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INTEGRATION OF THE GAUSS–CODAZZI EQUATIONS

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The Gauss–Codazzi equations imposed on the elements of the first and the second quadratic forms of a surface embedded in $\mathbb{R}^3$ are integrable by the dressing method. This method allows constructing classes of Combescure-equivalent surfaces with the same "rotation coefficients." Each equivalence class is defined by a function of two variables ("master function of a surface"). Each class of Combescure-equivalent surfaces includes the sphere. Different classes of surfaces define different systems of orthogonal coordinates of the sphere. The simplest class (with the master function zero) corresponds to the standard spherical coordinates.

1. Introduction

Bonnet's theorem asserts that a surface in three-dimensional Euclidean space is defined up to Euclidean motions if the components of the first and second quadratic forms are known. However, these components cannot be chosen arbitrarily. Indeed, the first quadratic form defines a metric on the surface, and the second quadratic form determines the field of normal vectors at any point on the surface. Given the components of both forms, we can construct the three-dimensional metric in a vanishingly thin layer near the surface. Insofar as the surface is embedded in an Euclidean space, the metric is flat, and the components of the curvature tensor in a neighborhood of the surface must be identically zero. This requirement imposes a set of differential relations (known as the Gauss–Codazzi equations (GCE)) on the components of the quadratic forms.

We show that the inverse scattering method can be used to integrate the GCE. Exactly speaking, we show that the components of both quadratic forms can be expressed through only one function of two variables, which we call the *master function of the surface*. Actually, the master function defines not a single surface but a whole class of *Combescure-equivalent surfaces*. This function is the kernel of a certain linear integral equation. To express the components of quadratic forms in terms of the master function, we must solve this equation. Given its solution, we can construct all Combescure-equivalent surfaces explicitly. The integral equation can be effectively solved only in some special cases where the kernel is generated by and can be represented as a superposition of binary products of functions of one variable. This "solitonic" case certainly deserves very serious attention because it allows the efficient study of some new classes of surfaces.

Moreover, we can hope that the results of this article could be more important. We actually reduce the problem of classifying surfaces to the problem of classifying their master functions and then subclassifying a given Combescure class. All previously known classes of surfaces have master functions satisfying some special conditions. To find these conditions, and also new conditions defining new special classes of surfaces, is an interesting program for future research. This article shows how deeply the theory of surfaces is connected with the theory of solitons.

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2. Formulation of the problem

Let Γ be a surface in $\mathbb{R}^3$. We can introduce coordinates x_1 and x_2 on Γ such that the first and second quadratic forms are diagonal,

$$\Omega_1 = p^2\,dx_1^2 + q^2\,dx_2^2, \qquad \Omega_2 = pA\,dx_1^2 + qB\,dx_2^2,$$

where the coordinates x_1 and x_2 are defined up to trivial transformations $x_1 = x_1(u_1)$ and $x_2 = x_2(u_2)$. We say that these two surfaces are *Combescure equivalent* if they have the same A and B for different p and q.

The coefficients of Ω_1 and Ω_2 cannot be chosen independently. The four functions p, q, A, and B are connected by three nonlinear PDEs known as the GCE. To find the GCE, we should embed the surface Γ in a special system of three-dimensional orthogonal curvilinear coordinates in $\mathbb{R}^3$ in the vicinity of Γ. The metric in this system is defined as

$$ds^2 = H_1^2\,dx_1^2 + H_2^2\,dx_2^2 + dx_3^2, \tag{2.1}$$

where H_1 and H_2 are the Lamé coefficients $H_1 = p + Ax_3$ and $H_2 = q + Bx_3$ and the third Lamé coefficient is $H_3 = 1$.

The GCE appear from the condition

$$R_{ijlm} = 0, \tag{2.2}$$

where R_{ijlm} is the Riemann curvature tensor for metric (2.1). Equation (2.2) takes a simple form in terms of the matrix given by $Q_{ij} = (1/H_j)(\partial H_i/\partial x_j)$, $i \neq j$. In the literature,

$$\beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial x_i} = Q_{ij}$$

are usually called *rotation coefficients*. By definition, we have

$$Q_{13} = \frac{1}{H_3}\frac{\partial H_1}{\partial x_3} = A, \qquad Q_{23} = \frac{1}{H_3}\frac{\partial H_2}{\partial x_3} = B,$$
$$Q_{31} = \frac{1}{H_1}\frac{\partial H_3}{\partial x_1} = 0, \qquad Q_{32} = \frac{1}{H_2}\frac{\partial H_3}{\partial x_2} = 0,$$

and for the remaining elements of Q, Eq. (2.2) becomes

$$\frac{\partial Q_{12}}{\partial x_3} = \frac{\partial Q_{21}}{\partial x_3} = 0,$$

$$\frac{\partial Q_{13}}{\partial x_2} = Q_{12}Q_{23}, \qquad \frac{\partial Q_{23}}{\partial x_2} = Q_{21}Q_{13}, \tag{2.3}$$

$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}Q_{23} = 0. \tag{2.4}$$

Obviously, some elements of Q are independent of the variable x_3, and we can set

$$Q_{12} = \alpha(x_1, x_2), \qquad Q_{21} = \beta(x_1, x_2).$$

Then system (2.3), (2.4) can be written as

$$\frac{\partial\alpha}{\partial x_2} + \frac{\partial\beta}{\partial x_1} + AB = 0, \qquad \frac{\partial A}{\partial x_2} = \alpha B, \qquad \frac{\partial B}{\partial x_2} = \beta A. \tag{2.5}$$

To express α and β in terms of the elements of the first quadratic form p and q, we use the definition of Q_{12} and Q_{21}. By definition,

$$\begin{aligned} \frac{\partial}{\partial x_2}(p + Ax_3) &= Q_{12}(q + Bx_3) = \alpha(q + Bx_3), \\ \frac{\partial}{\partial x_1}(q + Bx_3) &= Q_{21}(p + Ax_3) = \beta(p + Ax_3). \end{aligned} \tag{2.6}$$

Setting $x_3 = 0$ in (2.6), we obtain

$$\frac{\partial p}{\partial x_2} = \alpha q, \qquad \frac{\partial q}{\partial x_1} = \beta p, \tag{2.7}$$

and finally

$$\frac{\partial}{\partial x_2}\left(\frac{1}{q}\frac{\partial p}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{1}{p}\frac{\partial q}{\partial x_1}\right) + AB = 0, \qquad q\frac{\partial A}{\partial x_2} = B\frac{\partial p}{\partial x_2}, \qquad p\frac{\partial B}{\partial x_1} = A\frac{\partial q}{\partial x_1}. \tag{2.8}$$

System (2.8) of three equations imposed on the four functions A, B, p, and q is the Gauss–Codazzi system. It is suitable to study a simpler system of first-order equations (2.5), which form a system of three equations imposed on the four functions A, B, α, and β. Solving this system, we define a surface up to Combescure equivalence. To solve the GCE, we must solve linear system (2.7) for the components of the first quadratic form. Splitting Gauss–Codazzi system (2.8) into simpler systems (2.5) and (2.7) was done by Konopelchenko [1].

3. n-Dimensional orthogonal coordinate systems

Our approach to solving the GCE is based on the fact that they are a special degenerate case of Gauss–Lamé equations describing three-dimensional orthogonal curvilinear coordinate systems in $\mathbb{R}^3$. A method for solving Gauss–Lamé equations in a Euclidean space of arbitrary dimension was presented in [2]. Here, we give a different, but essentially equivalent, method for solving this problem.

Given a domain S in $\mathbb{R}^n$, the problem is to find all the orthogonal curvilinear coordinate systems in S. Let $x = (x_1, \dots, x_n)$ be these coordinates. In this coordinate system, the metric tensor is diagonal,

$$ds^2 = \sum H_i^2\, dx_i^2.$$

The coefficients $H_i = H_i(x)$ are the Lamé coefficients and are to be determined. They satisfy a heavily overdetermined system of nonlinear PDEs, the Gauss–Lamé equations,

$$\frac{\partial Q_{ij}}{\partial x_k} = Q_{ik}Q_{kj}, \quad i \neq j \neq k, \tag{3.1}$$

$$\frac{\partial Q_{ij}}{\partial x_j} + \frac{\partial Q_{jk}}{\partial x_i} + \sum_{k\neq i,j} Q_{ik}Q_{kj} = 0, \quad i \neq j, \tag{3.2}$$

where, as before, we have

$$Q_{ij} = \frac{1}{H_j}\frac{\partial H_i}{\partial x_j}. \tag{3.3}$$

We can verify that Eqs. (3.1)–(3.3) are equivalent to the condition $R_{ijkl} = 0$, where R_{ijkl} is the Riemann curvature tensor. To solve (3.1)–(3.3), we introduce a family of projection operators I_i in $\mathbb{R}^n$ satisfying the conditions $I_i^2 = I_i$ and $I_iI_j = 0$, $i \neq j$, and define

$$\Phi = \sum_{i=1}^{n} x_i I_i.$$

Let λ be a point on the complex plane $\mathbb{C}$ and χ with the elements $\chi_{ij}(\lambda, \bar{\lambda}, x)$, $i = 1, \ldots , n$, $j = 1, \ldots , n$, be a matrix-valued function on $\mathbb{C}$ that also depends on the coordinate x. We suppose that $\chi(\lambda, \bar{\lambda}, x)$ is a solution of the nonlocal $\bar{\partial}$-problem

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi * R = \int \chi(\nu, \bar{\nu}, x)R(\nu, \bar{\nu}, \lambda, \bar{\lambda}, x)\, d\nu\, d\bar{\nu} \tag{3.4}$$

normalized by the condition $\chi \to \delta_{ij}$ as $\lambda \to \infty$.

In (3.4),

$$R(\nu, \bar{\nu}, \lambda, \bar{\lambda}) = e^{\nu\Phi}Te^{-\lambda\Phi}, \tag{3.5}$$

where $T(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ is a matrix that is independent of x. We impose the restrictions

$$\overline{T}(\bar{\nu}, \nu, \bar{\lambda}, \lambda) = T(\nu, \bar{\nu}, \lambda, \bar{\lambda}), \qquad T^{\mathrm{tr}}(-\nu, -\bar{\nu}, -\lambda, -\bar{\lambda}) = \frac{\mu}{\lambda}T(\nu, \bar{\nu}, \lambda, \bar{\lambda}) \tag{3.6}$$

on T. Then $\bar{\partial}$-problem (3.4) is equivalent to the integral equation

$$\chi(\lambda, \bar{\lambda}, x) = \delta_{ij} + \frac{1}{\pi}\int \frac{\chi(\nu, \bar{\nu}, x)R(\nu, \bar{\nu}, \mu, \bar{\mu}, x)}{\lambda - \mu}\, d\nu\, d\bar{\nu}\, d\mu\, d\bar{\mu}. \tag{3.7}$$

We chose a matrix function $T(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ satisfying conditions (3.6) such that Eq. (3.7) has a unique regular solution. Then χ can be expanded at $\lambda \to \infty$ in the asymptotic series

$$\chi \to 1 + \frac{Q}{\lambda} + \frac{P}{\lambda^2} + \ldots , \tag{3.8}$$

where

$$Q = \frac{1}{\pi}\int \chi(\nu, \bar{\nu}, x)R(\nu, \bar{\nu}, \mu, \bar{\mu}, x)\, d\lambda\, d\bar{\lambda}\, d\mu\, d\bar{\mu}. \tag{3.9}$$

We note that the function χ satisfies the condition

$$\bar{\chi}(\bar{\nu}, \nu, x) = \chi(\nu, \bar{\nu}, x) \tag{3.10}$$

by virtue of (3.6) and that Q is a real matrix function on x. Using $\bar{\partial}$-problem (3.4) and the equivalent integral equation (3.8) to integrate the Gauss–Lamé system is based on the following two facts:

1. If conditions (3.6) are satisfied, then the matrix $Q(x)$ given by formula (3.10) satisfies the system of equations (3.1) and (3.2).
2. The matrix function $\phi(\lambda, \bar{\lambda}, x) = \chi(\lambda, \bar{\lambda}, x)e^{+\lambda\Phi(x)}$ satisfies the linear system

$$\frac{\partial \phi_{ik}}{\partial x_j} = Q_{ij}\phi_{jk}, \quad i \neq 0. \tag{3.11}$$

Let $\xi_i(\lambda, \bar{\lambda}) = \bar{\xi}_i(\bar{\lambda}, \lambda)$ be an arbitrarily chosen family of functions on $\lambda, \bar{\lambda}$, and let

$$H_i = \int \sum \chi_{ij}(\lambda, \bar{\lambda}, x)\xi_i(\lambda, \bar{\lambda})\, d\lambda\, d\bar{\lambda}. \tag{3.12}$$

The function $H_i(x)$ satisfies the linear system

$$\frac{\partial H_i}{\partial x_j} = Q_{ij}H_j \tag{3.13}$$

and can be chosen as the Lamé coefficients for some n-dimensional orthogonal coordinate system. Choosing different sets of $\xi_i(\lambda, \bar{\lambda})$, we obtain different sets of the Lamé coefficients corresponding to the same matrix Q. These sets are the so-called Combescure equivalents. The method for solving nonlinear equations via $\bar{\partial}$-problem (3.4) is called the *dressing method*, and the function $T(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ is a *dressing function*. The dressing procedure allows automatically finding all sets of Lamé coefficients associated with a given matrix function Q.

To prove the statements formulated above, we construct a family of operators $L_{ij} (i \neq j)$ acting on χ as

$$L_{ij}\chi = I_i\Big(\frac{\partial \chi}{\partial x_j} + \lambda\chi I_j - QI_j\chi\Big), \quad i \neq j.$$

Here, Q is given by (3.9), and we can easily verify that $L_{ij}\chi$ are solutions of the $\bar{\partial}$-problem

$$\frac{\partial}{\partial \bar{\lambda}} L_{ij}\chi = L_{ij}\chi * R$$

with zero normalization at infinity, $L_{ij}\chi \to 0$ as $\lambda \to \infty$. Each function $L_{ij}\chi$ satisfies the linear integral equation

$$L_{ij}\chi(\lambda, \bar{\lambda}) = \frac{1}{\pi}\int \frac{L_{ij}\chi(\nu, \bar{\nu})R(\nu, \bar{\nu}, \mu, \bar{\mu}, x)}{\lambda - \mu}\, d\nu\, d\bar{\nu}\, d\mu\, d\bar{\mu}. \tag{3.14}$$

Because Eq. (3.7) has a unique (regular) solution, homogeneous equation (3.14) has only the zero solution. Hence,

$$L_{ij}\chi = 0 \tag{3.15}$$

and

$$L_{ij}\chi I_k = 0. \tag{3.16}$$

Linear system (3.13) is equivalent to system (3.16). We substitute asymptotic approximation (3.8) in (3.15) and expand the result in an asymptotic series. All terms of the resulting asymptotic expansion should be identically zero. Setting the first nonvanishing term of the order $1/\lambda$ to zero yields system (3.1).

The proof of the validity of Eq. (3.2) is a little more difficult and is given in [3].

4. Integration of the GCE

In $\mathbb{R}^3$, system (3.1) is

$$\frac{\partial Q_{12}}{\partial x_3} = Q_{13}Q_{32}, \tag{4.1}$$

$$\frac{\partial Q_{21}}{\partial x_3} = Q_{23}Q_{31}, \tag{4.2}$$

$$\frac{\partial Q_{13}}{\partial x_2} = Q_{12}Q_{23}, \tag{4.3}$$

$$\frac{\partial Q_{23}}{\partial x_1} = Q_{21}Q_{13}, \tag{4.4}$$

$$\frac{\partial Q_{31}}{\partial x_2} = Q_{32}Q_{21}, \tag{4.5}$$

$$\frac{\partial Q_{32}}{\partial x_1} = Q_{31}Q_{12}. \tag{4.6}$$

To go to the Gauss–Codazzi system, we use the fact that the Q_{ij} are independent of x_3. Equations (4.1) and (4.2) then become

$$Q_{13}Q_{32} = 0, \qquad Q_{23}Q_{31} = 0. \tag{4.7}$$

We impose an additional condition compatible with (4.7),

$$Q_{31} = Q_{32} = 0. \tag{4.8}$$

Now, Eqs. (4.5) and (4.6) are satisfied automatically, and only Eqs. (4.3) and (4.4) survive in system (3.1).

System (3.2) for $n = 3$ consists of three equations:

$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}Q_{23} = 0,$$

$$\frac{\partial Q_{13}}{\partial x_3} + \frac{\partial Q_{31}}{\partial x_1} + Q_{12}Q_{32} = 0, \tag{4.9}$$

$$\frac{\partial Q_{23}}{\partial x_3} + \frac{\partial Q_{32}}{\partial x_2} + Q_{21}Q_{31} = 0. \tag{4.10}$$

By virtue of (4.7), Eqs. (4.9) and (4.10) are satisfied. Hence, the total system of equations resolving this special type of Gauss–Lamé system is reduced to the equations

$$\frac{\partial Q_{12}}{\partial x_2} = Q_{23}Q_{31}, \qquad \frac{\partial Q_{23}}{\partial x_1} = Q_{21}Q_{23}, \qquad \frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}Q_{23} = 0. \tag{4.11}$$

This system should be satisfied by virtue of the equations for the Lamé coefficients

$$\frac{\partial H_1}{\partial x_2} = Q_{12}H_2, \qquad \frac{\partial H_2}{\partial x_1} = Q_{21}H_1. \tag{4.12}$$

System (4.11), (4.12) coincides with Gauss–Lamé equations (2.5) and (2.7). To solve this system, we choose the dressing function $T_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ in a very special way. According to (3.5), we have

$$R_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda}) = e^{\nu x_i - \lambda x_j} T_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda}). \tag{4.13}$$

To eliminate the dependence of Q_{ij} on the coordinate x_3, we eliminate this dependence in $R_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda})$. Hence, we must set $T_{3i} \simeq \delta(\nu)\delta(\bar{\nu})$ and $T_{i3} \simeq \delta(\lambda)\delta(\bar{\lambda})$. Taking the second condition in (3.6) into account, we can construct T_{ij} uniquely,

$$\begin{aligned} &T_{12} = \mu F(\mu, \bar{\mu}, \lambda, \bar{\lambda}), && T_{21} = -\mu F(-\lambda, -\bar{\lambda}, -\mu, -\bar{\mu}), \\ &T_{13} = -\mu f_1(-\mu, -\bar{\mu})\, \delta(\lambda)\delta(\bar{\lambda}), && T_{23} = -\mu f_2(-\mu, -\bar{\mu})\, \delta(\lambda)\delta(\bar{\lambda}), \\ &T_{31} = \mu \delta(\mu)\delta(\bar{\mu})\, f_1(\lambda, \bar{\lambda}), && T_{32} = \mu \delta(\mu)\delta(\bar{\mu})\, f_2(\lambda, \bar{\lambda}). \end{aligned} \tag{4.14}$$

To satisfy the first condition in (3.6), we must set

$$f_1(\lambda, \bar{\lambda}) = \bar{f}_1(\bar{\lambda}, \lambda), \qquad f_2(\lambda, \bar{\lambda}) = \bar{f}_2(\bar{\lambda}, \lambda), \qquad R(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = \overline{R}(\bar{\mu}, \mu, \bar{\lambda}, \lambda). \tag{4.15}$$

Without loss of generality, we can set diagonal elements to zero,

$$T_{11} = T_{22} = T_{33} = 0.$$

The formulas for T_{31} and T_{32} include the product $\mu\delta(\mu)\delta(\bar{\mu})$. Normally, we must set this expression to zero. Actually, this is true for integration with a continuous test function,

$$\int f(\mu, \bar{\mu}) \mu \delta(\mu) \delta(\bar{\mu})\, d\mu\, d\bar{\mu} = 0,$$

but sometimes the test function has a simple pole at $\mu = 0$, and then $f(\mu) = g(\mu, \bar{\mu})/\mu$ at $\mu \sim 0$, where g is a continuous function. In this case,

$$\int f(\mu, \bar{\mu}) \mu \delta(\mu) \delta(\bar{\mu})\, d\mu\, d\bar{\mu} = \int g(\mu, \bar{\mu}) \delta(\mu) \delta(\bar{\mu})\, d\mu\, d\bar{\mu} = g(0, 0).$$

Integral equation (3.7) consists of a family of independent equations imposed separately on each row of the matrix χ (rows are indexed by the first subscript). We consider a system of equations for elements of the third row, χ_{31}, χ_{32}, and χ_{33}. By virtue of (4.13) and (4.14), χ_{33} drops out of the equations for χ_{31} and χ_{32}. As a result, these elements of χ_{ij} satisfy a homogeneous linear system with only the zero solution. Therefore,

$$\chi_{31} \equiv 0, \qquad \chi_{32} \equiv 0, \tag{4.16}$$

and $\chi_{33} \equiv 1$.

Hence, in accordance with (4.8), we have

$$Q_{31} = Q_{32} = 0, \qquad Q_{33} = 0. \tag{4.17}$$

To find expressions for χ_{13} and χ_{23}, we must use formulas (4.14) and (4.15). We have

$$\chi_{13} = \frac{A}{\lambda} = \frac{Q_{13}}{\lambda}, \qquad \chi_{23} = \frac{B}{\lambda} = \frac{Q_{23}}{\lambda},$$

$$A = Q_{13} = -\frac{1}{\pi}\int \nu\big[\chi_{11}(\nu,\bar\nu,x)\, f_1(-\nu,-\bar\nu)e^{\nu x_1} + \chi_{12}(\nu,\bar\nu,x)e^{\nu x_2} f_2(-\nu,-\bar\nu)\big]\, d\nu\, d\bar\nu,$$

$$B = Q_{23} = -\frac{1}{\pi}\int \nu\big[\chi_{21}(\nu,\bar\nu,x)\, f_1(-\nu,-\bar\nu)e^{\nu x_1} + \chi_{22}(\nu,\bar\nu,x)e^{\nu x_2} f_2(-\nu,-\bar\nu)\big]\, d\nu\, d\bar\nu.$$

We can use (4.16) and (4.17) to reduce system (3.4) to a system of equations imposed on the 2×2 matrix given by $X_{ij} = \chi_{ij}$, $i,j = 1,2$,

$$X_{ij}(\lambda,\bar\lambda,x) = \delta_{ij} + \frac{1}{\pi}\int \frac{X_{ik}(\nu,\bar\nu,x)e^{\nu x_k - \mu x_j} S_{kj}(\nu,\bar\nu,\mu,\bar\mu)}{\lambda-\mu}\, d\nu\, d\bar\nu\, d\mu\, d\bar\mu, \tag{4.18}$$

where the matrix S is a sum of two components,

$$S = U + V,$$

$$U = \begin{vmatrix} 0 & \mu F(\mu,\bar\mu,\lambda,\bar\lambda) \\ -\mu F(-\lambda,-\bar\lambda,-\mu,-\bar\mu) & 0 \end{vmatrix}, \tag{4.19}$$

$$V_{ij}(\mu,\bar\mu,\lambda,\bar\lambda) = -\frac{\mu}{\pi}\, f_i(-\mu,-\bar\mu) f_j(\lambda,\bar\lambda).$$

We note that $U(\mu,\bar\mu,\lambda,\bar\lambda)$ satisfies standard relation (3.7), and

$$V^{\mathrm{tr}}(-\lambda,-\bar\lambda,-\mu,-\bar\mu) = -\frac{\mu}{\lambda}\, V(\mu,\bar\mu,\lambda,\bar\lambda).$$

If the X_{ij} are known, we can find the Lamé coefficients (the elements of the first quadratic form) from

$$p(x_1,x_2) = H_1(x_1,x_2) = \int \big[\chi_{11}(\nu,\bar\nu,x)e^{\nu x_1} g_1(\nu,\bar\nu) + \chi_{12}(\nu,\bar\nu,x)e^{\nu x_2} g_2(\nu,\bar\nu)\big]\, d\nu\, d\bar\nu,$$

$$q(x_1,x_2) = H_2(x_1,x_2) = \int \big[\chi_{21}(\nu,\bar\nu,x)e^{\nu x_1} g_1(\nu,\bar\nu) + \chi_{22}(\nu,\bar\nu,x)e^{\nu x_2} g_2(\nu,\bar\nu)\big]\, d\nu\, d\bar\nu.$$

Different choices of g_1 and g_2 define different representatives of the same class of Combescure-equivalent surfaces. In particular, we can set

$$g_1(\nu,\bar\nu) = -\frac{\nu}{\pi} f_1(-\nu,-\bar\nu), \qquad g_2(\nu,\bar\nu) = -\frac{\nu}{\pi} f_2(-\nu,-\bar\nu). \tag{4.20}$$

In this case, $p = A$ and $q = B$.

We note that $k_1 = A/p$ and $k_2 = B/q$ are the main curvatures of the surface at a given point. In case (4.20), $k_1 = k_2 = 1$, and the surface is a sphere. We have an interesting fact: each class of Combescure-equivalent surfaces includes a sphere. Different classes just define different coordinate systems on the sphere.

5. Surfaces of type zero

We call the function $F(\mu,\bar{\mu},\lambda,\bar{\lambda})$ the *master function of the surface*. The surface belongs to a finite type N if its master function can be written in the form

$$F(\mu,\bar{\mu},\lambda,\bar{\lambda})=\sum_{k=1}^{N}a_k(\mu,\bar{\mu})b_k(\lambda,\bar{\lambda}).$$

In this case, a solution of the GCE can be found in a closed form. Surfaces of a finite type can also be called *solitonic surfaces*. In this paper, we describe surfaces of type zero where $F\equiv 0$. In this case,

$$S_{ij}=V_{ij}=-\frac{\mu}{\pi}f_i(-\mu,-\bar{\mu})\,f_j(\lambda,\bar{\lambda}).$$

We can set

$$\chi_{ij}=\delta_{ij}+\lambda_i(x_1,x_2)\,h_j(\lambda,\bar{\lambda},x_j), \tag{5.1}$$

where

$$h_j(\lambda,\bar{\lambda},x_j)=\frac{1}{\pi}\int\frac{f_j(-\mu,-\bar{\mu})e^{\mu x_j}}{\lambda+\mu}\,d\mu\,d\bar{\mu}.$$

We introduce the functions

$$c_i(x_i)=\frac{1}{\sqrt{2\pi}}\int f_i(-\mu,-\bar{\mu})e^{\mu x_i}\,d\mu\,d\bar{\mu}.$$

We can verify that

$$\lambda_i=-\frac{\sqrt{2\pi}c_i'(x_i')}{\Delta},\qquad \Delta=1+c_1^2(x_1)+c_2^2(x_2).$$

From (5.1), we easily obtain

$$\alpha=Q_{12}=-\frac{2c_1'(x_1)c_2(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)},\qquad \beta=Q_{21}=-\frac{2c_1(x_1)c_2'(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)}. \tag{5.2}$$

Substituting (5.1) in (4.18), we obtain

$$A=Q_{13}=-\frac{2c_1'(x_1)}{1+c_1^2(x_1)+c_2^2(x_2)},\qquad B=Q_{23}=-\frac{2c_2'(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)}. \tag{5.3}$$

We can verify that formulas (5.2) and (5.3) define the solution of the GCE.

For the elements of the matrix ϕ in (3.11), we have

$$\begin{aligned}\phi_{11}&=(1+\lambda_1h_1)e^{\lambda x_1}, & \phi_{21}&=\lambda_2h_1e^{\lambda x_1},\\ \phi_{12}&=\lambda_1h_2e^{\lambda x_2}, & \phi_{22}&=(1+\lambda_2h_2)e^{\lambda x_2}.\end{aligned}$$

To solve the GCE and find all possible sets of p and q compatible with (5.2) and (5.3), we introduce two arbitrary functions

$$\xi_i(\lambda,\bar{\lambda})=\bar{\xi}_i(\bar{\lambda},\lambda),\quad i=1,2.$$

Then p and q are given by the formulas

$$\begin{aligned} p &= \langle \xi_1 e^{\lambda x_1} \rangle + \lambda_1 \left[\langle h_1 e^{\lambda x_1} \xi_1 \rangle + \langle h_2 e^{\lambda x_2} \xi_2 \rangle \right], \\ q &= \langle \xi_2 e^{\lambda x_2} \rangle + \lambda_2 \left[\langle h_2 e^{\lambda x_1} \xi_1 \rangle + \langle h_2 e^{\lambda x_2} \xi_2 \rangle \right]. \end{aligned} \tag{5.4}$$

In (5.4), the brackets mean that we integrate with respect to λ and $\bar{\lambda}$.

Formulas (5.4) can be rewritten as

$$p = a(x_1) + \lambda_1(x_1, x_2)\, F, \qquad q = b(x_2) + \lambda(x_1, x_2)\, F,$$

$$F = \int_{\xi_1}^{x_1} c_1(x_1) a(x_1)\, dx_1 + \int_{\xi_2}^{x_2} c_2(x_2) b(x_2)\, dx_2,$$

where $a(x_1) = \langle \xi_1 e^{\lambda x_1} \rangle$ and $b(x_2) = \langle \xi_2 e^{\lambda x_3} \rangle$ are arbitrary functions of one variable.

6. Orthogonal coordinates on the sphere

Integrating the GCE is closely connected to another classical problem in differential geometry, classifying orthogonal coordinate systems on the sphere. This problem can be formulated as follows: find all the coordinate systems on the unit sphere in $\mathbb{R}^3$ such that the metric tensor is diagonal.

Let x_1 and x_2 be some coordinates on the sphere, and let the metric be given by the formula

$$ds^2 = p^2(x_1, x_2)\, dx_1^2 + q^2(x_1, x_2)\, dx_2^2.$$

The curvature tensor on the sphere has the form

$$R_{ij,lm} = g_{il} g_{jm} - g_{im} g_{jl}. \tag{6.1}$$

We can introduce α and β as

$$\alpha = \frac{1}{q} \frac{\partial p}{\partial x_2}, \qquad \beta = \frac{1}{p} \frac{\partial q}{\partial x_2}. \tag{6.2}$$

Equation (6.1) becomes

$$\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_1} + pq = 0. \tag{6.3}$$

Equations (6.2) and (6.3) are equivalent to "reduced" GCE (2.5). This equivalence leads to the following conclusions:

1. Each surface (at least locally) is Combescure equivalent to the sphere.
2. Each solution of "reduced" GCE (2.5) generates an orthogonal coordinate system on the sphere.
3. Each orthogonal coordinate system on the sphere generates a class of Combescure-equivalent surfaces given by the solution of "complete" GCE (2.5)–(2.7).

It is interesting to consider what kind of coordinate system corresponds to surfaces of type zero. Performing the change of variables

$$y_1 = c_1(x), \qquad y_2 = c_2(x)$$

in formulas (5.2) and (5.3), we obtain

$$A = B = -\frac{2}{1 + y_1^2 + y_2^2}, \qquad \alpha = -\frac{2y_2}{1 + y_1^2 + y_2^2}, \qquad \beta = -\frac{2y_1}{1 + y_1^2 + y_2^2}. \tag{6.4}$$

Assuming that $p = A$ and $q = B$, we can see that (6.4) corresponds to the coordinate system given by the stereographic projection of the sphere in the standard spherical coordinates.

The general expressions for p and q are

$$p = a(y_1) - \frac{2}{1 + y_1^2 + y_2^2} \left[\int_{\xi_1}^{y_1} y_1 a(y_1)\, dy_1 + \int_{\xi_2}^{y_2} y_2 b(y_2)\, dy_2 \right],$$

$$q = b(y_2) - \frac{2}{1 + y_1^2 + y_2^2} \left[\int_{\xi_1}^{y_1} y_1 a(y_1)\, dy_1 + \int_{\xi_2}^{y_2} y_2 b(y_2)\, dy_2 \right].$$

We can see that the classification of surfaces given by the classification of solutions of the GCE is quite different from all traditional classifications of the surfaces. Of course, the problem of classifying orthogonal coordinates on the sphere can be solved by elementary methods. We can construct such coordinates on the plane and perform the stereographic projection. But this way of constructing of orthogonal coordinates does not lead to a further construction of the classes of Combescure-equivalent surfaces.

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