

Decay of the Monochromatic Capillary Wave[†]

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It was demonstrated by direct numerical simulation that, in the case of weakly nonlinear capillary waves, one can get resonant waves interaction on the discrete grid when resonant conditions are never fulfilled exactly. The waves's decay pattern was obtained. The influence of the mismatch of resonant condition was studied as well.
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Nonlinear waves on the surface of a fluid are one of the most well known and complex phenomena in nature. Mature ocean waves and ripples on the surface of the tea in a pot, for example, can be described by very similar equations. Both these phenomena are substantially nonlinear, but the wave amplitude is usually significantly less than the wavelength. Under this condition, waves are weakly nonlinear.

To describe processes of this kind, weak turbulence theory was proposed [1, 2]. It results in Kolmogorov spectra as an exact solution of the Hasselman–Zakharov kinetic equation [3]. Many experimental results are in great accordance with this theory. In the case of gravity surface waves, the first confirmation was obtained by Toba [4], and the most recent data by Hwang [5] were obtained as a result of lidar scanning of the ocean surface. Recent experiments with capillary waves on the surface of liquid hydrogen [6, 7] are also in good agreement with this theory. On the other hand, some numerical calculations have been made to check the validity of the weak turbulent theory [8–10].

In this letter, we study one of the keystones of weak turbulent theory, the resonant interaction of weakly nonlinear waves. The question under study is the following:

How does a discrete grid for wavenumbers in numerical simulations affect the resonant interaction?

Can a nonlinear frequency shift broad resonant manifold to make discreteness unimportant?

We study this problem for nonlinear capillary waves on the surface of an infinitely depth incompressible ideal fluid. Direct numerical simulation can make the situation clear.

Let us consider a nonrotating flow of an ideal incompressible fluid of infinite depth. For the sake of

simplicity, let us suppose fluid density $\rho = 1$. The velocity potential ϕ satisfies the Laplace equation

$$\Delta\phi = 0 \quad (1)$$

in the fluid region bounded by

$$-\infty < z < \eta(\mathbf{r}), \quad \mathbf{r} = (x, y), \quad (2)$$

with the boundary conditions for the velocity potential

$$\left. \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x} \frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y} \frac{\partial\eta}{\partial y} = \frac{\partial\phi}{\partial z} \right|_{z=\eta}, \quad (3)$$

$$\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 \right) \Big|_{z=\eta} + \sigma(\sqrt{1 + (\nabla\eta)^2} - 1) = 0$$

on $z = \eta$ and

$$\phi_z|_{z=-\infty} = 0 \quad (4)$$

on $z \rightarrow -\infty$. Here $\eta = \eta(x, y, t)$ is the surface displacement. In the case of capillary waves, the Hamiltonian has the form

$$H = T + U, \quad (5)$$

$$T = \frac{1}{2} \int d^2r \int_{-\infty}^{\eta} (\nabla\phi)^2 dz,$$

$$U = \sigma \int (\sqrt{1 + (\nabla\eta)^2} - 1) d^2r, \quad (6)$$

where σ is the surface tension coefficient. In [11], it was shown that this system is Hamiltonian. The Hamiltonian variables are the displacement of the surface $\eta(x, y, t)$ and velocity potential on the surface of the fluid $\psi(x, y, t) = \phi(x, y, \eta(x, y, t); t)$. The Hamiltonian equations are

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \quad (7)$$

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Using the weak nonlinearity assumption [3], one can expand the Hamiltonian in the powers of surface displacement:

$$H = \frac{1}{2} \int (\sigma |\nabla \eta|^2 + \psi \hat{k} \psi) d^2 r + \frac{1}{2} \int \eta [|\nabla \psi|^2 - (\hat{k} \psi)^2] d^2 r. \quad (8)$$

The third order is enough for three-wave interactions. Here, \hat{k} is the linear operator corresponding to multiplication of Fourier harmonics by the modulus of the wavenumber \mathbf{k} . Using (7), one can get the following system of dynamical equations:

$$\begin{aligned} \dot{\eta} &= \hat{k} \psi - \operatorname{div}(\eta \nabla \psi) - \hat{k}[\eta \hat{k} \psi], \\ \dot{\psi} &= \sigma \Delta \eta - \frac{1}{2} [(\nabla \psi)^2 - (\hat{k} \psi)^2]. \end{aligned} \quad (9)$$

The properties of the \hat{k} operator suggest exploiting the equations in Fourier space for Fourier components of η and ψ ,

$$\psi_{\mathbf{k}} = \frac{1}{2\pi} \int \psi_{\mathbf{r}} e^{i\mathbf{k}\mathbf{r}} d^2 r, \quad \eta_{\mathbf{k}} = \frac{1}{2\pi} \int \eta_{\mathbf{r}} e^{i\mathbf{k}\mathbf{r}} d^2 r.$$

Let us introduce the canonical variables $a_{\mathbf{k}}$ as shown below:

$$a_{\mathbf{k}} = \sqrt{\frac{\omega_k}{2k}} \eta_{\mathbf{k}} + i \sqrt{\frac{k}{2\omega_k}} \psi_{\mathbf{k}}, \quad (10)$$

where

$$\omega_k = \sqrt{\sigma k^3}. \quad (11)$$

With these variables, the Hamiltonian (8) acquires the form

$$\begin{aligned} H &= \int \omega_k |a_{\mathbf{k}}|^2 d\mathbf{k} \\ &+ \frac{1}{6} \frac{1}{2\pi} \int E_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} (a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_0} + a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_0}^*) \\ &\times \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_0) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_0 \\ &+ \frac{1}{2} \frac{1}{2\pi} \int M_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} (a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_0}^* + a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_0}) \\ &\times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_0) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_0. \end{aligned} \quad (12)$$

Here,

$$\begin{aligned} E_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} &= V_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} + V_{\mathbf{k}_0 \mathbf{k}_2}^{\mathbf{k}_1} + V_{\mathbf{k}_0 \mathbf{k}_1}^{\mathbf{k}_2}, \\ M_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} &= V_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} - V_{-\mathbf{k}_0 \mathbf{k}_2}^{\mathbf{k}_1} - V_{-\mathbf{k}_0 \mathbf{k}_1}^{\mathbf{k}_2}, \\ V_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} &= \sqrt{\frac{\omega_{\mathbf{k}_1} \omega_{\mathbf{k}_2} k_0}{8k_1 k_2 \omega_{\mathbf{k}_0}}} L_{\mathbf{k}_1 \mathbf{k}_2}, \end{aligned} \quad (13)$$

$$L_{\mathbf{k}_1 \mathbf{k}_2} = (\mathbf{k}_1 \mathbf{k}_2) + |k_1| |k_2|.$$

The dynamic equations in these variables can be easily obtained by variation of Hamiltonian:

$$\begin{aligned} \dot{a}_{\mathbf{k}} &= -i \frac{\delta H}{\delta a_{\mathbf{k}}^*} = -i \omega_k a_{\mathbf{k}} \\ &- \frac{i}{2} \frac{1}{2\pi} \int M_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2 \\ &- \frac{i}{2\pi} \int M_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_0} a_{\mathbf{k}_2}^* a_{\mathbf{k}_0} \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_0) d\mathbf{k}_2 d\mathbf{k}_0 \\ &- \frac{i}{2} \frac{1}{2\pi} \int E_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned} \quad (14)$$

Each term in this equation has its own clear physical meaning. The linear term gives a periodic evolution of the initial wave. The first nonlinear term describes a merging of two waves \mathbf{k}_1 and \mathbf{k}_2 in \mathbf{k} . The second describes decay of the wave \mathbf{k}_0 to the waves \mathbf{k} and \mathbf{k}_2 . And the last term corresponds to the second harmonic generation process. It is useful to eliminate the linear term with the substitution

$$a_{\mathbf{k}} = A_{\mathbf{k}} e^{i\omega_k t}. \quad (15)$$

In these variables, the dynamical equations take the form

$$\begin{aligned} \dot{A}_{\mathbf{k}} &= -\frac{i}{2} \frac{1}{2\pi} \int M_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} A_{\mathbf{k}_1} A_{\mathbf{k}_2} e^{i\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} t} \\ &\times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_0) d\mathbf{k}_1 d\mathbf{k}_2 \\ &- \frac{i}{2\pi} \int M_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_2}^* A_{\mathbf{k}_0} e^{-i\Omega_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_0} t} \\ &\times \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_0) d\mathbf{k}_2 d\mathbf{k}_0, \end{aligned} \quad (16)$$

where

$$\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_0}. \quad (17)$$

Here we do not consider the harmonic generation term. The remaining terms give us the following conditions of resonance:

$$\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}} = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k} = 0. \quad (18)$$

All this theory is well known in the literature [3].

Now let us turn to the discrete grid. Also, from this point we assume periodic boundary conditions in x and

y with lengths L_x and L_y . One can easily obtain equations similar to (16):

$$\begin{aligned} \dot{A}_{\mathbf{k}} = & -\frac{i}{2} \frac{2\pi}{L_x L_y} \sum_{\mathbf{k}_1, \mathbf{k}_2} M_{\mathbf{k}\mathbf{k}_2}^{\mathbf{k}} A_{\mathbf{k}_1} A_{\mathbf{k}_2} e^{i\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}} t} \Delta_{(\mathbf{k}_1 + \mathbf{k}_2), -\mathbf{k}} \\ & - \frac{i2\pi}{L_x L_y} \sum_{\mathbf{k}_2, \mathbf{k}_0} M_{\mathbf{k}\mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_2}^* A_{\mathbf{k}_0} e^{-i\Omega_{\mathbf{k}\mathbf{k}_2}^{\mathbf{k}_0} t} \Delta_{(\mathbf{k} + \mathbf{k}_2), -\mathbf{k}_0}, \end{aligned} \quad (19)$$

where $\Delta_{\mathbf{k}_1, \mathbf{k}_2}$ is the Kronecker delta—the discrete analogue of the Dirac delta function.

Consider the decay of a monochromatic capillary wave $A_{\mathbf{k}_0}$ on two waves

$$\begin{aligned} \dot{A}_{\mathbf{k}_0} = & -\frac{i}{2} \frac{2\pi}{L_x L_y} M_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_1} A_{\mathbf{k}_2} e^{i\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} t}, \\ \dot{A}_{\mathbf{k}_1} = & -i \frac{2\pi}{L_x L_y} M_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_2}^* A_{\mathbf{k}_0} e^{-i\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} t}, \\ \dot{A}_{\mathbf{k}_2} = & -i \frac{2\pi}{L_x L_y} M_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_1}^* A_{\mathbf{k}_0} e^{-i\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} t}. \end{aligned} \quad (20)$$

Let $A_{\mathbf{k}_1}, A_{\mathbf{k}_2}$ be small ($|A_{\mathbf{k}_0}| \gg \max(|A_{\mathbf{k}_1}|, |A_{\mathbf{k}_2}|)$) at $t = 0$). In this case, the equations can be linearized. The solution of linearized (20) has the form ($A_{\mathbf{k}_0} \sim \text{const}$)

$$A_{\mathbf{k}_{1,2}}(t) = A_{\mathbf{k}_{1,2}}(0) e^{\lambda t}, \quad (21)$$

where

$$\lambda = -\frac{i}{2} \Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} + \sqrt{\left| \frac{2\pi}{L_x L_y} M_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_0} \right|^2 - \left(\frac{1}{2} \Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} \right)^2}. \quad (22)$$

In the case of a continuous media, resonant conditions (18) can be satisfied exactly. But on the grid, there is always a frequency mismatch $\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_0} \neq 0$ although if the amplitude of the initial wave is high enough there are resonances even on a discrete grid. But the width of this resonance is very important.

System of equations (9) can be solved numerically. This system is nonlocal in coordinate space due to the presence of the \hat{k} operator. The origin of this operator gives us a hint to solve (9) in wavenumber space (K space). In this case, we can effectively use the fast Fourier transform algorithm. Omitting the details of this numerical scheme, we reproduce only the final results of calculations.

We have solved system of equations (9) numerically in the dimensionless periodic domain $2\pi \times 2\pi$ (the wavenumbers k_x and k_y are integer numbers in this case). Correspondingly, all other variables also become dimensionless. It is convenient to use the surface tension $\sigma = 1$. The size of the grid was chosen as

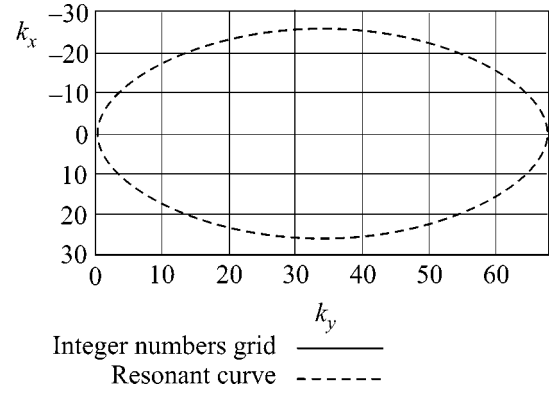


Fig. 1. The resonant manifold for $k_0 = 68$.

512×512 points. We have also included damping for waves with large wavenumbers. In K space, the damping terms for $\eta_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$, respectively, were the following: $\gamma_{\mathbf{k}} \eta_{\mathbf{k}}$ and $\gamma_{\mathbf{k}} \psi_{\mathbf{k}}$, where $\gamma_{\mathbf{k}}$ was of the form

$$\gamma_{\mathbf{k}} = 0, \quad |\mathbf{k}| < \frac{1}{2} |\mathbf{k}_{\max}|, \quad (23)$$

$$\gamma_{\mathbf{k}} = -\gamma_0 \left(|\mathbf{k}| - \frac{1}{2} |\mathbf{k}_{\max}| \right)^2, \quad |\mathbf{k}| \geq \frac{1}{2} |\mathbf{k}_{\max}|,$$

where γ_0 is some constant.

As an initial condition we used one monochromatic wave of sufficiently large amplitude with wavenumbers \mathbf{k}_0 ($k_{0x} = 0, k_{0y} = 68$). Along with that, there was a small random noise in all other harmonics.

Resonant manifold (18) for decaying waves

$$\mathbf{k}_0 = \begin{pmatrix} 0 \\ k_0 \end{pmatrix}, \quad (24)$$

$$\mathbf{k}_1 = \begin{pmatrix} -k_x \\ k_0 - k_y \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} k_x \\ k_0 + k_y \end{pmatrix}$$

is given in Fig. 1. Since the wavenumbers are integers, the resonant curve never coincides with grid points exactly. A detailed picture is given in Fig. 2. It is clear that some points are closer to the resonant manifold than others. This difference might be important in numerals.

In the beginning, one can observe exponential growth of resonant harmonics in accordance with (21) and (22). This is shown in Figs. 3 and 4. Here one can clearly see that some harmonics are in resonance and others are not.

Then almost all harmonics in the resonant manifold become involved in the decay process (Fig. 5). Later, the harmonics that are the closest to the resonant manifold (compare with Fig. 2) reach the maximum level, while the secondary decay process develops. Waves

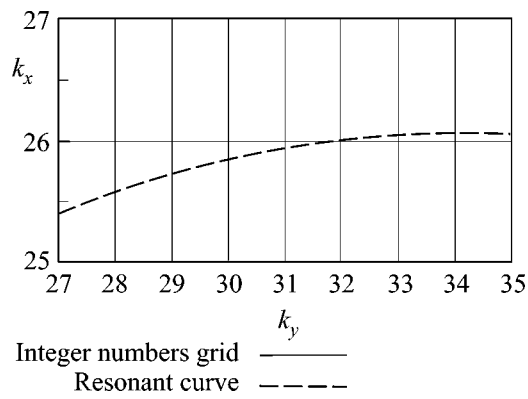


Fig. 2. Different mismatch is seen at different grid points.

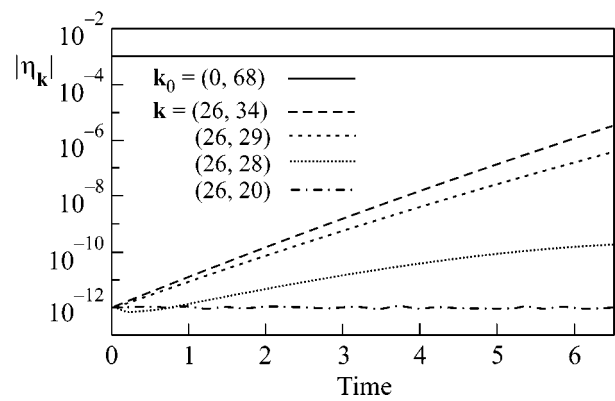


Fig. 3. Evolution of various harmonics for decaying wave $\mathbf{k}_0 = (0, 68)$.

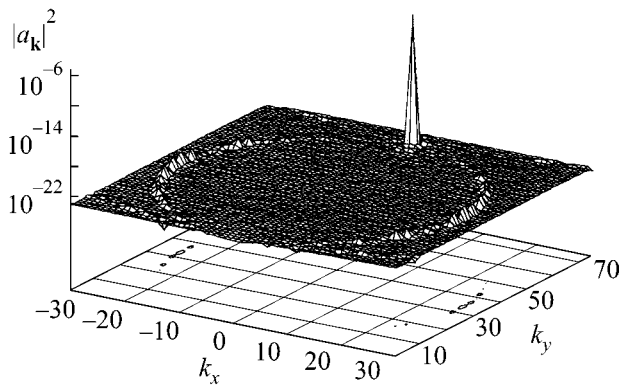


Fig. 4. Resonant harmonics starting to grow. At the basement there is a contour line for level $|a_k|^2 = 10^{-22}$. Time $t = 1.4$.

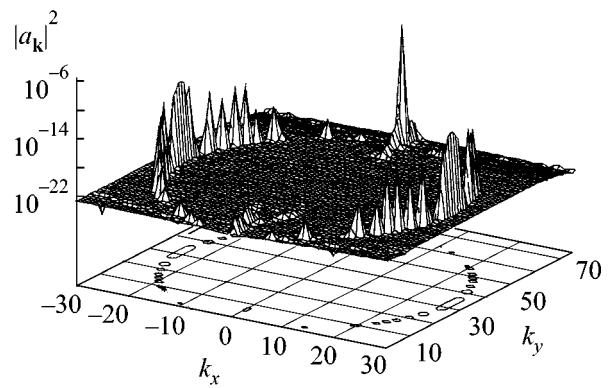


Fig. 5. Secondary decays start. At the basement there is a contour line for level $|a_k|^2 = 10^{-22}$. Time $t = 11$.

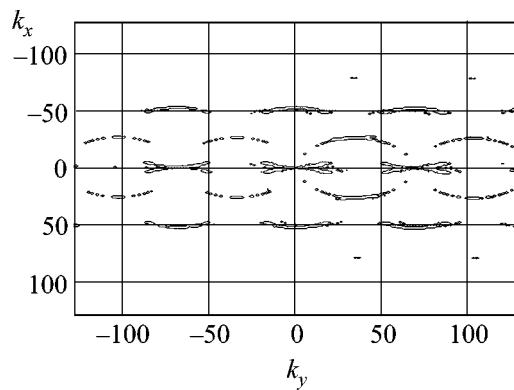


Fig. 6. The contour lines for $|a_k|^2 = 10^{-21}$. Secondary decays are clearly seen. Time $t = 14$.

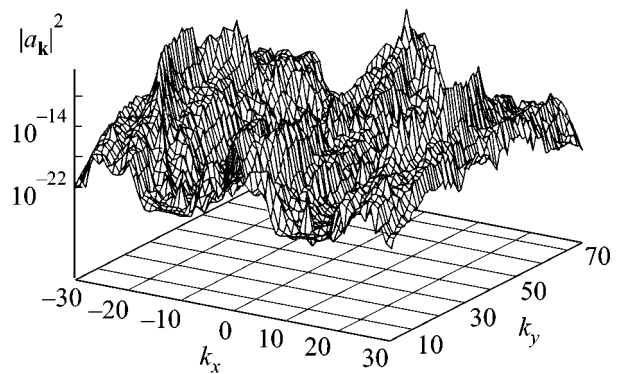


Fig. 7. Wavenumber spectrum at time $t = 57$.

amplitudes become significantly different. The largest amplitudes are for those waves with the maximum growth rate. One can see the regular structure generated by the \mathbf{k}_0 wave in Fig. 6. After a while, the whole \mathbf{k} space is filled by decaying waves, as shown in Fig. 7.

Direct numerical simulation has demonstrated that the finite width of the resonance makes the discrete grid very similar to a continuous one. Of course, this is true only if the amplitude of the wave is large enough, so that according to (22)

$$\left| \frac{2\pi}{L_x L_y} M_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_0} A_{\mathbf{k}_0} \right| > \left| \frac{1}{2} \Omega_{\mathbf{k}_1 \mathbf{k}_2}^{k_0} \right|. \quad (25)$$

As regards numerical simulation of the turbulence, namely, weak turbulence, condition (25) is very important. $A_{\mathbf{k}_0}$ has to be treated as the level of turbulence.

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REFERENCES

1. V. E. Zakharov and N. N. Filonenko, Dokl. Akad. Nauk SSSR **170**, 1292 (1966) [Sov. Phys. Dokl. **11**, 881 (1966)].
2. V. E. Zakharov and N. N. Filonenko, J. Appl. Mech. Tech. Phys. **4**, 506 (1967).
3. V. E. Zakharov, G. Falkovich, and V. S. Lvov, *Kolmogorov Spectra of Turbulence*, Vol. 1: *Wave Turbulence* (Springer, Berlin, 1992).
4. Y. Toba, J. Oceanogr. Soc. Jpn. **29**, 209 (1973).
5. P. A. Hwang, D. W. Wang, E. J. Walsh, *et al.*, J. Phys. Oceanogr. **30**, 2753 (2000).
6. M. Yu. Brazhnikov, G. V. Kolmakov, A. A. Levchenko, and L. P. Mezhev-Deglin, Pis'ma Zh. Éksp. Teor. Fiz. **74**, 660 (2001) [JETP Lett. **74**, 583 (2001)].
7. M. Yu. Brazhnikov, G. V. Kolmakov, and A. A. Levchenko, Zh. Éksp. Teor. Fiz. **122**, 521 (2002) [JETP **95**, 447 (2002)].
8. A. N. Pushkarev and V. E. Zakharov, Phys. Rev. Lett. **76**, 3320 (1996).
9. F. Dias, P. Guyenne, and V. E. Zakharov, Phys. Lett. A **291**, 139 (2001).
10. V. E. Zakharov, O. A. Vasilyev, and A. I. Dyachenko, Pis'ma Zh. Éksp. Teor. Fiz. **73**, 68 (2001) [JETP Lett. **73**, 63 (2001)].
11. V. E. Zakharov, J. Appl. Mech. Tech. Phys. **2**, 190 (1968).