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On the Formation of Freak Waves on the Surface of Deep Water[¶]

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Numerical simulation of the fully nonlinear water equations demonstrates the existence of giant breathers on the surface of deep water. The numerical analysis shows that this breather (or soliton of envelope) does not loose energy. The existence of such a breather can explain the appearance of freak waves.

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INTRODUCTION

The formation of giant freak waves from a relatively calm sea is one of the most enigmatic phenomena, not only in physical oceanography but also in the whole physics of nonlinear waves. At the moment, a vast literature is devoted to the explanation to this effect. However, practically all papers on this subject have the same weak point. For the description of wave trains, they use another form of the "envelope equation" based on an expansion in powers of a small parameter, the nonlinear Schrödinger equation for deep water derived in [1]:

$$\epsilon = \lambda/L.$$

Here, λ is the wavelength and L is the width of the wave pulse. However, in reality, $\epsilon \simeq 1$, and the freak wave is a single solitude event. Theories proposed thus far cannot explain this effect.

In the present article, we formulate the following suggestion. We consider the freak wave is a "giant breather." In other words, this is a solution of the Euler equation, which is periodic in a certain moving reference frame. We support this hypothesis by a numerical experiment. We would like to stress that we do not solve the envelope, but the exact fully nonlinear Euler equation for the potential flow of ideal incompressible fluid with a free surface. We study only a two-dimensional fluid, where one coordinate is horizontal and the other is vertical.

The question of the existence of breathers in the model a little beyond the "envelope equation" (namely, in the Dysthe equation [4, 5]) was studied numerically in [2, 3, 6]. Here, we study the fully nonlinear regime of the potential flow.

[¶] The text was submitted by the authors in English.

1. BASIC EQUATIONS

We assume that the fluid fills the area

$$-\infty < y < \eta(x, t), \quad -\infty < x < \infty.$$

The velocity field is the potential

$$V = \nabla\phi, \quad \nabla V = 0, \quad \Delta\phi = 0.$$

The hydrodynamic potential $\phi(x, y, t)$ and the shape of the surface $\eta(x, t)$ satisfy the equations

$$\begin{gathered} \frac{\partial\phi}{\partial t} + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = -\frac{P}{\rho}, \text{ at } y = \eta, \\ \frac{\partial\eta}{\partial t} + \eta_x\phi_x = \phi_y, \text{ at } y = \eta. \end{gathered} \tag{1}$$

One can perform the conformal transformation to map the domain that is filled with fluid on the Z plane ($Z = x + iy$), the lower half-plane

$$-\infty < u < +\infty, \quad -\infty < v < 0, \quad w = u + iv, \quad v < 0$$

in the w plane. This transformation is realized by the function

$$Z = Z(W), \quad Z = x + iy.$$

The potential $\phi(x, t)$ transforms to the complex velocity potential $\Phi(W, t)$. Both of them, Z and Φ are analytic functions in the lower half-plane.

Following [7], one can introduce the variables

$$R = \frac{1}{Z_w}, \text{ and } V = i\Phi_z = i\frac{\Phi_w}{Z_w}. \tag{2}$$

In the new variables, the Euler equation reads

$$\begin{gathered} R_t = i(UR_w - RU_w), \\ V_t = i(UV_w - RB_w) + g(R-1). \end{gathered} \tag{3}$$

Here, U and B are

$$\begin{aligned} U &= \hat{P}(V\bar{R} + \bar{V}R), \\ B &= \hat{P}(V\bar{V}), \end{aligned} \tag{4}$$

where $\hat{P} = \frac{1}{2}(1 + i\hat{H})$ is the projector operator. We must stress that

$$R \longrightarrow 1, \quad V \longrightarrow 0, \text{ at } v \longrightarrow -\infty,$$

where R and V are the periodic functions of u (or vanishing at $u \longrightarrow \pm\infty$).

2. RECALL NONLINEAR SCHRÖDINGER EQUATION

For weakly nonlinear flows, there is a well-known equation describing a weakly modulated wave train, or the nonlinear Schrödinger equation (NLSE):

$$i\left(\frac{\partial A}{\partial t} + C_g A_x\right) - \frac{\omega_0}{8k_0^2}A_{xx} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0, \tag{5}$$

where A is the envelope of the surface elevation $\eta(x, t)$, so that

$$\eta(x, t) = \frac{1}{2}(A(x, t)e^{i(\omega_0 t - k_0 x)} + \text{c.c.}) \tag{6}$$

and ω_0 and k_0 are the frequency and wavenumber of the carrier (wave train), respectively. The well-known solution for $A(x, t)$ is the envelope soliton

$$\begin{aligned} A(x, t) &= e^{-i\Lambda^2 t}\frac{\lambda}{\sqrt{2}k_0^2}\frac{1}{\cosh(\lambda(x - C_g t))}, \\ \Lambda^2 &= \omega_0\lambda^2/8k_0^2. \end{aligned} \tag{7}$$

For fully nonlinear equations (3), this $A(x, t)$ corresponds to the breather.

In this paper, we want to study the following problem: *does a similar solution exist for a strong nonlinear flow?* In other words, we have tried to find stationary breather solution with very few carrier wave and highest steepness.

It should be mentioned that "breather" (6) is relevant only for weakly nonlinear flows

$$\frac{\lambda}{k_0} < 0.07.$$

3. INITIAL CONDITIONS FOR CONFORMAL EQUATIONS

The initial condition for a soliton in physical variables is

$$\eta(x) = \frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0 x)}{\cosh(\lambda x)}. \tag{8}$$

Conformal mapping satisfies the equation

$$y(u) = \eta(u - \hat{H}y(u)). \tag{9}$$

One can suggest the following iterative procedure to solve equation (9):

$$y^{n+1}(u) = \eta(u - \hat{H}y^n(u)). \tag{10}$$

4. NUMERICAL SIMULATION OF GIANT BREATHER

For the initial condition, we have used an essentially nonlinear breather with the steepness

$$\lambda/k_0 > 0.4.$$

The idea for the numerical experiment is to study the possibility of the existence of breathers. We achieved this in the following manner.

Initial condition. For the initial conditions for equations (3), we have used a breather that came from the NLSE approximation. However, the parameters of breather (8) were chosen far beyond the applicability of the NLSE. The simulation was performed in the periodic domain $L = 2\pi$ with $k_0 = 50$. The value of λ varied from 30 to 50.

Damping. This initial NLSE breather radiates an excess energy (or whatever). In our simulation, we have been using damping for that radiation. It was carried out in the following way. First, the simulation was performed in the reference system moving with a group velocity. This velocity is calculated during the simulation, and is adjusted to keep the breather in the center of the domain. This trick allows us to introduce damping close to the edges of the domain to get rid of the excess radiation. After some time, when the radiation vanishes, the damping is switched off. Thus, this trick with damping looks like

$$\begin{aligned} R_t &\longrightarrow R_t + \gamma\cos^6\left(\frac{u}{2}\right)R, \\ V_t &\longrightarrow R_t + \gamma\cos^6\left(\frac{u}{2}\right)V, \end{aligned} \tag{11}$$

with $\gamma = 0.05$.

Long-time evolution. Since the damping is switched off, we observe a long-time evolution of the remaining structure. It should be mentioned here that we want to study the most possible nonlinear regime.

Below, the simulation with $k_0 = 50$ and $\lambda = 50$ is discussed. The initial condition is shown in Fig. 1. One can see that, under the envelope, there are very few wave periods for the carrier.

Formally, this is the exact solution for NLSE. However, the steepness is so high that the NLSE approximation fails. This means that, during the evolution, some

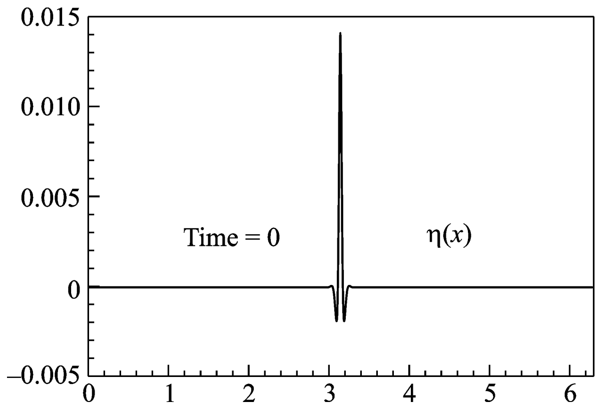

Fig. 1. Initial surface as the NLSE breather with $k_0 = 50$ and $\lambda = 50$.

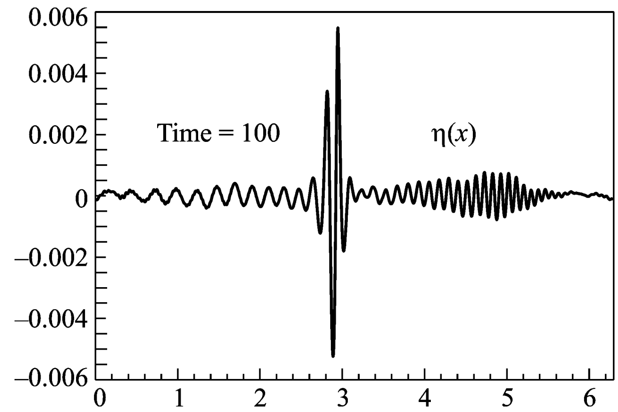

Fig. 2. Radiation from the breather.

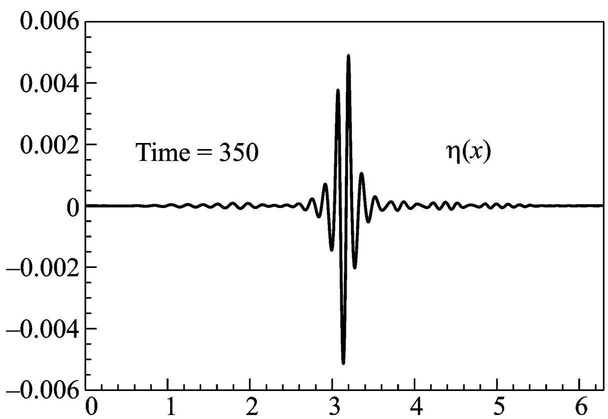

Fig. 3. The moment the damping is switched off.

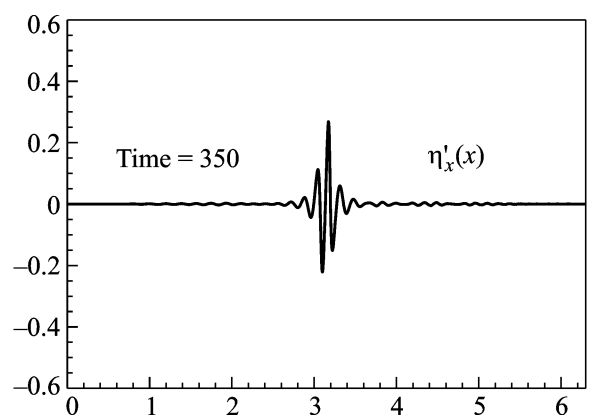

Fig. 4. Steepness profile at the moment of the switching off of the damping.

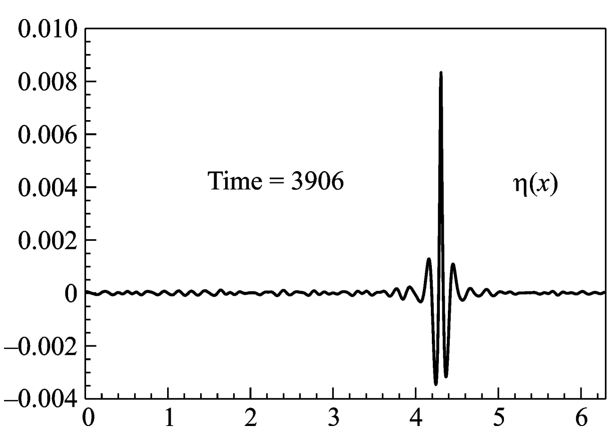

Fig. 5. Typical profile of a breather.

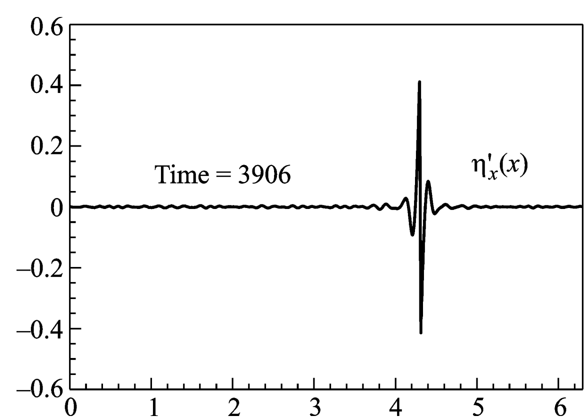

Fig. 6. Final steepness profile.

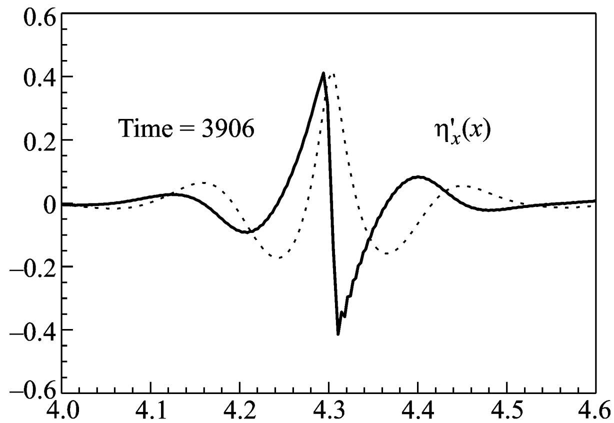

Fig. 7. Detailed steepness profile. The dotted line corresponds to the surface profile multiplied by $k_0 = 50$.

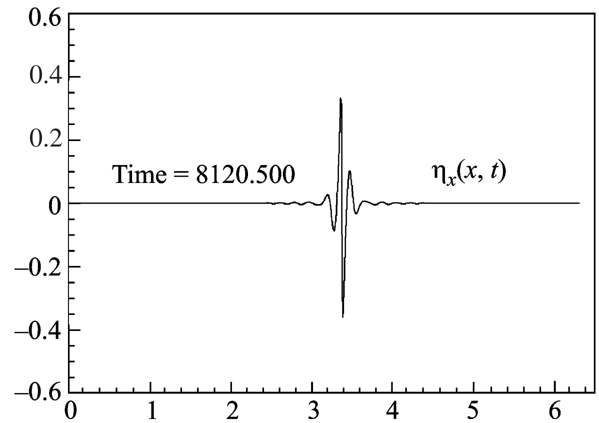

Fig. 8. Breather arising from other initial conditions.

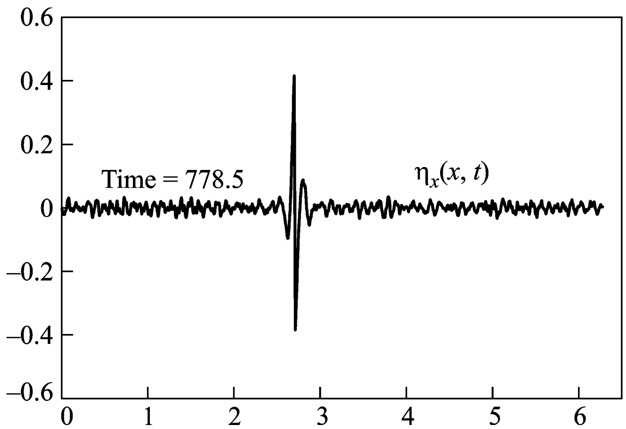

Fig. 9. Breather arising from other initial conditions.

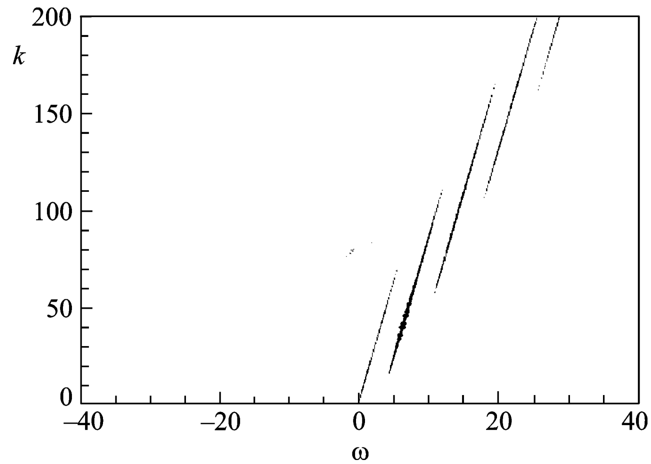

Fig. 10. k–ω spectrum of the breather.

essential changes must happen to this initial condition. After some time, the initial "hump" radiates during the time that artificial damping acts (see Fig. 2). It was switched off at T = 350, and the surface profile and steepness are shown in Figs. 3 and 4. We can assume that, at this time, we start a new simulation with the initial conditions as in Fig. 3. After more than 2000 wave periods, nothing happened when the breather propagates with the velocity that is approximately 10% larger than the linear group velocity. The surface profile is shown in Fig. 5. The local steepness is shown in Fig. 6. It should be pointed out that the steepness is always less than limiting value $1/\sqrt{3}$, but can be close to this value. Looking closely at the steepness as in Fig. 7, one can recognize the value $1/\sqrt{3} \simeq 0.57735$ for the limiting steepness. Those breathers arise from the localized initial conditions (see Fig. 8). For example, in Fig. 9, one can see the breather that arises from the NLSE soliton solution with

$$\lambda/k_0 = 0.6,$$

and no damping for the radiation. The excess radiation remains in the periodic domain of the simulation and does not affect the breather. This situation is very similar to the NLS equation which is integrable.

The hypothesis of integrability is also supported by the picture of the k–ω spectrum of the breather (see Fig. 10).

5. CONCLUSIONS

We have shown numerically that a strongly nonlinear localized breather can exist on the surface of deep water and propagate for a very long time without a loss of energy. We have every reason to think that the existence of such breathers is an explanation of the freak wave phenomenon. We offer the identification of freak waves with giant breathers. In this article, we do not discuss the problem of how freak waves appear from a relative calm sea. We must stress that the existence of high-amplitude breathers is a very exotic phenomenon in the world of nonlinear waves. Such breathers do exist in integrable systems, such as the nonlinear Schrödinger equation. However, in nonintegrable systems, they loose their energy due to radiation [8]. Thus, their existence might be considered as an indication of the integrability of the Euler equation for the potential flow with a free surface. But, we would like to be more cautious. To make a more definitive conclusion, we must compare the behavior of the giant breathers in guaranteed nonintegrable systems, such as the MMT model. These experiments are being carried out now.

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