. The reduced $\bar{\partial}$ problem for this equation is described by the conditions

$$\lambda^2 + \lambda\mu + \mu^2 = 1 \tag{40}$$

(time independence) and

$$\mathrm{Im}(\lambda - \mu) = 0 \tag{41}$$

(decreasing in the x direction). There are two possibilities to satisfy these conditions.

First, λ and μ are real ($\lambda^2 < \frac{4}{3}$, $\mu^2 < \frac{4}{3}$) and

$$\mu = -\tfrac{1}{2}\lambda \pm \sqrt{1 - \tfrac{3}{4}\lambda^2}. \tag{42}$$

We have a Riemann problem on the cut $-\sqrt{\frac{4}{3}} < \operatorname{Re}\lambda < \sqrt{\frac{4}{3}}$ with the twofold shift (42).

Second, λ and μ are complex, $\lambda = \nu + \mathrm{i}\xi$, $\mu = -\nu + \mathrm{i}\xi$, $\xi, \nu \in \mathbb{R}$

$$\nu^2 - 3\xi^2 = 1. \tag{43}$$

Both λ and μ are placed on the hyperbola (see figure 1, full curve). The shift, as for the 'plus' Boussinesq equation, is the reflection with respect to the imaginary axis.

Let us parameterize the curves, on which the solution, ψ, of the Riemann problem with a shift has a discontinuity, in the following way

$$\begin{aligned}
\gamma_+ &= \lambda_+(\xi) = \mathrm{i}\xi + \sqrt{1+3\xi^2} && -\infty < \xi < \infty \\
\gamma_- &= \lambda_-(\xi) = \mathrm{i}\xi - \sqrt{1+3\xi^2} && -\infty < \xi < \infty \\
\gamma_0 &= \lambda_0(\xi) = \xi && \xi^2 < \tfrac{4}{3}
\end{aligned}$$

and introduce the jumps $\rho_+(\xi), \rho_-(\xi), \rho_0(\xi)$ of the function ψ across the corresponding curves. Then the function ψ can be represented in the form

$$\begin{aligned}
\psi = 1 &+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \\
&+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'}\,\mathrm{d}\xi' + \frac{1}{2\pi\mathrm{i}} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\lambda - \xi')}\,\mathrm{d}\xi'.
\end{aligned}$$

The Riemann problem in this case is equivalent to the system of three integral equations

$$\begin{aligned}
\rho_0(\xi) = 1 &+ \Bigg(\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_+(\xi) - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \\
&+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_+(\xi) - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'}\,\mathrm{d}\xi' + \frac{1}{2\pi\mathrm{i}} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\mu_+(\xi) - \xi')}\,\mathrm{d}\xi' \Bigg) \\
&\times R_0^+(\xi) \exp\left\{ \mathrm{i}(\tfrac{1}{2}\xi - \sqrt{1 - \tfrac{3}{4}\xi^2})x + \frac{1}{\alpha}(\tfrac{3}{2}\xi^2 - \xi\sqrt{1 - \tfrac{3}{4}\xi^2} - 1)y \right\} \\
&+ \Bigg(\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_-(\xi) - \lambda_+(\xi'))} \frac{\mathrm{d}\lambda_+}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \\
&+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_-(\xi) - \lambda_-(\xi'))} \frac{\mathrm{d}\lambda_-}{\mathrm{d}\xi'}\,\mathrm{d}\xi' + \frac{1}{2\pi\mathrm{i}} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\mu_-(\xi) - \xi')}\,\mathrm{d}\xi' \Bigg) \\
&\times R_0^-(\xi) \exp\left\{ \mathrm{i}(\tfrac{1}{2}\xi + \sqrt{1 - \tfrac{3}{4}\xi^2})x + \frac{1}{\alpha}(\tfrac{3}{2}\xi^2 + \xi\sqrt{1 - \tfrac{3}{4}\xi^2} - 1)y \right\}
\end{aligned}$$

where

$$\mu_{\pm} = \tfrac{1}{2}\xi \pm \sqrt{1 - \tfrac{3}{4}\xi^2}$$

$$\begin{aligned}\rho_{+}(\xi) = 1 + \bigg(&\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_{+}(\xi')}{(\lambda_{-}(\xi) - \lambda_{+}(\xi'))} \frac{\mathrm{d}\lambda_{+}}{\mathrm{d}\xi'} \,\mathrm{d}\xi' \\ &+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_{-}(\xi')}{(\lambda_{-}(\xi) - \lambda_{-}(\xi'))} \frac{\mathrm{d}\lambda_{-}}{\mathrm{d}\xi'} \,\mathrm{d}\xi' + \frac{1}{2\pi \mathrm{i}} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_{0}(\xi')}{(\lambda_{-}(\xi) - \xi')} \,\mathrm{d}\xi' \bigg) \\ &\times R^{+}(\xi) \exp\left\{2\mathrm{i}\sqrt{1-3\xi^2}x + (4\mathrm{i}/\alpha)\xi\sqrt{1-3\xi^2}y\right\}\end{aligned}$$

$$\begin{aligned}\rho_{-}(\xi) = 1 + \bigg(&\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_{+}(\xi')}{(\lambda_{+}(\xi) - \lambda_{+}(\xi'))} \frac{\mathrm{d}\lambda_{+}}{\mathrm{d}\xi'} \,\mathrm{d}\xi' \\ &+ \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\rho_{-}(\xi')}{(\lambda_{+}(\xi) - \lambda_{-}(\xi'))} \frac{\mathrm{d}\lambda_{+}}{\mathrm{d}\xi'} \,\mathrm{d}\xi' + \frac{1}{2\pi \mathrm{i}} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_{0}(\xi')}{(\lambda_{+}(\xi) - \xi')} \,\mathrm{d}\xi' \bigg) \\ &\times R^{-}(\xi) \exp\left\{-2\mathrm{i}\sqrt{1-3\xi^2}x - (4\mathrm{i}/\alpha)\xi\sqrt{1-3\xi^2}y\right\}.\end{aligned}$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x}\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\left(\rho_{+}(\xi)\frac{\mathrm{d}\lambda_{+}}{\mathrm{d}\xi} + \rho_{-}(\xi)\frac{\mathrm{d}\lambda_{-}}{\mathrm{d}\xi}\right)\mathrm{d}\xi + \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \rho_{0}(\xi)\frac{\mathrm{d}\lambda_{0}}{\mathrm{d}\xi}\,\mathrm{d}\xi\right].$$

In this case the spectral data R_{γ} split into two parts; the short-wave part of the continuous spectrum is localized on the hyperbola (43), and the long-wave part of the spectrum is localized on the segment of the real axis (in fact on the covering of this segment); see figure 1, full curve. For $\alpha = 1$ the hyperbola corresponds to the stable part of the spectrum (where the exponent (2) for y is imaginary) and the segment to the unstable part (where the exponent is real), for $\alpha = \mathrm{i}$ the situation is reversed, i.e. for $\alpha = 1$ the long-wave instability takes place, and for $\alpha = \mathrm{i}$ the short-wave instability occurs.

Let us make a remark about the reduction. For $\alpha = 1$, $v(x, y)$ is real if the kernel of the problem (1) satisfies the condition

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu})$$

and, for $\alpha = \mathrm{i}$, it is real if

$$R(\lambda, \mu) = \bar{R}(\bar{\mu}, \bar{\lambda}).$$

6. The relativistically-invariant systems

The systems considered in this part were integrated first by Zakharov and Mikhailov [7], using the *matrix local* Riemann problem. In our work we use the *scalar non-local* problem and obtain some new spectral information about these equations.

Let us consider the solutions of the N-wave type system of equations (18) independent of the variable x_1. Such solutions are given by the $\bar{\partial}$-problem with a shift defined by the condition (27) and obey the relation

$$\psi_{jk}^{\alpha\gamma} = (K_1(\lambda_k^{\gamma}) - K_1(\lambda_j^{\alpha}))^{-1}\psi_{j1}^{\alpha\beta}a_1^{\beta}\psi_{1k}^{\beta\gamma}$$

where $j \neq k,\ j,k = 2,3$. Substituting this relation into equations (18) with the derivatives ∂_2 and ∂_3 we obtain the closed set of equations for the functions $\psi_{13}^{\alpha\gamma}$, $\psi_{12}^{\alpha\beta}$, $\psi_{31}^{\gamma\alpha}$, $\psi_{21}^{\gamma\beta}$

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_j} + (K_j(\lambda_k^\gamma) - K_j(\lambda_1^\alpha))\right)\psi_{1k}^{\alpha\gamma} = (K_1(\lambda_k^\gamma) - K_1(\lambda_j^\beta))^{-1}\psi_{1j}^{\alpha\beta} a_j^\beta \psi_{j1}^{\beta\alpha'} a_1^{\alpha'} \psi_{1k}^{\alpha'\gamma} \\
&\left(\frac{\partial}{\partial x_j} - (K_j(\lambda_k^\gamma) - K_j(\lambda_1^\alpha))\right)\psi_{k1}^{\gamma\alpha} \\
&\qquad = -(K_1(\lambda_k^\gamma) - K_1(\lambda_j^\beta))^{-1}\psi_{j1}^{\beta\alpha} a_1^\alpha \psi_{k1}^{\gamma\alpha'} a_1^{\alpha'} \psi_{1j}^{\alpha'\beta}
\end{aligned} \tag{44}$$

$k \neq j,\ k,j = 2,3$, summation is over β, α'. We consider the case $\mathrm{i}\alpha_i, \lambda_i \in \mathbb{R}$. In this case the system (44) admits a reduction

$$\psi_{ik}^{\alpha\gamma} = -\bar{\psi}_{ki}^{\gamma\alpha} \tag{45}$$

which corresponds to the following condition for the kernel of the problem (1)

$$R(\lambda, \mu) = \bar{R}(\bar{\mu}, \bar{\lambda}).$$

The linear part of equation (44) can be cancelled preserving the reduction by the substitution

$$\psi_{1k}^{\alpha\gamma} = \exp(K_j(\lambda_1^\alpha)x_j + K_k(\lambda_1^\alpha)x_k - K_j(\lambda_k^\gamma)x_j)\tilde{\psi}_{1k}^{\alpha\gamma} \tag{46}$$

and the coefficients a_i^α can be made equal to one by the change

$$\hat{\psi}_{1k}^{\alpha\gamma} = \tilde{\psi}_{1k}^{\alpha\gamma}(-a_k^\gamma a_1^\alpha)^{1/2} \tag{47}$$

if $\operatorname{Im} a_i^\alpha > 0$. We introduce the notation $\psi^{\alpha\gamma}$ for the function $\hat{\psi}_{13}^{\alpha\gamma}$ and $\varphi^{\alpha\beta}$ for the function $\hat{\psi}_{12}^{\alpha\beta}$. The set of equations (44) in this case reads

$$\begin{aligned}
\frac{\partial}{\partial x_2}\psi^{\alpha\gamma} &= (K_1(\lambda_3^\gamma) - K_1(\lambda_2^\beta))^{-1}\varphi^{\alpha\beta}\bar{\varphi}^{\alpha'\beta}\psi^{\alpha'\gamma} \\
\frac{\partial}{\partial x_3}\varphi^{\alpha\beta} &= -(K_1(\lambda_3^\gamma) - K_1(\lambda_2^\beta))^{-1}\psi^{\alpha\gamma}\bar{\psi}^{\alpha'\gamma}\varphi^{\alpha'\beta}
\end{aligned} \tag{48}$$

summation is over β, α' in the first equation and over γ, α' in the second equation. The Lagrangian density for this system is

$$\begin{aligned}
\mathcal{L}(x_2, x_3) = \mathrm{i}(&\bar{\psi}^{\alpha\gamma}\partial_2\psi^{\alpha\gamma} - \psi^{\alpha\gamma}\partial_2\bar{\psi}^{\alpha\gamma} - \bar{\varphi}^{\alpha\beta}\partial_3\varphi^{\alpha\beta} + \varphi^{\alpha\beta}\partial_3\bar{\varphi}^{\alpha\beta} \\
&+ 2(K_1(\lambda_2^\beta) - K_1(\lambda_3^\gamma))^{-1}\bar{\varphi}^{\alpha\beta}\bar{\psi}^{\alpha'\gamma}\psi^{\alpha\gamma}\varphi^{\alpha'\beta})
\end{aligned} \tag{49}$$

where summation is over $\alpha, \beta, \gamma, \alpha'$. In the special case where K_2 and K_3 have one pole, the system (48) reduces to the relativistically-invariant Nambu system [7]

$$\begin{aligned}
\partial_\eta\varphi^\alpha &= \mathrm{i}\psi^\alpha \sum_\beta \bar{\psi}^\beta\varphi^\beta \\
\partial_\xi\psi^\alpha &= \mathrm{i}\varphi^\alpha \sum_\beta \psi^\beta\bar{\varphi}^\beta
\end{aligned} \tag{50}$$

where $\eta = x_2$ and $\xi = -x_3$ are the light cone variables, $\varphi^\alpha = A\psi_{13}^{\alpha 1}$, $\psi^\alpha = A\psi_{12}^{\alpha 1}$ and $A = \mathrm{Im}(K_1(\lambda_3) - K_1(\lambda_2))^{1/2}$ (without loss of generality we suggest that $\mathrm{Im}(K_1(\lambda_3) - K_1(\lambda_2)) > 0$). In the general case the complete relativistic invariance of the system (48) can be obtained if K_2 and K_3 have an equal number of poles.

With an extra symmetry

$$\begin{aligned} K_2(\lambda) &= -K_2(-\lambda) \\ K_3(\lambda) &= -K_3(\lambda) \\ K_1(\lambda) &= \pm K_1(-\lambda) \end{aligned} \tag{51}$$

the system (48) admits a reduction

$$\begin{aligned} &\psi_{ik}^{\alpha\gamma} = -\bar{\psi}_{ik}^{\alpha'\gamma'} \\ &\lambda_i^\alpha = -\lambda_i^{\alpha'} \qquad \lambda_i^\gamma = -\lambda_i^{\gamma'} \end{aligned} \tag{52}$$

which corresponds to the condition

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu}). \tag{53}$$

For the Nambu system the symmetry (51) implies a special choice $K_2 = a_2\lambda$, $K_3 = a_3/\lambda$, $K_1(\lambda) = K_1(-\lambda)$ and leads to the reduction

$$\begin{aligned} &\psi^\alpha = \bar{\psi}^{\alpha'} \\ &\varphi^\alpha = \bar{\varphi}^{\alpha'} \qquad \lambda_1^\alpha = -\lambda_1^{\alpha'}. \end{aligned} \tag{54}$$

Thus we obtain the Gross–Neveu equations

$$\begin{aligned} \partial_\eta \varphi^\alpha &= \mathrm{i}\psi^\alpha \sum_\beta (\bar{\psi}^\beta \varphi^\beta + \psi^\beta \bar{\varphi}^\beta) \\ \partial_\xi \psi^\alpha &= \mathrm{i}\varphi^\alpha \sum_\beta (\psi^\beta \bar{\varphi}^\beta + \bar{\psi}^\beta \varphi^\beta). \end{aligned} \tag{55}$$

The Lagrangian density for the systems (50) and (55) is given by the expression (49).

Now let us investigate the problem allowing us to construct the solutions with the asymptotic behaviour resulting from the transform (46), (47); we find

$$\psi_{12}^{\alpha\beta} \to (\lambda_2^\beta - \lambda_1^\alpha)^{-1} \tag{56}$$

$$\psi^{\alpha\beta} \to A_{12}^{\alpha\beta} \exp((K_3(\lambda_2^\beta) - K_3(\lambda_1^\alpha))x_3 - K_2(\lambda_1^\alpha)x_2) \tag{57}$$

$$\psi^\alpha \to A'^\alpha \exp\left(\frac{a_3\xi}{\lambda_1^\alpha - \lambda_3} - \frac{a_2\eta}{\lambda_1^\alpha - \lambda_2} - \frac{a_3\xi}{\lambda_2 - \lambda_3}\right) \tag{58}$$

$$\psi^\alpha \to A''^\alpha \exp\left(\frac{a_3\xi}{\lambda_1^\alpha} - a_2\eta\lambda_1^\alpha\right) \tag{59}$$

for the systems (44), (48), (50), (55), respectively, as $\sqrt{\xi^2+\eta^2} \to \infty$ and where

$$A_{12}^{\alpha\beta} = \frac{\sqrt{-a_2^\alpha a_2^\beta}}{\lambda_2^\beta - \lambda_1^\alpha} \qquad A'^\alpha = \frac{\sqrt{-a_2^\alpha a_2}}{\lambda_2 - \lambda_1^\alpha} \qquad A''^\alpha = \sqrt{\frac{-a_2^\alpha a_2}{\mathrm{Im}(K_1(0))}}.$$

Taking into account that $ia_i^\alpha, \lambda_i^\alpha \in \mathbb{R}$, the system of conditions (20)

$$\begin{aligned} &\mathrm{Re}(K_2(\lambda) - K_2(\mu)) = 0 \\ &\mathrm{Re}(K_3(\lambda) - K_3(\mu)) = 0 \qquad \lambda \neq \mu \end{aligned} \tag{60}$$

has a solution $\lambda, \mu \in \mathbb{R}$ (which is unique in a generic case). Thus the Riemann problem with a shift is set on the real axis. The shift function is defined by the condition (27)

$$K_1(\lambda) - K_1(\mu) = 0 \qquad \lambda \neq \mu \tag{61}$$

$$K_1(\lambda) = \sum_{\alpha=1}^{n_1} \frac{a_1^\alpha}{\lambda - \lambda_1^\alpha} \tag{62}$$

and it can be rather complicated. This problem gives the solutions of the Nambu equations with the asymptotic behaviour (58).

In the presence of the extra symmetry (51) the equations (60) also have a solution

$$\lambda = -\bar{\mu}. \tag{63}$$

The substitution of this solution into the relation (62) gives the equation of an algebraic curve in the complex plane

$$K_1(\lambda) \pm K_1(\bar{\lambda}) = 0 \qquad \bar{K}_1(\bar{\lambda}) = -K_1(\lambda). \tag{64}$$

So in this case the Riemann problem with a shift is set on the curve consisting of the real axis and the algebraic curve (64), the shift functions are given by (62) and (63), respectively. This problem gives the solutions of the Gross–Neveu equations with the asymptotic behaviour (59).

7. Inverse problems for the differential operator of arbitrary order on the line

The developed method allows us to find a productive approach to a classic problem of analysis—the inverse problem for the differential operator of arbitrary order (see [6, 8]). We consider the spectral problem

$$L\psi = \zeta\psi$$

$$L = \partial^n + a_{n-1}\partial^{n-1} + \sum_{i=1}^{n-2} u_i(x)\partial^i$$

$\infty < x < \infty$, satisfying the condition

$$u_i(x) \to a_i \qquad x \to \pm\infty$$

where a_i are given complex constants. In other words, $L \to L_0,\ x \to \pm\infty$

$$L^0 = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i.$$

The problem is to restore potentials u_i through some properly defined 'scattering data'.

One can construct potentials with the corresponding wave functions, using non-local problems in the complex plane. Though we work in the frame of the dressing method and do not treat the direct scattering problem, we obtain information (which may be not complete) about the structure of the continuous spectrum. In fact we define *the inverse scattering transform* from the kernel of the Riemann problem with a shift to the small decreasing potentials of the corresponding operators, the wave functions are also given by this procedure.

Let us consider the non-local $\bar\partial$-problem (1) with $K_1 = \lambda$, $K_2 = \lambda^n + \sum_{i=1}^{n-1} a_i \lambda^i$, $a_i \in \mathbb{C}$, normalized by 1. This choice leads us to the relation

$$\left(D_2 - \sum_{i=1}^{n} a_i D_1^i\right)\psi(\lambda, x, y) = \sum_{i=1}^{n-2} u_i(x, y) D_1^i \psi(\lambda, x, y). \tag{65}$$

As usual, we can go from the 'prolonged' derivatives D_i to partial derivatives by the transform

$$\psi \to \psi \exp(K_i x_i).$$

The potentials u_i in the operator (65) can easily be expressed through the coefficients of expansion of the function $\psi(\lambda, x, y)$ as $\lambda \to \infty$. For the case $a_i \in \mathbb{R}$ the potentials are real if the kernel of the problem (1) satisfies the condition

$$R(\lambda, \mu) = \bar R(\bar\lambda, \bar\mu).$$

Now let us go to the one-dimensional case. To cancel the dependence on y, we should use the $\bar\partial$-problem (1) with the kernel $R(\lambda, \mu)$ localized on the manifold (27)

$$K_2(\lambda) - K_2(\mu) = 0$$

or

$$\lambda^n + \sum_{i=1}^{n-1} a_i \lambda^i - \mu^n - \sum_{i=1}^{n-1} a_i \mu^i = 0. \tag{66}$$

In this case we can solve the inverse scattering problem for the operator on the line

$$\left(\partial^n + \sum_{i=1}^{n-1} a_i \partial^i + \sum_{i=1}^{n-2} u_i(x) \partial^i\right)\psi(\lambda, x) = \zeta \psi(\lambda, x) \tag{67}$$

where

$$\zeta = \left(\lambda^n + \sum_{i=1}^{n-1} a_i \lambda^i\right).$$

Condition (66) defines a $\bar\partial$-problem with a shift

$$\bar\partial\psi(\lambda) = \sum_{i=1}^{n} R_i(\lambda)\psi(\mu_i(\lambda)). \tag{68}$$

The problem (68) gives the potentials with the wave functions locally near the point $x = 0$. To construct the decreasing potentials defined on all the line we should observe the condition (20)

$$\mathrm{Re}(\lambda - \mu) = 0. \tag{69}$$

This condition together with (66) defines a Riemann problem with a shift (10). The equation of the curve $\lambda(\xi)$, $\xi \in \mathbb{R}$ for this problem is given by the substitution of the expression $\mu = \lambda - \mathrm{i}\xi$ (compare (69)) into equation (66), the shift function is given by equation (66) (it is also useful to note that λ and μ have identical real parts).

Thus, small decreasing potentials for the operator (67), together with the corresponding wave functions, are given by the Riemann problem with a shift

$$\Delta(\psi(\lambda(\xi))) = R_\gamma(\lambda, \mu(\lambda))\psi\big(\mu(\lambda(\xi))\big)$$

which reduces to the integral equations (13); this problem defines the transform from the kernel R to the potentials $u_i(x)$ (the inverse scattering transform for the continuous spectrum).

Let us consider a simple example, $K_2 = \lambda^n$. In this case the operator (67) takes the form

$$\partial_1^n\psi(\lambda, x) = \sum_{i=1}^{n-2} u_i(x)\partial_1^i\psi(\lambda, x). \tag{70}$$

This class of operators was investigated in detail in [8]. We will show now how our technique works in this case. The shift function (66) for this case is

$$\lambda^n - \mu^n = 0 \tag{71}$$

and the problem (68) reads

$$\bar\partial\psi(\lambda) = \sum_{i=1}^{n} R_i(\lambda)\psi(e_i\lambda) \qquad e_i^n = 1. \tag{72}$$

If we take into account the condition (69), we obtain

$$\lambda - e_i\mu = 0$$

$$\lambda - \mu = \mathrm{i}\xi \qquad \xi \in \mathbb{R}.$$

The solution of this system is

$$\lambda = \mathrm{i}\xi(1 - e_i)^{-1} := \mathrm{i}\xi\alpha_i$$

$$\mu = -\mathrm{i}\xi(1 - e_i^{-1})^{-1} := -\mathrm{i}\xi\bar\alpha_i \qquad e_i \neq 1.$$

In this case ψ has a discontinuity on $(n - 1)$ lines with an angle π/n between them. Thus we have arrived at the Riemann problem with a shift for the function analytic in sectors, the shift function is $\mu = \bar\lambda$. The integral equations (13) for this case take the form

$$\rho_k(\xi) = \left(1 + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{n-1} \mathrm{v.p.}\int_{-\infty}^{\infty} \frac{\rho_j(\xi')}{\bar\alpha_k\xi - \alpha_j\xi'}\alpha_j\,\mathrm{d}\xi'\right) R^k(\xi)\exp((\alpha_k - \bar\alpha_k)\xi)$$

where ρ_k is the jump of the function ψ across the corresponding line. In the general case the Riemann problem with a shift may be defined on quite a general analytic curve in the complex plane. The symmetries of the function $K_2(\lambda)$ can simplify the investigation.

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