JETP Letters, Vol. 81, No. 6, 2005, pp. 255–259. From Pis'ma v Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 81, No. 6, 2005, pp. 318–322.

Modulation Instability of Stokes Wave ⟶ Freak Wave¶

A. I. Dyachenko[1] and V. E. Zakharov[1, 2]

[1] *Landau Institute for Theoretical Physics, Russian Academy of Sciences, Moscow, 117940 Russia*
e-mail: alexd@landau.ac.ru
[2] *Department of Mathematics, University of Arizona, Tucson, AZ 857201, USA*
Received February 24, 2005

Formation of waves of large amplitude (freak waves, killer waves) at the surface of the ocean is studied numerically. We have observed that freak waves have the same ratio of the wave height to the wave length as limiting Stokes waves. When a freak wave reaches this limiting state, it breaks. The physical mechanism of freak wave formation is discussed. *© 2005 Pleiades Publishing, Inc.*

PACS numbers: 02.60.Cb, 47.15.Hg, 92.10.–c

1. INTRODUCTION

Waves of extremely large size, which are alternatively called freak, rogue, or giant waves, are a well-documented hazard for mariners (see, for instance [1–4]). These waves are responsible for the loss of many ships and human lives. There are no doubts that freak waves are essentially nonlinear objects. They are very steep. In the last stage of their evolution, their steepness becomes infinite, thus, forming a wall of water. Before this moment, the steepness is higher than for the limiting Stokes wave. Moreover, a typical freak wave is a single event (see [5]). Before breaking, it has a crest three to four (or even more) times higher than the crests of neighboring waves. A freak wave is preceded by a deep trough, or hole in the sea. The characteristic life time of a freak wave is short—ten wave periods or so. If the wave period is fifteen seconds, this is just a few minutes. Freak waves appears almost instantly from a relatively calm sea. Certainly, these peculiar features of freak waves cannot be explained by a linear theory. The focusing of ocean waves creates only the preconditions for formation of freak waves, which are a strongly nonlinear effect.

It is natural to associate the appearance of freak waves with the modulation instability of Stokes waves. This instability is usually named after Benjamin and Feir; however, it was first discovered by Lighthill in [6]. The theory of instability was developed independently in [7] and in [8]. Feir (see [9]) was the first to observe the instability experimentally in 1967.

A slowly modulated weakly nonlinear Stokes wave is described by the nonlinear Shrödinger equation (NLSE), which is derived in [10]. This equation is integrable (see [11]) and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves. The second term in this hierarchy was calculated by Dysthe in [12], and the next one was found a few years ago in [13]. The Dysthe equation was solved numerically by Ablovitz and his collaborates (see [14]).

Since the first work of [1], many authors have tried to explain freak wave formation in terms of NLSE and its generalizations such as the Dysthe equation. A vast amount scientific literature is devoted to this subject. The list presented below is long but incomplete: [13–23]. A survey of the different possible mechanisms of freak wave formation is given in [24, 25].

One cannot deny some advantages achieved by the use of the envelope equations. The results of many authors agree on one important point: nonlinear development of modulation instability leads to concentration of wave energy in a small spatial region. This is a clue about the possible formation of freak waves. On the other hand, it is clear that the freak wave phenomenon cannot be explained in terms of envelope equations. Indeed, NLSE and its generalizations are derived by expansion in series on powers of the parameter $\lambda \simeq (Lk)^{-1}$, where k is the wave number and L is the length of the modulation. For a real freak wave, $\lambda \sim 1$, and any slow modulation expansion fails. However, the analysis of the NLS-type equations gives some valuable information about the formation of freak waves.

Modulation instability leads to the decomposition of an initially homogeneous Stokes wave into a system of envelope quasi-solitons [26, 27]. This state can be called quasi-solitonic turbulence. In this model, solitons can merge, thus, increasing the spatial intermittency and leading to the establishment of chaotic intense modulations of energy density. So far, this model cannot explain the formation of freak waves with $\lambda \sim 1$.

Freak wave phenomenon could be explained if the envelope solutions of a certain critical amplitude are unstable and can collapse. While, in the framework of

¶ This article was submitted by the authors in English.

1D focusing NLSE solitons are stable, the improved model must have some threshold in amplitude for soliton stability. The instability of a soliton of large amplitude and its further collapse could be a proper theoretical explanation of the origin of freak waves.

This scenario was observed in a numerical experiment using the heuristic one-dimensional Maida–McLaughlin–Tabak (MMT) model (see [28]) of one-dimensional wave turbulence [27]. In the experiments described in the cited paper, the instability of a moderate amplitude monochromatic wave leads first to the creation of a chain of solitons, which merge due to inelastic interaction into one soliton of large amplitude. This soliton sucks energy from neighboring waves and becomes unstable and collapses up to $\lambda \sim 1$, thus, producing a freak wave.

In our experiments, a different scenario is observed. Namely, a freak wave appears inside of a slightly modulated wave train. A freak wave looks like the development of some defect on the periodic grid, which is a Stokes wave train.

The most direct way to prove the validity of the scenario outlined above for freak wave formation is a straight numerical solution of the Euler equation describing the potential oscillations of an ideal fluid with a free surface in a gravitational field. This solution can be found using the method published in several articles [29–31]. This method is applicable in 2 + 1 geometry; it includes conformal mapping of a fluid bounded by the surface to the lower half-plane together with an optimal choice of variables, which guarantees the well-posedness of the equations [32].

In the present article, we perform experiments for wave trains of steepness $\mu \simeq 0.15$. This experiment can be considered as a simulation of a realistic situation. If the typical steepness of a swell is $\mu \simeq 0.06$–0.07, in a caustic area, it could easily be two to three times more. In the experiments, we start with the Stokes wave train perturbed by a long wave with twenty times less amplitude. We observe the development of modulation instability and, finally, the explosive formation of a freak wave that is pretty similar to the waves observed in nature.

2. BASIC EQUATIONS

Suppose that an incompressible fluid covers a two-dimensional domain

$$-\infty < y < \eta(x, t), \tag{1}$$

where $\eta(x, t)$ is the shape of the surface. The flow is potential; hence,

$$V = \nabla\phi, \quad \Delta V = 0, \quad \nabla^2\phi = 0. \tag{2}$$

Let $\psi = \phi|_{y=\eta}$ be the potential at the surface and $H = \mathcal{T} + U$ be the total energy. The terms

$$\mathcal{T} = -\frac{1}{2}\int_{-\infty}^{\infty}\psi\phi_n \mathrm{d}x, \quad U = \frac{g}{2}\int_{-\infty}^{\infty}\eta^2(x, t)\mathrm{d}x, \tag{3}$$

are, correspondingly, the kinetic and potential parts of the energy, where g is the gravity acceleration and ϕ_n is the normal velocity at the surface. The variables ψ and η are canonically conjugated; in these variables, the Euler equation of the hydrodynamics reads

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{4}$$

One can perform a conformal transformation to map the domain that is filled with fluid

$$-\infty < x < \infty, \quad -\infty < y < \eta(x, t), \quad Z = x + iy$$

in the Z plane to the lower half-plane

$$-\infty < u < -\infty, \quad -\infty < v < 0, \quad w = u + iv$$

in the w plane. After the conformal mapping, it is convenient to introduce, along with the conformal mapping $Z(w, t)$, the complex velocity potential $\Phi(w, t)$. Next, in [33], equations (4) were transformed into a simple form, which is convenient both for the numerical simulation and analytical study. Namely, by introducing the new variables

$$R = \frac{1}{Z_w}, \quad V = i\Phi_z = i\frac{\Phi_w}{Z_w} \tag{5}$$

one can transform system (4) into the following one:

$$\begin{aligned} R_t &= i(UR_w - RU_w), \\ V_t &= i(UV_w - RB_w) + g(R-1). \end{aligned} \tag{6}$$

Now, the complex transport velocity U and B

$$U = \hat{P}(V\bar{R} + \bar{V}R), \quad B = \hat{P}(V\bar{V}). \tag{7}$$

In (7), $\hat{P}$ is the projector operator generating a function that is analytical in a lower half-plane. Here, we have omitted all the details, which can be found in [29, 33].

3. NUMERICAL APPROACH

Many numerical schemes have been developed for the solution of Euler equations describing the potential flow of a free-surface fluid in a gravity field. Most of them use integral equations that solve the boundary-value problem for a Laplace equation [34–36]. A survey of the method can be found in [37].

In this article, we study the modulation instability of Stokes waves. As the initial condition, we use a slightly modulated stationary nonlinear wave train. This train is unstable with respect to growing long-scale modulation. This remarkable fact was first established in [6], where the authors calculated the growth-rate of instability in the limit of long-wave perturbation. As far as Lighthill's growth-rate coefficient was proportional to the wave number of the perturbation length, the result was in principle incomplete: somewhere at short scales, the instability must be arrested. The complete form of the growth-rate coefficient was found independently in [7, 8, 10].

We apply the spectral code to solve equations (6). We should mention that conformal mapping is a routine approach for studying a stationary Stokes wave. The equations for the Fourier coefficients were solved numerically by many authors (see, for instance [38]). The idea to implement conformal mapping for simulation of essentially nonstationary wave dynamics emerged in the beginning of the eighties (see [39]). Since equations (6) were not derived at that time, the authors used the quasi-Lagrangian approach to fluid dynamics. After some experiments and discussion of their results, the idea to use the conformal mapping was abandoned for the following reason: conformal mapping is not good for resolution of wedge-type singularities naturally appearing on the free surface of a fluid. This reason is important if the spatial mesh is sparse. However, modern computers make it possible to use very fine meshes consisting of more than a million points or spectral modes. Thus, this argument is not pertinent any more.

Our recent experiments are sufficiently accurate: we use 10^5 to 2×10^6 harmonics. We solve equations (6) in the periodic domain $0 < x < 2\pi$, putting $g = 1$. The initial data are chosen as a combination of the exact Stokes wave (wave number $k = 10$; steepness $ka = \mu = 0.15$) and a long monochromatic wave with the wave number $k = 1$ and a moderate amplitude 5×10^{-2}. This relatively high level of perturbation is deliberately chosen to make the period of exponential instability growth that is not interesting for us shorter. At given conditions, the maximum growth-rate is

$$\gamma_{\max} \simeq \frac{\sqrt{10}}{2} \times 0.15^2 \simeq 0.035$$

and $\gamma_{\max}^{-1} \simeq 28.6$. The period of the initial wave is $T_0 = 2\pi/\sqrt{10}$. The simulation is continued until $T \simeq 458.842$, that is, until more than sixteen inverse growth-rates have been completed. We performed the computations with double precision with the number of modes doubled as far as the amplitude of the last mode reached 10^{-15}. The maximum number of modes was two millions.

We observed a short period of exponential growth of perturbation, then, some intermediate regime of intensive modulation, which ends up with explosive formation of one single freak wave. Pictures of the surface shape before breaking at the times $T = 442$ and $T = 458.56$ are presented in Fig. 1 and Fig. 2.

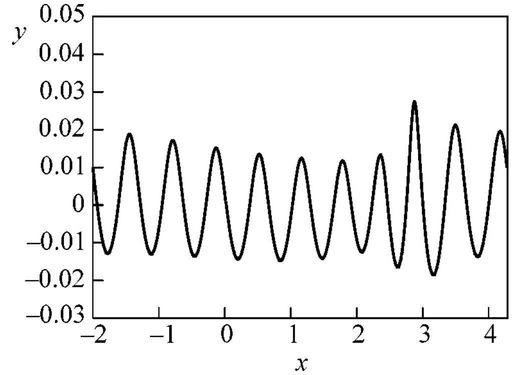

Fig. 1. The shape of the surface at $T = 422$.

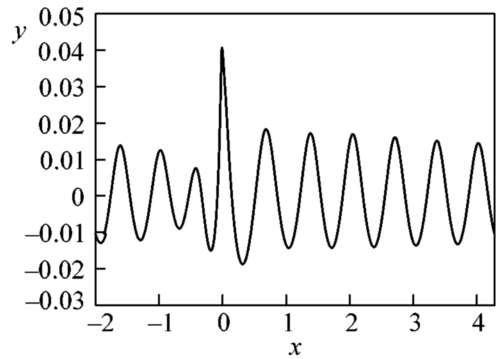

Fig. 2. The shape of the surface at $T = 458.56$.

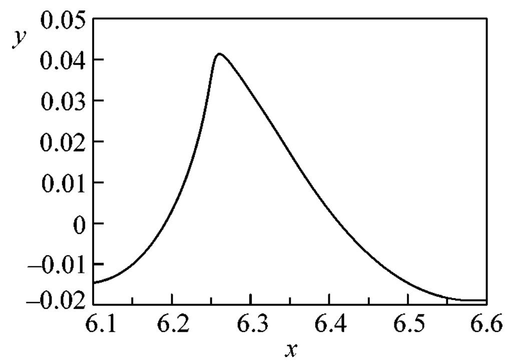

Fig. 3. The shape of the surface near the wave crest at $T = 458.61$.

The time interval from $T = 442$ to $T = 458.56$ contains seven periods of the initial wave only. One can see fast, nonmonotonic formation of the freak wave. At this moment, the freak wave is more steep than the Stokes wave of limiting amplitude. The amplitudes of the waves preceding the freak wave are relatively small (three times less). One can see a trough just ahead of the freak wave. This is the so-called hole in the water (marine folklore) that precedes a freak wave. Figure 3

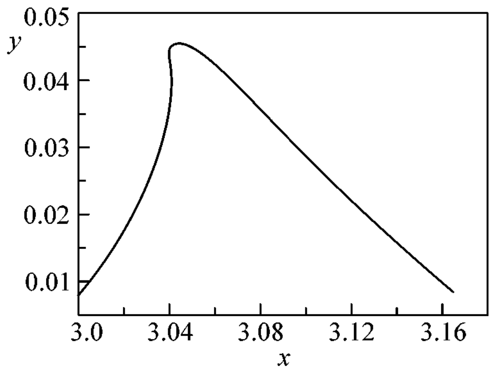

Fig. 4. The shape of the surface near the wave crest at T = 458.842.

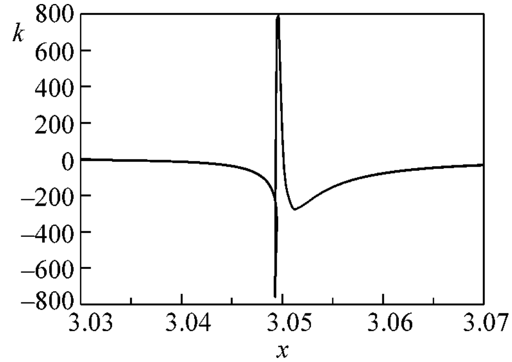

Fig. 5. Curvature (k) of the surface at T = 458.842.

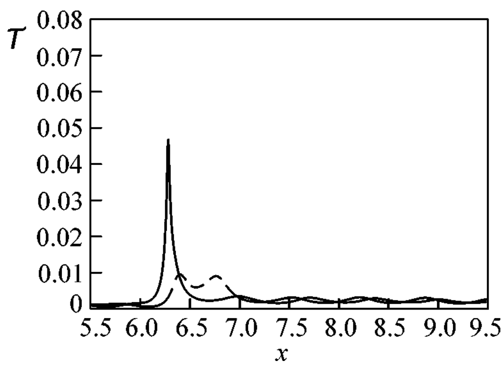

Fig. 6. The density of the kinetic energy just before breaking at T = 456 (dashed line) and at the moment of breaking at T = 458.5 (solid line).

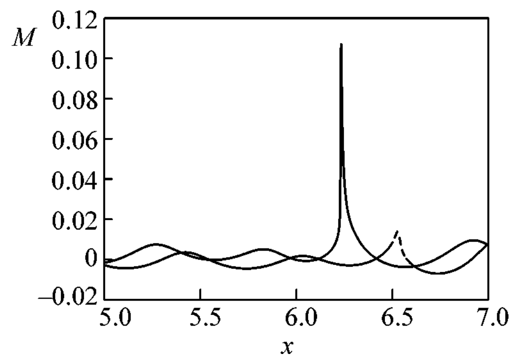

Fig. 7. Distribution of momentum (M) before (dashed line) and after (solid line) breaking.

demonstrates the fine structure of the surface shape near the wave crest.

We managed to continue our simulation until the moment T = 458.842. The zoomed shape of the surface at that time is presented in Fig. 4.

One can see that, near the crest, the front face of the wave is very steep. This is really a wall of water. In some regions, the steepness is even negative. The curvature of the shape is plotted in Fig. 5.

This is actually a breaking wave. Moreover, in all our experiments, at the moment of breaking, we observed that the ratio of the wave height to the wavelength is practically the same and close to that of the limiting Stokes wave, 0.141.

Note that the maximum value of the freak wave height is three times higher than the height of the initial wave. The growing of the wave height up to this level from the level of insignificant wave height takes less than ten wave periods. This is a really fast process; it is three times faster than the development of modulation instability.

Figure 6 displays the evolution of the spatial density of the kinetic energy (in the domain [5.5–9.5]) where the breaking takes place.

One can see that this evolution is nonmonotonous. The density oscillates in time and finally condensates in one very narrow wave crest. In general, the whole process of freak wave formation is nonmonotonous. We can say that the freak wave runs over wave crests until it reaches an extremely high amplitude. This behavior can be easily explained by the difference of the phase and group velocities: the energy propagates with a group velocity that is twice less than the phase velocity. Figure 7 demonstrates the distribution of the horizontal momentum before and after breaking at T = 455 and T = 456. One can see that the process of momentum concentration in a moving but localized area is monotonous. Definitely, this behavior can be explained by the fact that momentum is a conserved quantity.

4. CONCLUSIONS

Let us summarize our numerical experiments. Certainly, they reproduce the most apparent features of freak waves: single wave crests of very high amplitude, exceeding of the significant wave height by more than three times, appearing from nowhere and reaching full height in a very short time, less than ten periods of surrounding waves. A singular freak wave is proceeded by an area of diminished wave amplitudes. The final fate of a freak wave is breaking. The ratio of the freak wave height to its wavelength is practically the same, being close to the limiting Stokes wave, 0.141. A freak wave moves with the group velocity.

In our experiments, the freak wave appears as a result of the development of modulation instability (if the threshold of the instability is not exceeded, no freak waves appear at all). Then, it takes a long time for the onset of instability to create a freak wave. Meanwhile, a freak wave appears only after the fifteenth inverse growth rate of instability. What happens after the development of instability but before the formation of a freak wave? This stage could be considered as the development of some defect on the periodic grid. This grid is just the initial Stokes wave train. A similar picture was observed in [40], where the breaking of a wave in the group was studied.

We thank E.A. Kuznetsov for helpful discussions. This work was supported by the Russian Foundation for Basic Research (project no. 03-01-00289), the Program "Mathematical Problems in Nonlinear Dynamics" of the RAS Presidium, by the grant for Leading Scientific Schools of Russia, by the United States Army (grant DACA, no. 42-00-C-0044), and by the NSF (under DMS grant no. 0072803).

REFERENCES

1. R. Smith, J. Fluid Mech. **77**, 417 (1976).
2. R. G. Dean, in *Water Wave Kinetics*, Ed. by A. Torum and O. T. Gudmestad (Kluwer Academic, Dordrecht, 1990), p. 609.
3. I. V. Lavrenov, Nat. Hazards **17**, 117 (1998).
4. G. A. Chase, http://bell.mma.edu/achase/NS-221-Big-Wave.html (2003).
5. V. D. Divinsky, B. V. Levin, L. I. Lopatikhin, *et al.*, Dokl. Earth Sci. **395**, 438 (2004).
6. M. J. Lighthill, J. Inst. Math. Appl. **1**, 269 (1965).
7. T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).
8. V. E. Zakharov, Zh. Éksp. Teor. Fiz. **51**, 668 (1966) [Sov. Phys. JETP **24**, 455 (1967)].
9. J. E. Feir, Proc. R. Soc. London, Ser. A **299**, 54 (1967).
10. V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
11. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
12. K. B. Dysthe, Proc. R. Soc. London, Ser. A **369**, 105 (1979).
13. K. Trulsen and K. B. Dysthe, Wave Motion **24**, 281 (1996).
14. M. I. Ablovitz, D. Hammack, J. Henderson, and C. M. Scholder, Phys. Rev. Lett. **84**, 887 (2000); Physica D (Amsterdam) **152–153**, 416 (2001).
15. M. Onorato, A. R. Osborne, M. Serio, and T. Damiani, in *Rogue Waves 2000: Proceedings of a Workshop, Brest, France, 2000*, Ed. by M. Olagnon and G. A. Athanassoulis (Ifremer, Plouzanu, France, 2001), p. 181.
16. M. Onorato, A. R. Osborne, and M. Serio, Phys. Lett. A **275**, 386 (2000).
17. M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, Phys. Rev. Lett. **86**, 5831 (2001).
18. M. Onorato, A. R. Osborne, and M. Serio, Phys. Fluids **14**, L25 (2002).
19. D. H. Peregrine, J. Aust. Math. Soc. B, Appl. Math. **25**, 16 (1983).
20. D. H. Peregrine, D. Skyner, M. Stiassnie, and J. W. Dold, in *Proceedings of 21st International Conference on Coastal Engineering* (1988), Vol. 1, Chap. 54, p. 732.
21. K. Trulsen and K. B. Dysthe, in *Proceedings of 21st Symposium on Naval Hydrodynamics* (1997), p. 550; http://www.nap.edu/books/0309058791/html/550.html.
22. K. Trulsen, in *Rogue Waves 2000: Proceedings of a Workshop, Brest, France, 2000*, Ed. by M. Olagnon and G. A. Athanassoulis (Ifremer, Plouzanu, France, 2001), p. 265.
23. K. Trulsen, I. Kliakhandler, K. B. Dysthe, and M. G. Velarde, Phys. Fluids **24**, 32 (2000); D. Clamond and J. Grue, C. R. Acad. Sci., Ser. Mec. **330**, 575 (2002).
24. C. Kharif and E. Pelinovsky, Eur. J. Mech. B/Fluids **22**, 603 (2003).
25. A. A. Kurkin and E. N. Pelinovsky, *Freak Waves: Facts, Theory and Modeling* (Nizhni Novgorod Univ., 2004).
26. V. E. Zakharov and E. A. Kuznetsov, JETP **86**, 1035 (1998).
27. V. E. Zakharov, F. Dias, and A. N. Pushkarev, Phys. Rep. **398**, 1 (2004).
28. A. Majda, D. McLaughlin, and E. Tabak, J. Nonlinear Sci. **7**, 9 (1997).
29. A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, Phys. Lett. A **221**, 73 (1996).
30. V. E. Zakharov, A. I. Dyachenko, and O. A. Vasilyev, Eur. J. Mech. B/Fluids **21**, 283 (2002).
31. V. E. Zakharov, Am. Math. Soc. Trans., Ser. 2 **182**, 167 (1998).
32. A. I. Dyachenko, in *Proceedings of the II International Conference on Frontiers of Nonlinear Physics* (Nizhny Novgorod–St. Petersburg, Russia, 2004).
33. A. I. Dyachenko, Dokl. Math. **63**, 115 (2001).
34. J. W. Dold and D. H. Peregrine, in *Proceedings of 20th International Conference on Coastal Engineering* (1986), Vol. 1, Chap. 13, p. 163.
35. D. G. Dommermuth and D. K. P. Yue, J. Fluid Mech. **184**, 267 (1987).
36. J. W. Dold, J. Comput. Phys. **103**, 90 (1992).
37. W. Tsai and D. Yue, Annu. Rev. Fluid Mech. **28**, 249 (1996).
38. J. A. Zufiria and P. G. Saffman, Stud. Appl. Math. **74**, 259 (1986).
39. D. Meison, S. Orzag, and M. Izraely, J. Comput. Phys. **40**, 345 (1981).
40. J. Song and M. L. Banner, J. Phys. Oceanogr. **32**, 2541 (2002).