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ORIGINAL ARTICLE

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Dynamics of vortex line in presence of stationary vortex

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Abstract The motion of a thin vortex with infinitesimally small vorticity in the velocity field created by a steady straight vortex is studied. The motion is governed by non-integrable PDE generalizing the Nonlinear Schrodinger equation (NLSE). Situation is essentially different in a co-rotating case, which is analog of the defocusing NLSE and a counter-rotating case, which can be compared with the focusing NLSE. The governing equation has special solutions shaped as rotating helixes. In the counter-rotating case all helixes are unstable, while in the co-rotating case they could be both stable and unstable. Growth of instability of counter-rotating helix ends up with formation of singularity and merging of vortices. The process of merging goes in a self-similar regime. The basic equation has a rich family of solitonic solutions. Analytic calculations are supported by numerical experiment.

Keywords Vortex · Helix · Instability · Soliton

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1 Basic equations

One of the most important and unresolved questions in Hydrodynamics is formation of singularity for velocity and vorticity in a finite time. The most perspective candidate for this blow-up solution is the non-linear stage of antiparallel vortex pair instability, first studied by Crow [1]. It was shown [2,3], see also [4–6] that in the limit, when sizes of vortex cores are much less than distance between them, this instability leads to formation of singularity and reconnection of vortices in a finite time. However, this approximation fails as soon as the distance between vortices becomes comparable with core sizes. The scenario of further evolution for the vortex pair is disputable [7,8].

In the study presented in [2,3], we considered approximate solutions of the Euler equation. Now, we study the exact solution of Euler equation, up to logarithmic accuracy. Our consideration has its own weak point: the solutions are singular from the beginning, however, we believe that the model offered in this article is interesting by itself.

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Let the stationary vortex of intensity q be posed at $x = 0$, $y = 0$ along the z-axis. It creates the velocity field

$$V_x = -\frac{qy}{x^2+y^2}, \quad V_y = \frac{qx}{x^2+y^2}, \quad V_z = 0. \tag{1}$$

Another vortex of intensity Γ and core width a moves in this field. We assume that $\Gamma \to 0$, $a \to 0$, but the local induction parameter is

$$\lambda = \frac{\Gamma}{4\pi} \log \frac{R}{a} = 1. \tag{2}$$

In the limit $\Gamma \to 0$, the moving vortex does not disturb the stationary vortex. Let $\psi = x + iy$, then ψ satisfies the equation

$$\frac{\partial \psi}{\partial t} = i \left(\frac{\partial}{\partial z} \frac{\psi'}{\sqrt{1+|\psi'|^2}} + \frac{q}{\bar{\psi}} \right). \tag{3}$$

In the limit $|\psi'| \to 0$, Eq. 3 goes to the NLSE with an exotic nonlinearity (this question was discussed in [3–6]):

$$\frac{\partial \psi}{\partial \tau} = i \left(\frac{\partial^2 \psi}{\partial y^2} \pm \frac{1}{\bar{\psi}} \right) = i \left(\frac{\partial^2 \psi}{\partial y^2} \pm \frac{\psi}{|\psi|^2} \right). \tag{4}$$

Here, $y = z|q|^{\frac{1}{2}}$, $\tau = |q|t$. Sign of q is crucial: if $q > 0$, we have co-rotating case, if $q < 0$, the counter-rotating case takes place.

Equation 3 is a Hamiltonian system

$$\frac{\partial \psi}{\partial t} = i \frac{\partial H}{\partial \bar{\psi}}, \qquad H = \int_{-\infty}^{\infty} \left\{ 2\left(\sqrt{1+|\psi'|^2} - \sqrt{1+c^2}\right) + q \log \frac{|\psi|^2}{|\psi_0|^2} \right\} dz. \tag{5}$$

We assume that at $z \to \infty$, $|\psi|^2 \to |\psi_0|^2$, $|\psi'|^2 \to c^2$.

Hamiltonian H is a constant of motion. Other constants of motion are the following:

$$N = \int_{-\infty}^{\infty} |\psi|^2 dz, \qquad P = i \int_{-\infty}^{\infty} (\bar{\psi}\psi' - \psi\bar{\psi}')dz. \tag{6}$$

In the degenerate case $q = 0$, Eq. 5 is completely integrable. It is just another version of the Landau–Lifshitz (or local induction) equation. By the Hashimoto transformation [9], see also [10], it can be transformed to the focusing NLSE. The Lax pair for Eq. 3 exists if $q = 0$; it is presented in Appendix.

After separation of amplitude and phase

$$\psi = Ae^{i\Phi}, \qquad n = A^2, \qquad v = \Phi_z$$
$$R = \sqrt{1+|\psi'|2} = \sqrt{1 + A_z^2 + A^2 v^2}$$

Equation 3 takes form

$$\frac{\partial n}{\partial t} + 2\frac{\partial}{\partial z}\frac{nv}{R} = 0, \qquad A\left(\frac{\partial \Phi}{\partial t} + \frac{v^2}{R}\right) = \frac{\partial}{\partial z}\frac{A_z}{R} + \frac{q}{A} \tag{7}$$

Equation 7 are Hamiltonian

$$\frac{\partial n}{\partial t} = \frac{\partial H}{\partial \Phi}, \qquad \frac{\partial \Phi}{\partial t} = -\frac{\partial H}{\partial n}. \tag{8}$$

In the long wave semiclassical limit

$$\frac{A_z}{A} \ll v, \qquad R \to \sqrt{1 + A^2 v^2}$$

the second of Eq. 7 simplifies up to the form

$$\frac{\partial \Phi}{\partial t} + \frac{v^2}{A} = \frac{q}{n}. \tag{9}$$

Now (7), (9) is a system of hydrodynamic type. In the NLSE limit $R = 1$ and these equations turn to the gas dynamic equations with an exotic dependance of pressure on density:

$$P = q \log n.$$

2 Helix solutions and their stability

Equation 3 has an exact helix solution

$$\begin{aligned} &\psi = \psi_0 = Ae^{ikz - i\Omega t}, \\ &A_z = 0, \quad v = k, \quad R = \sqrt{1 + A^2k^2}, \quad \Omega = \frac{k^2}{\sqrt{1 + A^2k^2}} - \frac{q}{A^2}. \end{aligned} \tag{10}$$

It is a rotating helix. If

$$q < \frac{k^2A^2}{\sqrt{1 + k^2A^2}},$$

the helix rotates in positive direction ($\Omega > 0$). In the opposite case the helix rotates in negative direction ($\Omega < 0$). In the marginal case

$$q = \frac{k^2A^2}{\sqrt{1 + k^2A^2}} < 1,$$

the helix is stationary. If $q = 0$, the Hashimoto transformation converts the rotating helix to a stationary monochromatic wave that is an exact solution of NLSE.

Studying the rotating helix stability, let us assume

$$\psi = \psi_0(1 + \delta\psi e^{ipz - i\omega t}), \qquad |\delta\psi| \ll 1$$

and linearize the equations. After solving the linearized equations, we end up with formula

$$\omega_p^2 = \frac{p^4}{(1 + k^2A^2)^2} + p^2 \left[-\frac{k^4A^2}{(1 + k^2A^2)^2} + \frac{2q}{A^2\sqrt{1 + k^2A^2}} \right]. \tag{11}$$

If $q \le 0$, the helix is unstable. In the case $q = 0$, the helix is unstable if $p^2 < k^4 A^2$. Now

$$\omega_p^2 = \frac{p^2}{(1 + k^2A^2)^2} \left[p^2 - k^4A^2 \right].$$

This is just modulational instability of monochromatic wave in the focusing NLSE.

For $q > 0$, the co-rotating helix is stable if $A^2 < x/k^2$, where x is the solution of equation

$$\frac{x^2}{(1 + x^2)^{\frac{3}{2}}} = 2q.$$

In this case helixes, which are close to the steady vortex, are stable, while the "remote" helixes are unstable.

If $k = 0$ and $q < 0$, the instability of helix is a generalization of Crow instability for two antiparallel vortices.

3 Self-similar collapse of counter-rotating vortices

One can look for self-similar solutions of Eq. 3

$$\psi = (t_0 - t)^{\frac{1}{2}+i\xi} f\left(\frac{z}{\sqrt{t_0 - t}}\right), \tag{12}$$

where ξ is some unknown real constant. Here, we implicitly state that in this system the Leray scaling takes place, i.e., the domain of vortices interaction shrinks proportionally to $(t_0 - t)^{\frac{1}{2}}$, where t_0 is the time of singularity formation.

Let us denote self-similar variable $\eta = \frac{z}{\sqrt{t_0 - t}}$. We obtain the following equation for the self-similar solution:

$$-i\xi f - \frac{1}{2}(f - \eta f) = i\left(\frac{\partial}{\partial \eta}\frac{f'}{\sqrt{1 + |f'|^2}} + \frac{q}{\bar{f}}\right). \tag{13}$$

Here, ξ is an eigenvalue of the nonlinear boundary problem with

$$f'(0) = 0, \qquad f(\eta) \to \eta^{1+2i\xi} \quad \text{at} \quad \eta \to \infty.$$

Equation 13 has reasonable solutions in the counter-rotating case $q < 0$. The eigenvalue ξ is a function on q. This is a subject of determination from the numerical experiment. We applied periodic boundary conditions $\psi(0, t) = \psi(2\pi, t)$ and used the Strang splitting algorithm to solve Eq. 3 numerically. We took $k = 0$, $\psi(z, 0) = 1.25 - 0.05e^{-(z-\pi)^2}\cos(z - \pi)$. With this choice of parameters,

$$\omega_p = p^4 + 2qp^2, \qquad p = 1, 2, \ldots$$

we studied development of instability for $q < -1/2$. The case $q = -1/2$ is a marginal one, however, the instability develops even in this case. On Fig. 1 are presented different shapes of instability development for $q = -1$.

The instability ends up with merging of the vortices at the moment of time $t = T = 1.4721$.

4 Solitonic solutions

Solitons are the following solutions of Eq. 3:

$$\psi = \psi(z - ct)e^{i\lambda t}. \tag{14}$$

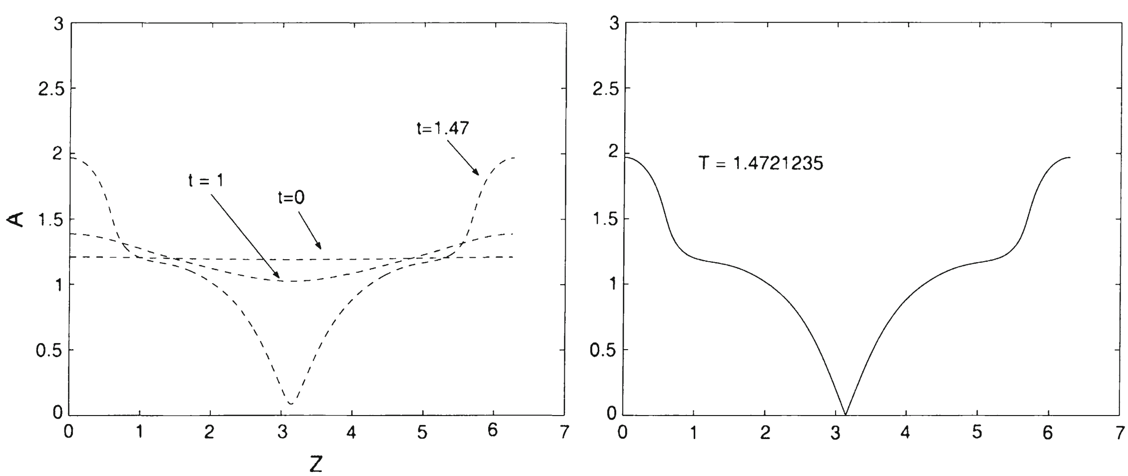

Fig. 1 Development of instability at $q = -1$

Here c and λ are constants. One can use hydrodynamic version of Eq. 3. Then Eq. 7 can be integrated

$$-cn + 2\frac{nv}{R} = Q = -cn_0 + 2\frac{n_0 v_0}{R_0} \tag{15}$$

$n \to n_0$, $v \to v_0$, $R \to R_0$ as $|z| \to \infty$.

One can find v and R

$$v^2 = \frac{(1 + A'^2)(Q + cn)^2}{n\left(4n - (Q + cn)^2\right)} \qquad R = \frac{2A\sqrt{1 + A'^2}}{\left(4n - (Q + cn)^2\right)^{\frac{1}{2}}}. \tag{16}$$

and we end up with a pretty complicated second order nonlinear equation for $A(z)$:

$$\frac{\partial}{\partial z}\frac{A'}{R} + \frac{q}{A} - \lambda A = \frac{Q^2 - c^2 A^4}{2A^2}\frac{\sqrt{1 + A'^2}}{\left(4A^2 - (Q + cA^2)^2\right)^{\frac{1}{2}}}. \tag{17}$$

This equation can be integrated as follows

$$\frac{1}{\sqrt{1 + A'^2}} = \frac{2A}{(4A^2 - (Q + cA^2)^2)^{\frac{1}{2}}}\left(q \log A - \frac{\lambda}{2}A^2 + E\right). \tag{18}$$

Here, E is a constant of integration. If the soliton is steady ($c = 0$) and asymptotic is not a helix but a straight vortex ($Q = 0$), we get

$$\frac{1}{\sqrt{1 + A'^2}} = q \log \frac{A}{A_0} - \frac{\lambda}{2}(A^2 - A_0{}^2) + 1.$$

"Free" soliton ($q = 0$) is given by equation

$$A'^2 = \left(\frac{1}{1 - \frac{\lambda}{2}A^2}\right)^2 - 1$$

$A' = 0$, if $A = 0$ or $A' = \infty$, if $A = \sqrt{\frac{2}{\lambda}}$. In this case $A_0 = 0$.

The total, very rich family of solutions depends on three parameters: c, Q, and λ. Detailed description of solitons and their stability will be published separately.

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Appendix

Equation 13 at $q = 0$ is a compatibility condition for the following overdeterminated linear system of equations imposed on a complex 2×2 matrix function Φ

$$\Phi_x = \lambda\, A\, \Phi, \tag{19}$$

$$\Phi_t = 2\left(\lambda^2 \frac{A}{R} + \lambda\, B\right)\Phi \tag{20}$$

Here, λ is a spectral parameter

$$A = \begin{bmatrix} i & \Psi' \\ -\overline{\Psi}' & -i \end{bmatrix} \qquad R = \sqrt{1 + |\Psi'|^2} \tag{21}$$

$$B = \begin{bmatrix} 0 & v \\ -\overline{v} & 0 \end{bmatrix} \qquad v = -\frac{i}{2}\frac{\partial}{\partial z}\frac{\Psi'}{\sqrt{1 + |\Psi'|^2}} \tag{22}$$

Equations 19 and 20 form the "Lax pair" for (13). Equation 13 is the first non-trivial term in the infinite integrable hierarchy, generated by Eq. 19. This equation is the Gauge equivalent to the Nonlinear Schrodinger equation.

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