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# Probability of the Occurrence of Freak Waves¶

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The statistics of the occurrence of freak waves on a surface of an ideal heavy liquid is studied. The freak (rogue, extreme) waves arise in the course of evolution of a statistically homogeneous random Gaussian wave field. The mean steepness of initial data varies from small ($\mu^2 = 1.54 \times 10^{-3}$) to moderate ($\mu^2 = 3.08 \times 10^{-3}$) values. The frequency of the occurrence of extreme waves decreases with an increase in the spectral width of the initial distribution, but remains relatively high even for broad spectra ($\Delta_k/\Delta \sim 1$).

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Today it is already clear that a freak wave is a natural result of evolution spectrally narrow trains of gravity waves on the surface of a fluid (see [1–8]). It is possible to tell that the formation of freak waves is a nonlinear stage of modulation instability.

We present the first results on quantitative analysis of this effect. We solved numerically the Euler equation that describes a deep ideal liquid with a free surface in 2D-geometry $0 < x < 2\pi$, $-\infty < y < \eta(x)$. Boundary conditions on the interval ends ($x = 0, 2\pi$) are assumed periodic.

The flow is potential and the liquid is incompressible

$$v = \nabla\phi, \quad \mathrm{div}\, v = 0.$$

The potential satisfies the Laplace equation

$$\Delta\phi = 0.$$

We carried out conformal mapping of the area occupied by the liquid onto the lower half-plane (with coordinates $w = u + iv$). This mapping is set by function $z = z(w)$, $z = x + iy$.

The dynamic equations are formulated for Dyachenko's variable

$$R = \frac{1}{z'_w}, \quad V = i\frac{\partial\Phi}{\partial z}$$

and have the form

$$\begin{aligned} R_t(u, t) &= i(UR_u - U_uR), \\ V_t(u, t) &= i(UV_u - B_uR) + g(R - 1), \\ U &= P(VR^* + RV^*), \\ B &= P(VV^*), \end{aligned} \tag{1}$$

where $P$ is the projector operator generating a function that is analytic on the lower half-plane $P = 1/2(1 + iH)$; $H$ is the Hilbert operator for periodic case

$$H[f](y) = \frac{1}{2\pi} v.p. \int_0^{2\pi} \frac{f(u')}{\tan\left(\frac{u' - u}{2}\right)} du'.$$

The system (1) is widely used now. Strict mathematical results on the resolvability of system (1) and the description of methods of its numerical simulation are presented in works [9–12].

In our experiments, the initial conditions were defined as an ensemble of waves traveling in the same direction with the average wavenumber $K_0 = 25$.

We assumed that initial perturbation of the surface is set by the sum of harmonics with random phases

$$\eta_0(x) = \sum_{-\frac{1}{2}K_{\max}}^{\frac{1}{2}K_{\max}} \phi(k - k_0)\cos(kx - \xi_k). \tag{2}$$

Here, $K_{\max}$ is the total number of spectral modes and $\xi_k$ is the random variable distributed on the interval $-\frac{1}{2}K_{\max} < k < \frac{1}{2}K_{\max}$.

The initial velocities were assumed to relate to Eq. (2) by formulas of the linear theory. Conformal mapping was carried out by means of the iterative algorithm offered by A.I. Dyachenko and described in details in [10].

¶The article was translated by the authors.

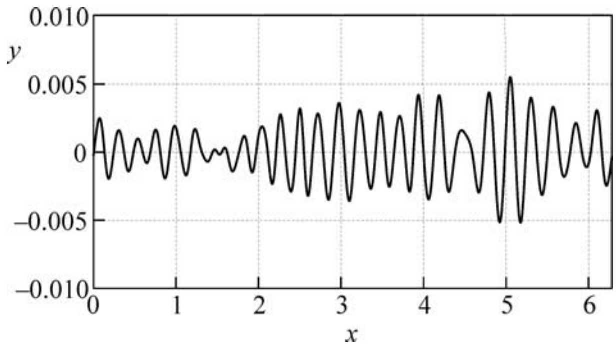


**Fig. 1.** Profile of an initial wave, the average steepness is $\mu^2 = 2.56 \times 10^{-3}$, and the standard deviation is $D = 4$.

The function $\phi(k)$ was defined by formula

$$\phi(k) = \begin{cases} \delta_k, & |k| > K_w; \\ \kappa \exp(-\alpha k^2) + \delta_k, & |k| \le K_w. \end{cases} \tag{3}$$

Here, $\delta_k$ are the independent random parameters uniformly distributed in the interval $-\frac{1}{2}K_{\max} < k < \frac{1}{2}K_{\max}$.

The number $1 \le K_w \le 10$ defines the spectral width, $\kappa$, $\alpha$ are "internal" parameters of the spectrum. They are defined so that external parameters accept the preset values: average steepness $\mu$

$$\mu^2 = \frac{1}{2\pi}\int_0^{2\pi} \eta_x^2 dx,$$

and dispersion $D$,

$$D = \left(\int_{-K_w}^{K_w} k^2 e^{-\alpha k^2} dk\right)\left(\int_{-K_w}^{K_w} e^{-\alpha k^2} dk\right)^{-1}.$$

Further, we calculate the exact values of the total energy $E$ and we observe that the contribution of random noise is no more than three percent. 5000 individual experiments have been done. In each experiment time varied in the range of $0 < t < 200$ that corresponded approximately to 500 periods of waves. If there was a collapse of waves, the experiment was stopped ahead of time. In calculations the full number of harmonics was $K_{\max} = 2048$ or $K_{\max} = 4096$ depending on the total energy, which varied within $1.5 \times 10^{-4} \le E \le 4 \times 10^{-4}$.

Freak waves were detected as follows. After the termination of each experiment, the value $\nu$ was calculated by the formula

$$\nu = \frac{\max\eta(x, t)}{\langle|\eta|\rangle}.$$

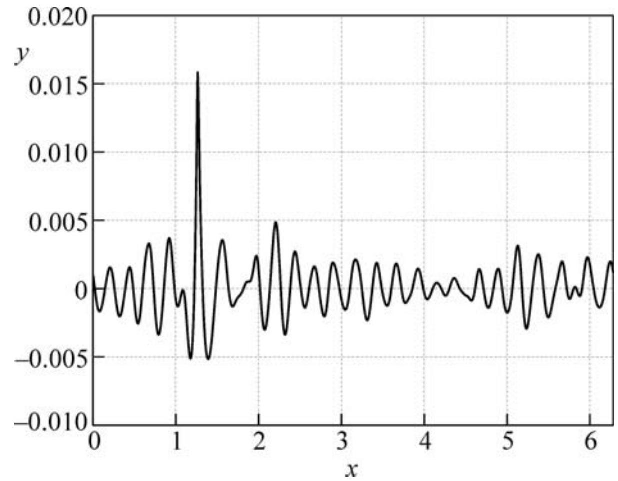


**Fig. 2.** Profile of a freak wave, time is $t = 67.2$, $\nu = 2.13$, and the maximum steepness is 0.558.

Here, the maximum in numerator is taken on coordinate and on time in the interval $0 < t < T$,

$$\langle|\eta|\rangle = \frac{1}{T}\int_0^T \max_{x \in (0, 2\pi)} |\eta(x, t)| dt.$$

The freak wave was fixed if the parameter $\nu$ exceeded the critical value $\nu = 1.8$. This definition quantitatively does not differ essentially from the standard definition where it is considered that freak waves twice exceed significant wave height. It was required also that the local steepness of the wave $|\eta_x|$ exceeded critical value $\max_{0 < x < 2\pi} |\eta_x| \le 0.3$. This requirement is caused by obvious physical reasons and is rather essential.

The results of experiments are presented in the table, where the dispersion and steepness squared values are shown along with the number of the active modes of the initial condition for each experiment. It follows from our data that even for waves of enough moderate steepness ($\mu^2 \simeq 2.06 \times 10^{-3}$, $\mu \simeq 0.045$), the formation of an extreme wave for a time interval as short as 500 periods (at the period of 10 seconds it is less than one and a half hours) is a rather probable event even if the spectral width on wavenumbers is comparable with the carrier wavenumber. Actually, this experiment underlines the fact that the formation of extreme waves is an ordinary event. Figure 1 shows the initial profile of the wave whose evolution led to the occurrence of the freak wave whose profile is shown in Fig. 2. In Fig. 3, the density of an impulse at the moment of the formation of this freak wave is presented.

It is interesting that probability of occurrence of the extreme waves, considered as function from an average steepness at the set dispersion has the maximum at

**Table**

| $D$ | $\mu^2 = 1.54 \times 10^{-3}$ | $\mu^2 = 2.06 \times 10^{-3}$ | $\mu^2 = 2.56 \times 10^{-3}$ | $\mu^2 = 3.08 \times 10^{-3}$ |
|---|---|---|---|---|
| $D = 0.07$ $K_w = 1$ | 0.141 | 0.638 | 0.828 | 0.849 |
| $D = 2$ $K_w = 1$ | 0.152 | 0.457 | 0.616 | 0.554 |
| $D = 4$ $K_w = 2$ | 0.011 | 0.231 | 0.346 | 0.272 |
| $D = 6$ $K_w = 3$ | 0.000 | 0.192 | 0.305 | 0.246 |
| $D = 8$ $K_w = 4$ | 0.011 | 0.154 | 0.280 | 0.195 |
| $D = 10$ $K_w = 5$ | 0.022 | 0.125 | 0.247 | 0.186 |
| $D = 12$ $K_w = 6$ | 0.010 | 0.173 | 0.256 | 0.172 |
| $D = 14$ $K_w = 7$ | 0.000 | 0.058 | 0.216 | 0.170 |
| $D = 16$ $K_w = 8$ | 0.000 | 0.136 | 0.208 | 0.151 |
| $D = 18$ $K_w = 9$ | 0.000 | 0.118 | 0.219 | 0.134 |
| $D = 20$ $K_w = 10$ | 0.034 | 0.127 | 0.206 | 0.099 |

rather moderate steepnesses ($\mu^2 = 2.0 \times 10^{-3}$) and then decreases at steepness increase. This fact is explained by an increase in the competing effect—wave breaking. The used calculation scheme allows us to conduct the experiment only to the first wave breaking. We believe that with the use of more perfect techniques, the steepness dependence of the probability of the occurrence of extreme waves will remain monotonic.

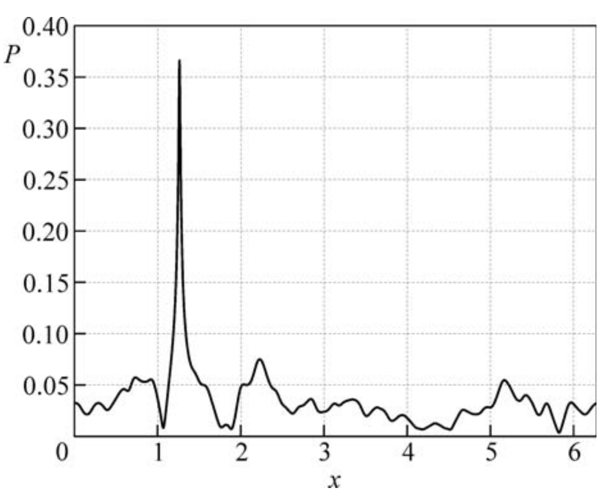


**Fig. 3.** Density of an impulse at the time of the formation of the freak wave presented at Fig. 2.

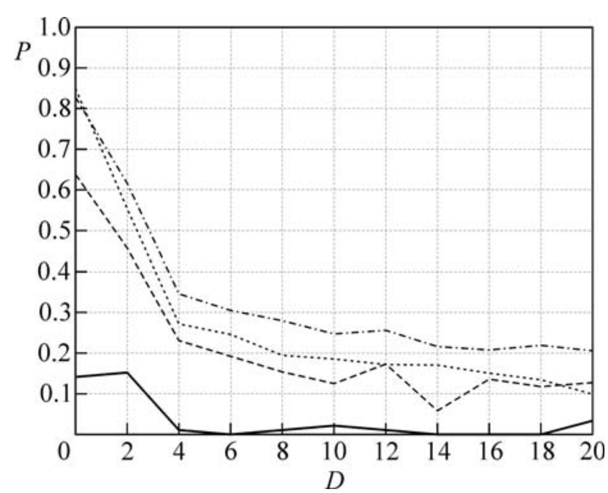


**Fig. 4.** Frequencies of occurrence of freak waves versus the dispersion at $\mu^2$ = (solid line) $1.54 \times 10^{-3}$, (dashed line) $2.56 \times 10^{-3}$, (dotted line) $2.06 \times 10^{-3}$, and (dash–dotted line) $3.08 \times 10^{-3}$.

Figure 4 shows the dispersion dependence of the frequency of occurrence of freak waves.

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