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Energy balance in a wind-driven sea

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Abstract
In this paper, we offer the answers to certain questions extremely important for the development of a self-consistent analytical theory for the wind-driven sea. (i) We discuss the separation into ‘resonant’ and ‘slave’ harmonics in an ensemble of weakly nonlinear gravity waves on the surface of deep water, and we construct an explicit form of the generation function for canonical transformation that eliminates the slave harmonics. (ii) When two waves compiling a quadruple are short in comparison with two others, we find an asymptotic form for the four-wave coupling coefficient. This result makes it possible to reduce the Hasselmann equation to the nonlinear diffusion equation, whose solution describes the well-known effect of angular spreading of wave spectra on its rear face. (iii) Studying the isotropic Kolmogorov–Zakharov solution of the Hasselmann equation, we find numerically the values of Kolmogorov constants. (iv) We calculate the nonlinear damping of surface waves appearing due to four-wave interaction and compare the damping with the growth rate of the instability of the wave surface induced by the wind. It is found that for all known models of wind input, the nonlinear damping surpasses the instability at least in order of magnitude. This result, supported by numerical simulation of the Hasselmann equation, leads to the conclusion: in a real sea, except for the case of very young waves, four-wave interaction is the dominant process. This statement opens the way for the development of a well-justified analytical theory for the wind-driven sea.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

In our opinion, some important theoretical aspects of the physics of the wind-driven sea have not been clarified enough and need to be elucidated. The clarification is necessary for providing an adequate comparison of theory and experiment; without the clarification, the costly and laborious field and laboratory measurements cannot be properly interpreted and understood.

The first question is about the correct definition of wave action $N_k(t)$, which obeys the Hasselmann kinetic equation

$$\frac{dN}{dt} = S_{\text{d}} + S_{\text{n}} + S_{\text{ds}},$$

(1.1)

augmented by the source and the dissipation terms. How to find the current action spectrum $N_k(t)$ from experimental data? What is measured in the best experiments is the space–time spectrum

$$Q_{k\omega} = \langle |\eta_{k\omega}|^2 \rangle.$$  

(1.2)

Here $\eta_{k\omega}$ is the Fourier transform of the surface elevation. The most advanced definition of wave action, used in many research papers (see, for example, [1, 2]), is the following:

$$N_k = \frac{2}{\omega k} \int_0^\infty Q_{k\omega} d\omega.$$  

(1.3)

Equation (1.3) is certainly correct for waves of very small amplitude in the limit $\mu \to 0$, where $\mu$ is the characteristic average steepness of the surface. At a finite steepness, it can be treated as the first term in the expansion

$$N_k = N_0(k) + \mu^2 N_1(k) + \cdots.$$  

(1.4)
Now $N_0(k)$ is given by equation (1.3), whereas $N_1(k)$ is to be determined. One may assume that this question is not very important because even for the steepest young waves, $\mu^2 \simeq 0.01$, and the accuracy of equation (1.3) is good. However, our preliminary estimates show that the ratio $N_1(k)/N_0(k)$ is a fast growing function of $k$; thus for spectral tails, the difference between $N_1$ and $N_0(k)$ might be essential.

Now we formulate the inverse problem. Suppose we know $N_0$. How to find $Q_{\kappa\omega}$?

In the linear approximation, at $\mu \to 0$, the answer is known:

$$Q_{\kappa\omega} = \frac{\omega_k}{2} \left( N_0 \delta(\omega - \omega_k) + N_{-k} \delta(\omega + \omega_k) \right).$$

(1.5)

What happens if $\mu$ is finite? In the neighborhood of $\omega = \omega_k$, we should perform the replacement

$$\delta(\omega - \omega_k) \to \frac{1}{\pi} \frac{\Gamma_k}{(\omega - \omega_k)^2 + \Gamma_k^2},$$

(1.6)

where $\omega_k = \omega_k + \mu^2 \omega_k + \cdots$ is the renormalized frequency and $\Gamma_k \simeq \mu^4 \Gamma_0 + \cdots$ is the effective dissipation due to four-wave processes. As long as $\mu^2$ is small, one may assume that both the shifting of $\omega_k$ and the blurring of $\delta$-function are weak effects. However, the quotients $\omega_k/\omega_k$ and $\Gamma_k/\omega_k$ are growing functions of $k$; thus for $k \gg k_p$ ($k_p$ is the wave number of a spectral peak), derivation from the simple equation (1.5) could be essential. There is one more important effect. In a real sea, all waves can be separated into two classes: ‘resonant’ waves with $\omega \sim \omega_k$ and ‘slaves harmonics’ caused by quadratic nonlinearity of primitive dynamic equations. The slave waves do not obey dispersion relations; as a result, their frequency spectrum for the given $k$ is a broad function, not concentrated at $\omega \simeq \omega_k$.

Accurate determination of $N_1(k)$ at given $Q_{\kappa\omega}$ and $Q_{\omega\omega}$ at given $N(k)$ is possible but is technically a cumbersome problem. In sections 2 and 3 we are taking the first but important steps to obtain their solution. In section 4 we study the axial asymmetric solutions of the equation

$$S_{nl} = 0,$$

(1.7)

which has been known since 1966 ([3]; see also [4, 5]). This equation has exactly two powerlike solutions:

$$N_1(k) = c_p \left( \frac{P}{g^2} \right)^{1/3} \frac{1}{k^4},$$

(1.8)

$$N_2(k) = c_q \left( \frac{Q}{g^{5/2}} \right)^{1/2} \frac{1}{k^{23/6}}.$$ 

(1.9)

Equation (1.8) is known as the Zakharov–Filonenko spectrum [4]. Here $P$ is the flux of energy from small wave numbers and $Q$ is the flux of wave action from high wave numbers. The Kolmogorov constants $c_p$ and $c_q$ were not known, but have now been calculated:

$$c_p = 0.219, \quad c_q = 0.227.$$ 

(1.10)

General isotropic solutions of equation (1.7) depend on two constants $P$ and $Q$. In section 5, we discuss the general anisotropic solution of this equation. We show that the solution is defined by an arbitrary constant, the flux of wave action from high wave numbers, and an arbitrary function of angle. In the axially symmetric case this function degenerates to the constant $P$. The general anisotropic solution of equation (1.7) describes the angular spreading of a spectrum growing with frequency. Section 6 is most important from the practical viewpoint. We discuss the balance equation in the universal domain $\omega \gg \omega_p$,

$$S_{nl} + S_m + S_{dis} = 0.$$ 

(1.11)

Apparently, in some domain on the $k$-plane, $S_m + S_{dis} > 0$. Suppose that $S_m = \gamma(k) N_k$. We note that $S_{nl}$ can be presented in the form

$$S_{nl} = F_k - \Gamma_k N_k,$$

(1.12)

and the nonlinear wave interaction process is predominant if $\Gamma_k > \gamma_2 k$. We show that this condition is satisfied in a majority of realistic cases if the waves are not very young. This means that, as we claimed before, nonlinear wave interaction is the dominant process in the wind-driven sea.

2. What is wave action?

Consider the widely used Hasselmann equation:

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial k} \frac{\partial N}{\partial \tilde{\omega}} = S_{nl},$$

(2.1)

$$S_{nl} = \pi g^2 \int |T_{kk;kk}k^4|^2 \delta(k + k_1 - k_2 - k_3) \times$$

$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})$$

$$\times \delta(\omega_{k_1} + \omega_{k_2} - \omega_{k_3}) \times (N_{k_1} N_{k_2} N_{k_3} + N_{k_1} N_{k_2} N_{k_3}) \, dk_1 \, dk_2 \, dk_3.$$ 

(2.2)

Here $\omega_k = \sqrt{g k \tanh kH}$, $H$ is the depth, $T_{kk;kk} = T_{kk;kk} = T_{kk;kk} = T_{kk;kk}$ are the coupling coefficients, and

$$\tilde{\omega}(k) = \omega(k) + 2g \int T_{kk;kk} N_{k_1} \, dk_1$$ 

(2.3)

is the renormalized frequency.

As mentioned earlier, the nonlinear interaction term $S_{nl}$ can be presented in the form

$$S_{nl} = F_k - \Gamma_k N_k,$$

(2.4)

where

$$F_k = \pi g^2 \int |T_{kk;kk}|^2 \delta(k + k_1 - k_2 - k_3) \times$$

$$\times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times (N_{k_1} N_{k_2} N_{k_3}) \, dk_1 \, dk_2 \, dk_3,$$

(2.5)

and $\Gamma_k$, the dissipation rate due to the presence of four-wave processes, is the following:

$$\Gamma_k = \pi g^2 \int |T_{kk;kk}|^2 \delta(k + k_1 - k_2 - k_3) \times$$

$$\times (N_{k_1} N_{k_2} N_{k_3}) \, dk_1 \, dk_2 \, dk_3.$$ 

(2.6)
One can say that in a real nonlinear sea, the dispersion relation \( \omega = \omega_k \) is renormalized and becomes a complex function
\[
\omega_k \to \tilde{\omega}_k + \frac{i}{2} \mu \Gamma_k.
\] (2.7)

Equations (2.1) and (2.2) are written for the wave action spectrum \( N_k(\vec{r}, t) \). What is the exact definition for the wave action? How can \( N_k(\vec{r}, t) \) be expressed using the observable measurable quantities? These are not very simple questions.

Taking a snapshot of the surface from two points, one can get its stereoscopic image and restore the shape of elevation \( \eta(\vec{r}) \). If we perform nonsymmetric Fourier transform and define
\[
\eta_k = \frac{1}{(2\pi)^2} \int \eta(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} d\vec{r},
\] (2.8)
we can introduce the spatial spectrum
\[
Q_k = \langle |\eta_k|^2 \rangle.
\] (2.9)

Taking a series of snapshots at consecutive moments of time, one can restore the full space–time spectrum
\[
Q_{kw} = \langle |\eta_{kw}|^2 \rangle.
\] (2.10)

Apparently,
\[
Q_k = \int_{-\infty}^{\infty} Q_{kw} dw.
\] (2.11)

What is wave action \( N_k \)? In some papers and monographs, we can find the following definition:
\[
N_k = \frac{Q_k}{\tilde{\omega}_k}.
\] (2.12)

This is just carelessness. Spectrum \( Q_k \) is an even function, \( Q_{-k} = Q_k \), while \( N_k \) certainly does not obey this restriction. One can present the spatial spectrum in the form
\[
Q_k = \frac{\omega_k}{2} (n_k + n_{-k}),
\] (2.13)
where \( n_k \) is the wave action. We have deliberately denoted it by a lower case letter, because \( n_k \) and \( N_k \) are different wave actions.

The wave field consists of ‘resonant’ and ‘slave’ harmonics. The resonant harmonic with wave vector \( \vec{k} \) has a frequency close to the renormalized frequency \( \tilde{\omega}_k \). The strongest slave harmonics appear as a result of the interaction of two resonant harmonics. Suppose that they have wave vectors \( \vec{k}_1, \vec{k}_2 \). In the first order of nonlinearity, they generate four slave harmonics with wave vectors \( \vec{p}_1, \vec{p}_2, -\vec{p}_1, -\vec{p}_2 \) and frequencies \( \Omega_1, \Omega_2, -\Omega_1, -\Omega_2 \). Here \( \vec{p}_1 = \vec{k}_1 - \vec{k}_2, \vec{p}_2 = \vec{k}_1 + \vec{k}_2, \Omega_1 = \omega_1 - \omega_2, \Omega_2 = \omega_1 + \omega_2 \). There is no definite relationship between the wave vector and frequency for slave harmonics.

Returning to the wave action, let us now explain the difference between \( n_k \) and \( N_k \). \( N_k \) is the ‘refined’ wave action that includes resonant harmonics and slave harmonics of higher order only, and \( n_k \) is the ‘total’ wave action that includes both the resonant and the slave harmonics. Apparently, \( n_k > N_k \) and is directly connected with the experimentally measurable spatial spectrum by equation (2.13). But \( n_k \) does not obey the Hasselmann equation. On the contrary, the ‘purified’ wave action \( N_k \) in principle cannot be measured in any kind of experiment. But exactly this sort of wave action satisfies the Hasselmann equation. As a result, all operational models solve the Hasselmann equation augmented with additional terms: \( S_{\text{in}} \), the input from the wind, and \( S_{\text{dis}} \), the dissipation due to wave breaking. Hence, the operational models do predict \( N_k \). At the same time, experimentalists can measure \( n_k \) only.

At first glance, we see a serious discrepancy; however, nobody pays any attention to it. Why does this happen?

To give an answer, we should estimate the relative difference between \( n_k \) and \( N_k \). Let us denote
\[
\alpha(k) = \frac{n_k - N_k}{n_k}.
\] (2.14)

In a typical observed spectrum of the wind-driven sea, we should separate the spectral area near the peak frequency \( \omega \sim \omega_p \) and the tail \( \omega \gg \omega_p \). In the energy capacitive spectral band close to \( \omega_p \), \( \alpha \) is small:
\[
\alpha \sim \mu^2.
\]

The characteristic steepness \( \mu \) is defined as
\[
\mu^2 \simeq \frac{\omega^4}{g^2 \sigma^2},
\]
where \( \sigma \) is the total energy of waves. Even for young waves, \( \mu^2 \ll 0.01 \); thus the relative difference between \( n \) and \( N \) for deep water is not more than 1% and can easily be neglected. However, \( \alpha(k) \) is a fast growing function of \( k \). An accurate estimate of the dependence of \( \alpha \) on frequency at \( \omega \gg \omega_p \) is not the subject of this paper. An article on this topic will be submitted for publication soon; however, our preliminary results show that this dependence is fast growing:
\[
\alpha \simeq \mu^2 \left( \frac{\omega}{\omega_p} \right)^3.
\] (2.15)

As mentioned above, in the area \( \omega \sim \omega_p \) one can neglect the difference between \( n_k \) and \( N_k \). In this area, we can replace equation (2.9) by
\[
Q_k = \frac{\omega_k}{2} (N_k + N_{-k}).
\] (2.16)

There is an essential difference between equations (2.13) and (2.16). Because \( n_k > 0 \) for any \( k \), wave vectors of slave harmonics cover all of the \( k \)-plane; thus the determination of \( n_k \) from \( Q_k \) is impossible in principle. In contrast, in many practical cases, \( N_k \) is nonzero only inside the bounded domain \( G \) on the \( k \)-plane. At the same time, \( N_{-k} \neq 0 \) inside the domain \( \bar{G} \) only, which is radially symmetric to \( G \). In other words, if the vector \( \vec{k} \) belongs to \( G \), the vector \( -\vec{k} \) belongs to \( \bar{G} \). Suppose that \( G \) and \( \bar{G} \) have no intersection. In this case, in the domain \( G \) we have \( N_k = 2Q_k/\omega_k. \) Despite the presence of factor 2 in (2.13), the integral identity \( \int Q_k \, dk = \int \omega_k N_k \, dk \) remains the same as if we had used the naive and blatantly incorrect equation (2.12).

In some important cases, domains \( G \) and \( \bar{G} \) have intersection. In this case, we face an ambiguity in the determination of \( N_k \) from equation (2.16). To overcome this
ambiguity, one should use the space–time spectrum $Q_{k,\omega}$ and define
\[ n_k = \frac{2}{\omega_k} \int_0^\infty Q(k, \omega) d\omega. \tag{2.17} \]
An equivalent formula is presented in the monograph by Monin and Krasitsky [1] published in Russia in 1985. It was also used by Rosental \textit{et al} [2] at approximately the same time. In this case again
\[ \int \omega_k n_k dk = \int_{-\infty}^{\infty} Q(k, \omega) d\omega dk. \tag{2.18} \]
Let us note that equations (2.13) and (2.17) account for slave harmonics and can be used in the comparison of spectral tails obtained from the experiment and those obtained from the solution of the Hasselmann equation, both numerical and analytical, with caution. They work out to an accuracy of $\mu^2$ in the neighborhood of the spectral peak, but can lead to essential errors in the area of spectral tails. A preliminary estimation for the accuracy of expression (2.17) will be made in the next section.

3. How to separate resonant and slave harmonics?

To perform an accurate separation into resonant and slave harmonics and find an explicit formula that connects $Q(k, \omega)$ and $N_k$, one should use Hamiltonian formalism and implement the canonical transformation, excluding cubic harmonics and find an explicit formula that connects $\phi_1$ and $\Psi$. This pair of variables is canonical; thus the evolution equations for $\eta$ and $\Psi$ take the form [6]
\[ \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{3.8} \]
After non-symmetric Fourier transform,
\[ \Psi(r) = \int \Psi(k) e^{ikr} dk, \quad \Psi(k) = \frac{1}{(2\pi)^{2}} \int \Psi(r) e^{-ikr} dr, \tag{3.9} \]
equation (3.8) reads
\[ \frac{\partial \eta}{\partial t} = \frac{\delta \tilde{H}}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta \tilde{H}}{\delta \eta}. \tag{3.10} \]
\[ \tilde{H} = \frac{1}{4\pi^2} H = H_0 + H_1 + H_2 + \cdots. \tag{3.11} \]
In [7–9], it was shown that the Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of $k\eta_k$:
\[ H_0 = \frac{1}{2} \int \{ A_k |\Psi_k|^2 + g |\eta_k|^2 \} dk, \quad A_k = k \tan kH, \]
\[ H_1 = \frac{1}{2} \int L^{(1)}(k, k_2) \Psi_k \psi_{k_2} \eta_k\eta_{k_2} \delta(k_1 + k_2 + k_3) dk_1 dk_2 dk_3, \]
\[ H_2 = \frac{1}{2} \int L^{(2)}(k, k_2, k_3, k_4) \Psi_k \psi_{k_2} \psi_{k_3} \psi_{k_4} \eta_k \eta_{k_2} \eta_{k_3} \eta_{k_4} \] \[ \times \delta(k_1 + k_2 + k_3 + k_4) dk_1 dk_2 \eta_k \eta_{k_2} \eta_{k_3} \eta_{k_4}. \tag{3.12} \]
Here
\[ L^{(1)}(k_1, k_2) = -(k_1, k_2) - A_k A_{k_2}, \]
\[ L^{(2)}(k_1, k_2, k_3, k_4) = \frac{1}{2} (k_1^2 A_2 + k_2^2 A_1) + \frac{1}{4} A_1 A_2(A_{1+4} + A_{2+4} + A_{1+4}) \]
\[ + A_{1+4} + A_{2+4}). \tag{3.13} \]
Now we can introduce normal variables $a_k$:
\[ \eta_k = \frac{1}{\sqrt{2}} \left( A_k g \right)^{1/4} (a_k + a_k^*), \tag{3.14} \]
\[ \Psi_k = \frac{i}{\sqrt{2}} \left( \frac{g}{A_k} \right)^{1/4} (a_k - a_k^*). \]
Normal variables obey the following Hamiltonian equations:
\[ \frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0. \tag{3.15} \]
All terms in the expansion of Hamiltonian (3.11) must be expressed in terms of $a_k$:
\[ H_0 = \int \omega_k |a_k|^2 dk, \tag{3.16} \]
\[ H_1 = \frac{1}{2} \int V^{(1,2)}_{k_k k_2} (a_k a_k^* a_{k_2} a_{k_2}^* + a_{k_2} a_k a_{k_2} a_k^*) \]
\[ \times \delta(k - k_1 - k_2) dk \] \[ dk_1 dk_2 + \frac{1}{6} \int V^{(0,3)}_{k_k k_2} (a_k a_k a_{k_2} + a_k^* a_k^* a_k a_k) \]
\[ \times \delta(k + k_1 + k_2) dk \] \[ dk_1 dk_2. \]
\[ V^{(1,2)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left( \frac{A_k}{A_{k_1} A_{k_2}} \right)^{1/4} L^{(1)}(k_1, k_2) - \left( \frac{A_{k_1}}{A_k A_{k_2}} \right)^{1/4} L^{(1)}(-k_1, -k_2) \right\}. \]

(3.17)

\[ V^{(0,3)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}} \left\{ \left( \frac{A_k}{A_{k_1} A_{k_2}} \right)^{1/4} L^{(1)}(k_1, k_2) + \left( \frac{A_{k_1}}{A_k A_{k_2}} \right)^{1/4} L^{(1)}(k, k_2) \right\}. \]

(3.18)

Now we can define the ‘total’ or rough action:

\[ n_k \delta(k - k') = g(a_k a_k^*). \]

(3.19)

It is clear that the fundamental relation (2.13) is satisfied. Then, we perform the Fourier transform in time

\[ a_{k_1} = \frac{1}{2\pi} \int a(k, t)e^{-i\omega t} dt \]

(3.20)

and introduce

\[ n_{k_1} \delta(k - k') \delta(\omega - \omega') = g(a_{k_1} a_{k_1}^* \omega). \]

(3.21)

The space–time spectrum of elevation is simply

\[ Q_{k, \omega} = \frac{\omega_k}{2} (n_{k_1} \omega + n_{-k_1} \omega). \]

(3.22)

To separate resonant and slave harmonics, we must perform a canonical transformation to new variables, excluding cubic terms in the Hamiltonian. This is a standard procedure known in celestial dynamics down to the nineteenth century. However, in our case, this procedure is rather cumbersome. It was first performed by Krasitski [9]. He found that initial canonical variables \( a_k \) transform to new canonical variables \( b_k \), which contain first-order slave harmonics only. Variables \( a_k \) are presented by an infinite series as new variables \( b_k \):

\[ a_k = b_k + a_k^{(1)} + a_k^{(2)} + a_k^{(3)}. \]

(3.23)

He calculated the first two terms in this expansion and found the following expressions:

\[ a_k^{(1)} = \int \Gamma^{(1)}(k, \bar{k}, \bar{k_1}, \bar{k_2}) b_{k_1} b_{k_2} \delta(\bar{k} - \bar{k_1} - \bar{k_2}) d\bar{k}_1 d\bar{k}_2 \]

\[ -2 \int \Gamma^{(1)}(\bar{k}, \bar{k_2}, \bar{k}, \bar{k_1}) b_{k_1}^* b_{k_2}^* \delta(\bar{k} + \bar{k_1} - \bar{k_2}) d\bar{k}_1 d\bar{k}_2 \]

\[ + \int \Gamma^{(2)}(\bar{k}, \bar{k_1}, \bar{k_2}) b_{k_1}^* b_{k_2}^* \delta(\bar{k} + \bar{k_1} + \bar{k_2}) d\bar{k}_1 d\bar{k}_2, \]

\[ a_k^{(2)} = \int B(\bar{k}, \bar{k_1}, \bar{k_2}) b_{k_1}^* b_{k_2}^* \delta(\bar{k} - \bar{k_1} - \bar{k_2} - \bar{k_3}) d\bar{k}_1 d\bar{k}_2 d\bar{k}_3 + \cdots, \]

(3.24)

where

\[ \Gamma^{(1)}(\bar{k}, \bar{k_1}, \bar{k_2}) = \frac{1}{2} \left\{ \frac{V^{(1,2)}(\bar{k}, \bar{k_1}, \bar{k_2})}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \right\}, \]

(3.25)

\[ \Gamma^{(2)}(\bar{k}, \bar{k_1}, \bar{k_2}) = \frac{1}{2} \left\{ \frac{V^{(0,3)}(\bar{k}, \bar{k_1}, \bar{k_2})}{\omega_k + \omega_{k_1} + \omega_{k_2}}, \right\}. \]

(3.26)

In our opinion, Krasitski used a rather long method for the calculation of terms in expansion (3.23). He directly checked the validity of the canonicity condition

\[ \{a_k, a_k^* \} = \int \left\{ \frac{\delta a_k}{\delta b_k} \frac{\delta a_k}{\delta b_k^*} - \frac{\delta a_k}{\delta b_k^*} \frac{\delta a_k}{\delta b_k} \right\} d\omega = 0, \]

(3.27)

Calculation of \( a_k^{(3)} \) by this method is just an impossibly complicated task. The canonical transformation can be found using more sophisticated methods; the first one was offered in [7] in 1998. Let us consider that \( a_k \) is a solution of the Hamiltonian system

\[ \frac{\partial a_k}{\partial \tau} + i \frac{\delta R}{\delta a_k^*} = 0, \]

(3.28)

where \( \tau \) is ‘artificial time’ and \( R \) is an efficient Hamiltonian:

\[ R = i \int \Gamma^{(1)}_{kk_1k_2} a_k a_k^* a_k^* a_k - a_k a_k^* a_k^* a_k \]

\[ \times \delta(k - k_1 - k_2) d\bar{k}_1 d\bar{k}_2 \]

\[ + \frac{i}{2} \int \Gamma^{(2)}_{kk_1k_2} a_k^* a_k^* a_k^* a_k^* a_k a_k^* \delta(k + k_1 + k_2) d\bar{k}_1 d\bar{k}_2. \]

(3.29)

Equations (3.28) and (3.29) must be augmented with the initial condition

\[ a_k|_{\tau=0} = b_k. \]

(3.30)

The needed canonical transformation will be obtained if we put \( \tau = 1 \). Expanding the solution in Taylor series of \( \tau \) and putting \( \tau = 1 \) at the end, we reproduce the result of Krasitski (3.24)–(3.26) in a much more economical way.

Now we demonstrate another, more traditional method for constructing canonical transformation, which is based on finding the generating function. We present \( a_k \) in the form

\[ a_k = \frac{1}{\sqrt{2}} (q_k + ip_k), \quad q_{-k} = q_k^*, \quad p_{-k} = p_k^*. \]

The functions \( q_k, p_k \) obey the equations

\[ \frac{\partial q_k}{\partial \tau} = \frac{\delta H}{\delta p_k}, \quad \frac{\partial p_k}{\partial \tau} = \frac{\delta H}{\delta q_k}, \]

(3.31)
where $H$ is the same Hamiltonian expressed through $q_k$, $p_k$. Now
\begin{equation}
H_0 = \frac{1}{2} \int \omega_k (|q_k|^2 + |p_k|^2) \, dk, \tag{3.32}
\end{equation}
\begin{equation}
H_1 = \frac{1}{2} \int L_{kk,k} q_k p_k \delta(k + k_1 + k_2) \, dk \, dk_1 \, dk_2, \tag{3.33}
\end{equation}
\begin{equation}
L_{kk,k} = \frac{L_{k1}^{1/4} A_k^{1/4}}{A_{k1}^{1/4} A_k^{1/4}} L_{k1,k}^{1/4}. \tag{3.34}
\end{equation}
We will perform transformation to new variables $R_k$, $\xi_k$ using the following generation function (see also [10]):
\begin{equation}
S = \int R_k q_k \, dk + \frac{1}{2} \int A_{kk,k} \, q_k q_{k_1} \, R_k \, R_{k_1} \, \delta(k - k_1 - k_2) \, dk \, dk_1 \, dk_2 \\
\times \left[ \frac{1}{3} \int B_{kk,k} R_k R_{k_1} R_{k_2} \delta(k + k_1 + k_2) \, dk \, dk_1 \, dk_2 \right]. 
\end{equation}
The ‘old momentum’ $p_k$ and the ‘new coordinates’ $\xi_k$ are expressed as follows:
\begin{equation}
p_k = \frac{\delta S}{\delta q_{-k}} = R_k + \frac{1}{2} \int A_{-kk,k} q_{k_1} R_{k_1} \delta(k - k_1 - k_2) \, dk_1 \, dk_2, \tag{3.36}
\end{equation}
\begin{equation}
\xi_k = \frac{\delta S}{\delta R_{-k}} = q_k + \frac{1}{2} \int A_{k,k_2,-k} q_{k_1} q_{k_2} \delta(k - k_1 - k_2^2) \, dk_1 \, dk_2 \\
+ \frac{1}{3} \int B_{k,k_2,-k} R_{k_1} R_{k_2} \delta(k - k_1 - k - 2) \, dk_1 \, dk_2. \tag{3.37}
\end{equation}
Apparent $B_{kk,k}$ is symmetric with respect to all permutations and $A_{kk,k} = A_{kk,k}$. To find $A$, $B$, we note that in the first approximation
\begin{equation}
q_k = \xi_k = \frac{1}{2} \int A_{kk,k} x_k, \xi_k, \xi_k \delta(k - k_1 - k_2) \, dk_1 \, dk_2 \\
+ \frac{1}{3} \int B_{kk,k} R_{k_1} R_{k_2} \delta(k - k_1 - k_2) \, dk_1 \, dk_2, \tag{3.38}
\end{equation}
and in equation (3.36) we can replace $q_k \rightarrow \xi_k$. Now we plug $q_k$, $p_k$ into equation (3.32). In equation (3.33), we can just replace $q_k \rightarrow \xi_k$ and $p_k \rightarrow R_k$. From the condition for eliminating cubic terms that are proportional to $\xi_k \xi_k, \xi_k x_k$, and the symmetry conditions, we find, after some calculations, the following nice and elegant expressions for $A$, $B$:
\begin{equation}
A_{kk,k} = -\frac{1}{4} \left( \frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 + L_1 - L_2}{\omega_0 - \omega_1 - \omega_2} \right), \tag{3.39}
\end{equation}
\begin{equation}
B_{kk,k} = -\frac{1}{4} \left( \frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 - L_1 - L_2}{\omega_0 - \omega_1 - \omega_2} \right) \\
- \frac{1}{4} \left( \frac{L_1 - L_0 - L_2}{\omega_0 - \omega_1 - \omega_2} + \frac{L_2 - L_0 - L_1}{\omega_2 - \omega_0 - \omega_1} \right). \tag{3.40}
\end{equation}
Here
\begin{equation}
L_0 = L_{kk,k}, \quad L_1 = L_{k,k_1}, \quad L_2 = L_{k_1,k}, \quad \omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2}. \tag{3.41}
\end{equation}
To reproduce the results of Krasitski one has to expand old variables $q_k$, $p_k$ in powers of new variables $\xi_k$, $R_k$, and then $b_k$ will be as follows:
\begin{equation}
b_k = \frac{1}{\sqrt{2}} \left( \left( \frac{g}{A_k} \right)^{1/4} \xi_k - i \left( \frac{A_k}{g} \right)^{1/4} R_k \right). \tag{3.42}
\end{equation}
New normal variables $b_k$ satisfy Zakharov’s equation [6]
\begin{equation}
\frac{\partial b_k}{\partial t} + i o_k b_k + \frac{i}{2} \int T_{kk,k} b_k^* b_k b_{k_1} \delta(k + k_1 - k_2) \, dk_1 \, dk_2 \, dk_3 = 0. \tag{3.43}
\end{equation}
Here $T_{kk,k}$ is the same as in equation (2.2). An explicit expression for $T_{kk,k}$ is too complicated to be presented here. Note that now we can calculate $n_k = |a_k|^2$ by the use of expansion (3.23). We will assume that triple correlations of new variables are zero
\begin{equation}
\langle b_k b_k^* b_{k_1} \rangle = 0, \quad \langle b_k^* b_{k_1} b_{k_2} \rangle = 0. \tag{3.44}
\end{equation}
We also use the Gaussian closure for quartic variables
\begin{equation}
\langle b_k^* b_k^* b_k b_{k_1} \rangle = N_k N_{k_1} (\delta_{k_1-k_2} \delta_{k-k_1} - \delta_{k-k_1} \delta_{k_1-k_2} \delta_{k_2-k_3}). \tag{3.45}
\end{equation}
Here $N_k$ is the ‘refined’ action. After some calculations, we find that $n_k$ and $N_k$ are connected by the following relation (it can be found in [8]):
\begin{equation}
n_k = N_k + \frac{1}{2} \left( \frac{|V^{(1,2)}(k, k_1, k_2)|^2}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2} \right) \\
\times (N_{k_1} N_{k_2} - N_{k_1} N_{k_2} N_{k_3} - N_{k_1} N_{k_2} N_{k_3} N_{k_4} \delta(k - k_1 - k_2)) \, dk_1 \, dk_2 \\
+ \frac{1}{2} \left( \frac{|V^{(1,3)}(k, k_1, k_2)|^2}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2} \right) \\
\times (N_{k_1} N_{k_2} + N_{k_1} N_{k_2} - N_{k_1} N_{k_2} N_{k_3} + N_{k_1} N_{k_2} N_{k_3} N_{k_4} \delta(k - k_1 - k_2)) \, dk_1 \, dk_2 \\
+ \frac{1}{2} \left( \frac{|V^{(1)}(k_1, k_2)|^2}{(\omega_{k_1} - \omega_{k_2})^2} \right) \\
\times (N_{k_1} N_{k_2} + N_{k_1} N_{k_2} - N_{k_1} N_{k_2} N_{k_3} + N_{k_1} N_{k_2} N_{k_3} N_{k_4} \delta(k - k_1 - k_2)) \, dk_1 \, dk_2. \tag{3.46}
\end{equation}
The difference between $n_k$ and $N_k$,
\begin{equation}
\Delta_k = \frac{n_k - N_k}{N_k},
\end{equation}
is essential on the surface of shallow water. However, even on the surface of deep water $\Delta_k$ is a fast growing function of $k$.

The relationship between space–time spectra of the ‘total’ $n_{k,ω}$ and ‘purified’ $N_{k,ω}$ versions of wave action is not known.
so far. This is a subject for future research. However, $N_{kw}$ can be presented in the form

$$N_{kw} = \frac{1}{\pi} \frac{\Gamma_k N_k}{(\omega - \omega_k)^2 + \Gamma_k^2}$$  \hspace{1cm} (3.47)$$

and we can put approximately

$$Q_{kw} = \frac{1}{2} \omega_k (N_{kw} + N_{-k,-\omega}) = \frac{1}{2\pi} \left\{ \Gamma_k N_k \left(\frac{1}{(\omega - \omega_k)^2 + \Gamma_k^2} + \frac{\Gamma_{-k} N_{-k}}{\omega - \omega_k}\right) \right\}.$$  \hspace{1cm} (3.48)$$

After integration by $\omega$ and assuming that arctan $\Gamma_k/\omega_k \sim \Gamma_k/\omega_k$, one gets the following relationship:

$$N_k = \int_0^\infty N(k, \omega) d\omega + \frac{1}{\pi} \left( \frac{N_k \Gamma_k}{\omega_k} - \frac{N_{-k} \Gamma_{-k}}{\omega - \omega_k} \right).$$  \hspace{1cm} (3.49)$$

From (3.48), we see that the identity

$$N_k = \int_0^\infty N(k, \omega) d\omega$$  \hspace{1cm} (3.50)$$

is valid up to a relative accuracy $\Gamma_k/\omega_k$. The value of this accuracy will be discussed in section 6. Near the spectral peak it is of the order of $4\pi \mu^2$. Identity (2.17) is satisfied with much less accuracy. Even near the spectral peak, the accuracy is of the order of $\mu^2$ and becomes worse at $k \gg k_p$. An explicit expression for $Q(k, \omega)$ through $N_k$ will be the subject of a separate forthcoming paper.

4. Stationary solutions: the isotropic case

In this section, we address the following question: How to solve the stationery kinetic equation

$$S_{ml} \equiv 0? \hspace{1cm} (4.1)$$

Formally speaking, this equation has thermodynamically equilibrium solutions

$$N_k = \frac{T}{\omega_k + \mu},$$  \hspace{1cm} (4.2)$$

where temperature $T$ and $\mu$ are constants. It might sound like a paradox, but in fact spectrum (4.2) is not a real solution of equation (4.1). From this moment we discuss only the case of deep water and consider $\omega = \sqrt{gk}$. Also we denote that $k = |\vec{k}|$.

To justify this statement, we note that in two particular cases, $\mu = 0$ and $T = c\mu$, $\mu \to \infty$, solution (4.2) takes the form

$$N = \frac{T}{\omega_k} = \frac{T}{\sqrt{g}} k^{-1/2},$$  \hspace{1cm} (4.3)$$

Both these solutions are isotropic powerlike functions

$$N_k = k^{-x}$$  \hspace{1cm} (4.4)$$

with particular values $x = 1/2$ and 0. Let us study the general powerlike solution of equation (4.1). By plugging equation (4.4) into equation (4.1) we find that each particular term in $S_{ml}$ is diverging, but in different terms the divergence can be cancelled; thus there is a 'window of opportunity' for the exponent $x$. As a result,

$$S_{ml} = \frac{1}{2} k^{-3x+19/2} F(x).$$  \hspace{1cm} (4.5)$$

Here $F(x)$ is a dimensionless function, defined inside the interval $x_1 < x < x_2$. The edges of the window, $x_1$ and $x_2$, are to be determined. Outside the 'window of opportunity', at $x < x_1$ and $x > x_2$, $F(x) = \infty$. Thus all admitted values of $x$ must be posed between $x_1$ and $x_2$.

Let the quadruplet of waves be formed of wave vectors satisfying resonant conditions

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4,$$

$$\omega_k + \omega_\omega = \omega_{k_3} + \omega_{k_4}.$$  \hspace{1cm} (4.6)$$

Suppose that $|k_1| \ll |k|$. The three-wave resonant condition,

$$\vec{k} = \vec{k}_3, \hspace{0.5cm} \omega_k = \omega_{k_3} + \omega_{k_3},$$  \hspace{1cm} (4.7)$$

cannot be satisfied; thus one of the vectors $\vec{k}_2, \vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\vec{k}_2 = \vec{k} + \vec{k}_3,$$

$$\omega(k_2) = \sqrt{g k} \left( \frac{1}{2} \frac{k}{k^2} \right).$$  \hspace{1cm} (4.8)$$

In the first approximation by a small parameter $|k_1|/|k|$, one can put $\omega(k_2) = \omega(k), \omega(k_1) = \omega(k_3)$ and $|k_3| \sim |k_1|$. In other words, vectors $\vec{k}_1, \vec{k}_3$ are small and have approximately the same length $k_1$. If vector $k$ is directed along the axis $x$, the coupling coefficient $T_{kk,k_3}$ depends on four parameters $k, k_1, \theta_1, \theta_3$. Here $\theta_1, \theta_3$ are angles between $k_1, k_3$ and $\vec{k}$. Recalling that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation presented in [11] leads to the following compact result:

$$T_{kk,k_3} \simeq \sqrt{g} k \Gamma_3,$$

$$T_{\theta_1,\theta_3} = 2 \cos(\theta_1 + \cos \theta_3) - \sin(\theta_1 - \theta_3) \sin(\theta_1 - \theta_3).$$  \hspace{1cm} (4.9)$$

On the diagonal $k_3 = k_1, \theta_1 = \theta_3$, we get the following very simple expression, published in 2003 in [29]:

$$T_{kk} \simeq 2\sqrt{g} k \cos \theta_1.$$  \hspace{1cm} (4.10)$$

Suppose that the spectrum is separated into the low-frequency component $N_0(k)$ and the high-frequency component $N_1(k)$. We assume that $N_1 \ll N_0$ and take into account the interaction between $N_0$ and $N_1$ only. One can see that $N_1$ satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_3} D_{ij} k^2 \frac{\partial}{\partial k_j} N_1,$$  \hspace{1cm} (4.11)$$

where $D_{ij}$ is the tensor of diffusion coefficients,

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq q^{17/2} \int_0^{2\pi} d\theta_i \times \int_0^{2\pi} d\theta_j T(\theta_i, \theta_j)^2 p_1 p_2 N(\theta, q) N(\theta_3, q),$$  \hspace{1cm} (4.12)$$

$$p_1 = \cos \theta_1 - \cos \theta_3, \hspace{0.5cm} p_2 = \sin \theta_1 - \sin \theta_3.$$
If the spectrum is isotropic and does not depend on angle $\theta$, we get the further simplification:

$$D_{ij} = D \delta_{ij}, \quad D = \frac{5}{8} \pi^3 g^{3/2} \int_0^\infty q^{17/2} N^2(q) \, dq.$$ (4.13)

The diffusion coefficient $D$ diverges at $k \to 0$ if $x > 19/4$. Thus, $x_2 = 19/4$.

Let us find the behavior of the function $F(x)$ near $x = x_2$.

In the isotropic case, equation (3.9) reads as

$$\frac{\partial N_k}{\partial t} = \frac{D}{k} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} N_k.$$ (4.14)

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4} - \frac{11}{4} \frac{5\pi^3}{16} \frac{1}{19/4 - x} \simeq \frac{126.4}{19/4 - x}.$$ (4.15)

To find $x_1$, the lower end of the window, we should study the influence of short waves on the long ones. Let us suppose that $|k_1|, |k_2| \gg k$. In the first approximation, $|k_1| = |k|$, and the resonant interaction $S_{nl}$ can be separated into two groups of terms: $S_{nl} = S_{nl}^{(1)} + S_{nl}^{(2)}$. For $S_{nl}^{(1)}$ the integrand includes the product $N_{k_1} N_{k_2}$. If we put $k_1 = k_2$, we get the following expression for the low-frequency tail of the spectrum:

$$S_{nl}^{(1)} = 2\pi g^2 \int |T_{kk,k} k| \delta(\omega - \omega_{k_1}) (N_{k_1} - N_{k_2}) N_{k_1}^2 \, dk_1.$$ (4.16)

Note that if $|k_1| \gg |k|$, then $|T_{kk,k} k| \simeq k_1^2$, and the integrand in (4.16) is proportional to $k_1^2 N_{k_1}^2$. If $x < 2$, the integral diverges.

The group of terms linear with respect to the high-frequency tail of the spectrum is more complicated:

$$S_{nl}^{(2)} = 2\pi g^2 N_k \int |T_{kk,k} k| k N_{k_1} (N_{k_1} - N_{k_2}) \times \delta(\omega + \omega_{k_2} - \omega_{k_1}) \times \delta(k + k_1 - k_2) \partial \omega_{k} \, dk_1 \partial \omega_{k_2}.$$ (4.17)

We can perform the expansion

$$N_{k_1} - N_{k_2} = p_k \frac{\partial N}{\partial k_1}, \quad p_k = (k - k_1).$$ (4.18)

In the general anisotropic case, the integrand is proportional to $k_1^2 (\partial \omega_{k} / \partial N_{k_1})$ and the divergence occurs if $x = x_1 = 3$. However, in the isotropic case, this term, the most divergent one, is canceled after integration by angles. In this case, we should study quadratic terms in the expansion of the integrand in powers of parameter ($P, k_1$)$/k_1^2$. The most aggressive term appears from the expansion of $\delta$-function on frequencies $\delta(\omega_{k_1} - \omega_{k_1 + p} + \omega_{k_2} - \omega_{k_2})$. Performing integration by angles, we end up with the equation

$$\frac{\partial N_k}{\partial t} = q k^7 N_k \frac{\partial N}{\partial k},$$

$$q = \frac{25}{16} \pi^3 g^{3/2} E = \frac{25}{8} \pi^3 g^{3/2} \int_0^\infty k^{3/2} N_k \, dk.$$ (4.19)

Here $E$ is the total energy. Thus, in the isotropic case, $x_1 = 5/2$, and we get, for the function $F(x)$, the following estimate:

$$F = \frac{5}{2} \frac{25}{8} \pi^3 \frac{1}{(5/2) - x} = \frac{241.86}{(5/2) - x}.$$ (4.20)

In figure 1(a) is presented a plot of the function $F(x)$ for the isotropic case, which we calculated numerically. One can see that in the interval $x_1 < x < x_2$, the function $F(x)$ has exactly two zeros at

$$x = y_1 = 4, \quad x = y_2 = \frac{23}{6}.$$ (4.21)

To prove this result, let us consider that the spectra are isotropic, and present the conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k \omega_{k} \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k},$$ (4.22)

$$P = 2\pi \int_0^k k \omega_{k} S_{nl} \, dk,$$ (4.23)

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{\partial b},$$ (4.24)

$$Q = 2\pi \int_0^k k S_{nl} \, dk.$$ (4.25)

Here, $P$ is the flux of energy directed to high wave numbers, while $Q$ is the flux of wave action directed to small wave numbers. The equations

$$P = P_0 = \text{const}, \quad Q = Q_0 = \text{const}.$$ (4.26)
apparently are solutions of the stationary equation \( S_{nl} = 0 \). We will look for the solution in the powerlike form \( N = \lambda k^{-s} \); then equations (4.23) and (4.25) read as

\[
P_0 = 2\pi g^2 \lambda^{3/2} \frac{F(x)}{3(x - 4)} k^{-3(x - 4)},
\]

\[
Q_0 = -2\pi g^2 \lambda^{3/2} \frac{F(x)}{3(x - 26/3)} k^{-3(x - 26/3)}.
\]

One can see that \( P_0 \) and \( Q_0 \) are finite only if \( F(4) = 0 \) and \( F(26/3) = 0 \) and if \( F'(4) > 0 \) and \( F'(26/3) < 0 \). We conclude that equation \( S_{nl} = 0 \) has the following solutions:

\[
N^{(1)}_k = c_p \left( \frac{P_0}{g^2} \right)^{1/3} \frac{1}{K^4},
\]

\[
N^{(2)}_k = c_q \left( \frac{Q_0}{g^{1/2}} \right)^{1/3} \frac{1}{K^{23/6}}.
\]

Here, \( c_p, c_q \) are dimensionless Kolmogorov constants

\[
c_p = \left( \frac{3}{2\pi F(4)} \right)^{1/3}, \quad c_q = \left( \frac{3}{2\pi |F(23/6)|} \right)^{1/3}.
\]

In figure 1(b) is presented a zoom of the function \( F(x) \) in the vertical coordinate. The numerics gives \( F'(4) = 45.2 \) and \( F'(23/6) = -40.4 \). In the area of zeros, \( F(x) \) can be approximated by a parabola,

\[
F(x) \simeq 256.8(x - 23/6)(x - 4).
\]

Let us note that

\[
F(9/2) = 85.6;
\]

thus, we obtain

\[
c_p = 0.219, \quad c_q = 0.227,
\]

and find that both the Kolmogorov constants are numerically small.

In the isotropic case, the energy spectrum \( F(\omega) \) can be expressed through \( N_k \),

\[
F(\omega) d\omega = 2\pi \omega N_k k dk,
\]

and the energy spectrum corresponding to solution (4.29) has the following form, called the Zakharov–Filonenko spectrum:

\[
F^{(1)}(\omega) = 4\pi c_p \left( \frac{P}{g^2} \right)^{1/3} \frac{g^2}{\omega^3}.
\]

This spectrum was found to be a solution of the equation \( S_{nl} = 0 \) [3].

For the spatial spectrum

\[
I_k dk = 2\pi \omega N(k) k dk,
\]

solution (4.30) transforms to

\[
I_k^{(1)} = 2\pi c_p \left( \frac{P}{g^2} \right)^{1/3} \frac{g^{1/2}}{K^{5/2}} k^{-2.5}.
\]

Spectra (4.29), (4.35) and (4.37) are realized if we have a source of energy that is concentrated at a small wave number and generates the amount of energy \( P \) in one unit of time. For spectrum (4.30), first reported by Zakharov in 1966 [3],

\[
F_k^{(2)} = 2\pi c_q Q^{1/3} k^{-7/3} \simeq 2\pi c_q Q^{1/3} k^{2.33}
\]

(4.38)

\[
F^{(2)}(\omega) = 4\pi c_q Q^{1/3} \frac{g^{5/3}}{\omega^{1/3}}.
\]

(4.39)

Spectra (4.30) and (4.38) can be realized in the case of a source of wave action in the high wave number area.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation \( S_{nl} = 0 \). It is important to stress that thermodynamical solutions \( N = \text{const} \) and \( N = c/k^{1/2} \) are not the solutions of this equation, because their exponents \( x = 0 \) and \( x = 1/2 \) are far below the lower end of the 'window of possibility' \( x_1 = 5/2 \). This fact means that thermodynamics has nothing in common with the theory of the wind-driven sea.

Solutions (4.29) and (4.30) are not unique stationary solutions of \( S_{nl} = 0 \). The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

\[
N_k^{(3)} = c_p \left( \frac{P}{g^2} \right)^{1/3} \frac{1}{k^4} L \left( \frac{g^{1/2} Q k^{1/2}}{P} \right).
\]

(4.40)

Here \( L \) is an unknown function of one variable,

\[
L \rightarrow 1 \quad \text{at} \quad k \rightarrow 0, \quad L(\xi) \rightarrow \frac{c_p}{c_p} \xi^{1/3} \quad \text{at} \quad k \rightarrow \infty.
\]

(4.41)

Let us note that if there is no flux of wave action from infinity, we must set \( Q = 0 \). Under this constraint, the general isotropic solution is the Zakharov–Filonenko spectrum (4.29), parameterized by a single arbitrary constant \( P \), which is a flux of energy to \( k \rightarrow \infty \).

Frequency spectra with tails in the form \( F(\omega) \simeq \omega^{-d} \) were observed in numerous field experiments [11–16] and were obtained in numerical experiments as well [17–19]. Spatial spectra with asymptotics \( I_k \simeq k^{5/2} \) were also observed in many experiments [20–22]. A more careful study of the experimental results shows that in a majority of cases, the spectral area right behind the spectral peak can be better approximated by the tail \( \omega^{-11/3} \) in the frequency spectrum and by the tail \( k^{-7/3} \) in the spatial spectrum. This is especially clear from experiments of Huang et al [20]. Figure 2 taken from [20] demonstrates the coexistence of both types of Kolmogorov–Zakharov (KZ) spectra.

5. Stationary solutions: the anisotropic case

To study the anisotropic solutions of equation (4.1), we introduce the polar coordinates on the \( k \)-plane and put \( k^2 = \omega/g \). Thereafter, we will use the notation

\[
N(\omega, \phi) d\omega d\phi = N(k) dk.
\]

(5.1)

\[
N(\omega, \phi) = \frac{2\omega^3}{g^4} N(k).
\]
To introduce the elliptic differential operator

\[ L f(\omega, \phi) = \left( \frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) f(\omega, \phi) \]  

with the following parameters: \( 0 < \omega < \infty \), \( 0 < \phi < 2\pi \). The equation

\[ L G = \delta(\omega - \omega') \delta(\phi - \phi') \]  

with the boundary conditions

\[ G|_{\omega=0} = 0, \quad G_{\omega=\infty} < \infty, \quad G(2\pi) = G(0) \]  

can be resolved as

\[ G(\omega, \omega', \phi - \phi') = \frac{1}{4\pi} \sqrt{\omega' \omega} \sum_{n=-\infty}^{\infty} e^{i(\phi - \phi')} \times \left[ \left( \frac{\omega}{\omega'} \right)^{\Delta_4} \Theta(\omega' - \omega) + \left( \frac{\omega'}{\omega} \right)^{\Delta_4} \Theta(\omega - \omega') \right], \]  

where \( \Delta_n = 1/2 + 8n^2 \). Now we present \( s_{nl} \) in the form

\[ A(\omega, \phi) = \int_0^\infty d\omega' \int_0^{2\pi} d\phi' G(\omega, \omega', \phi - \phi') s_{nl}(\omega', \phi'). \]  

Note that \( A(\omega, \phi) \) is a regular integral operator and suppose that \( N(\omega, \phi) = \omega^{-\zeta} \). Then

\[ A[\omega^{-\zeta}] = \omega^{-3\zeta+15} H(z), \]  
\[ H(z) = \frac{G(z)}{9(z-5)(z-14/3)}. \]  

The function \( H(z) \) is positive and has no zeros. If \( G(z) \) is presented by a parabola \((5.5)\), \( H(z) \) is just a constant:

\[ H(z) = H_0 = 16.05/9 = 1.83. \]  

This fact leads us to a bold idea. If we assume that

\[ A = \frac{H_0}{8^2} \omega^{15} N^3, \]  

the nonlinear term \( s_{nl} \) turns into the elliptic operator:

\[ s_{nl} = \frac{H_0}{8^2} \left( \frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) \omega^{15} N^3. \]  

This is the so-called ‘diffusion approximation’, introduced in [23]. Being very simple, it grasps the basic features of the wind-driven sea theory. We will refer mostly to this model, bearing in mind that the real case \((5.9)\) does not differ much from it, at least qualitatively.

Let us integrate equation \((5.2)\) by angles. We get

\[ \frac{\partial N(\omega, t)}{\partial t} = \frac{\partial Q}{\partial \omega}. \]  

Here \( N(\omega, t) = \int_0^{2\pi} N(\omega, \phi) d\phi \). Then

\[ B(\omega, t) = \frac{g}{2\omega} \int_0^{2\pi} \cos \phi N(\omega, \phi) d\phi, \]  

and the flux of wave action is

\[ Q = \frac{\partial K}{\partial \omega}, \quad K = \int_0^{2\pi} A(\omega, \phi) d\phi. \]
After multiplying equation (5.14) by $\omega$, one obtains the equation
\[
\frac{\partial F(\omega, t)}{\partial t} + \frac{\partial P}{\partial \omega} = 0, \tag{5.17}
\]
where $P = K - \omega \partial K / \partial \omega$ is the flux of energy.

Let us introduce now the following definitions: the integrated by angle spectral density of momentum
\[
M_s(\omega, t) = \frac{\omega^2}{g} \int_0^{2\pi} \cos \phi B(\omega, \phi) \, d\phi, \tag{5.18}
\]
the quantity
\[
C_s(\omega, t) = \frac{\omega}{2g} \int_0^{2\pi} \cos^2 \phi N(\omega, \phi) \, d\phi, \tag{5.19}
\]
and the flux of momentum
\[
R_s = \int_0^{2\pi} \cos \phi \left( A \omega - \frac{\omega^2}{2} \frac{\partial A}{\partial \omega} \right) \, d\phi. \tag{5.20}
\]
All these quantities are connected by the equation
\[
\frac{\partial M_s}{\partial t} + \frac{\partial R_s}{\partial \omega} = 0. \tag{5.21}
\]
Equations (5.14), (5.17) and (5.21) are averaged by angle balance equations for the basic conservative quantities.

Now we can return to the question formulated above. How many solutions do the stationary kinetic equations (1.5) and (4.1) have? Note that we simplified it to the linear equation
\[
\left( \frac{\partial^2}{\partial t^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A = 0. \tag{5.22}
\]
In particular, the kinetic equation has the anisotropic KZ solution
\[
A = \frac{1}{2\pi} \left( P + \omega Q + \frac{R_s}{\omega} \cos \phi \right), \tag{5.23}
\]
where $P$ and $R_s$ are fluxes of energy and momentum at $\omega \to \infty$ and $Q$ is the flux of wave action directed to small wave numbers. In the general case, equation (5.23) is a nonlinear integral equation; however, in the diffusion approximation the KZ solution can be found in the explicit form
\[
N(\omega, \phi) = \frac{1}{(2\pi H_0)^{1/3}} \frac{g^{4/3}}{\omega^{5}} \left( P + \omega Q + \frac{R_s}{\omega} \cos \phi \right)^{1/3}. \tag{5.24}
\]
By comparing with equations (4.35) and (4.38), one can easily find that, in this case,
\[
ce_p = c_q = \frac{1}{2(2\pi H_0)^{1/3}} = 0.223, \quad H_0 = 1.83.
\]
This is exactly the arithmetic mean between the values of Kolmogorov constants given by equation (3.31).

By multiplying equation (5.24) by $2\pi \omega$, we get the general KZ spectrum in the diffusion approximation:
\[
F(\omega) = 2.78 \frac{g^{4/3}}{\omega^{5}} \left( P + \omega Q + \frac{R_s}{\omega} \cos \phi \right)^{1/3}. \tag{5.25}
\]
We must be sure that in the isotropic case $R_s = 0$, the expression
\[
F(\omega) = 2.78 \frac{g^{4/3}}{\omega^{5}} (P + \omega Q)^{1/3} \tag{5.26}
\]
approximates the generic KZ spectrum with accuracy up to a few per cent.

If we somehow know the value of $A(\omega, \phi)$ on the circle $\omega = \omega_0$, we can solve the external and internal Dirichlet boundary problems for equation (5.22) with the boundary condition $A(\omega, \phi) < \infty$ at $\omega \to \infty$. Suppose that
\[
A(\omega, \phi) = A_0(\phi)
\]
and
\[
= A_0 + \frac{A_1}{\omega} \cos \phi + \sum_{n=2}^{\infty} A_n \left( \frac{\omega_0}{\omega} \right)^{-1/2+\sqrt{1+4n^2}} \cos n\phi. \tag{5.27}
\]
The first two terms in equation (5.27) present the KZ spectrum with $Q = 0$, $P = 2\pi A_0$, $R_s = 2\pi \omega_0 A_1$. The next terms describe the fast stabilization of any arbitrary solution to the KZ spectrum at $\omega/\omega_0 \to \infty$. The first additional term in (5.27) decays as $(\omega_0/\omega)^{3.53} \cos 2\phi$.

This stabilization to the KZ spectrum is actually the ‘angular spreading’ of wind-driven wave spectra that is usually observed in field experiments (see, for instance, [12]). If $Q = 0$, the general KZ solution (5.25) at $\omega \to 0$ is the following spectrum:
\[
F(\omega) \to \frac{2.78}{\omega^5} g^{4/3} \pi^{1/3} \left( 1 + \frac{3}{5} \frac{R_s}{P \omega} \cos \phi + \cdots \right). \tag{5.28}
\]
Similar results were predicted by Kontorovich and Kats [30] and Balk [31].

From equation (5.27), one can see that $A(\omega, \phi)$ is parametrized by the function of one variable, $A_0(\phi)$. In the presence of flux of action $Q$ from infinity, one should add to equation (5.27) an additional term $Q_\phi$. Thus, in the general case, the uncertainty for the determination of $A$ consists of a function that has one variable and one constant. We implicitly assume that the mapping $N \to A$ is uniquely inversible. This fact has not been proven, but it is plausible.

6. Damping due to nonlinear interaction

How must we compare $S_{in}$ and $S_{is}$?

In this section, we show that $S_{in}$ is the leading term in the balance equation (1.11). In fact, the forcing terms $S_{in}$ and $S_{is}$ are not sufficiently accurately known; thus it is reasonable to accept the simplest models of both terms assuming that they are proportional to the action spectrum:
\[
S_{in} = \gamma_{in}(k) N(k), \tag{6.1}
\]
\[
S_{dis} = -\gamma_{dis}(k) N(k). \tag{6.2}
\]
Hence
\[
\gamma(k) = \gamma_{in}(k) - \gamma_{dis}(k). \tag{6.3}
\]
In reality, $\gamma_{\text{int}}(k)$ depends strongly on the overall steepness $\mu$. So far, let us note that the balance kinetic equation (1.24) can be written in the form

$$S_{nl} + y(k) N_k = 0$$  \hspace{1cm} (6.4)

and present the $S_{nl}$ term as

$$S_{nl} = F_k - \Gamma_k N_k.$$  \hspace{1cm} (6.5)

The definitions of $\Gamma_k$ and $F_k$ are given by equations (2.5) and (2.6).

The solution of stationary equation (6.4) is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}.$$  \hspace{1cm} (6.6)

A positive solution exists if $\Gamma_k > \gamma_k$. The term $\Gamma_k$ can be treated as the nonlinear damping that appears due to four-wave interaction. This damping has a very powerful effect. A ‘naive’ dimensional consideration gives

$$\Gamma_k \approx \frac{4\pi g^2}{\omega_k} k^{10} N_k^2;$$ \hspace{1cm} (6.7)

however, this estimate works only if $k \approx k_p$, with $k_p$ being the wave number of the spectral maximum.

Let $k \gg k_p$. Now for $\Gamma_k$ one gets

$$\Gamma_k = 2\pi g^2 \int |T_{kk},k,k| \delta(\omega_k - \omega_k) N_k N_k \, dk, \, dk_2.$$  \hspace{1cm} (6.8)

The main source of $\Gamma_k$ is the interaction between long and short waves. To estimate integral (2.6) more accurately, we assume that the spectrum of long waves is narrow in angle, $N(k_1, \theta_1) = \tilde{N}(k_1) \delta(\theta_1)$. Long waves propagate along the axis $x$ and $k$ is the wave vector of the short wave propagating in the direction $\theta$. For the coupling coefficient we must put $T_{kk},k,k \approx 2k^2 \cos \theta$. Then

$$\Gamma_k = 8\pi g^2 k^2 \theta \int_0^{\infty} k_1^{1/2} \tilde{N}(k_1) \, dk_1.$$  \hspace{1cm} (6.9)

Even for the most mildly decaying KZ spectrum, $N_k \approx k^{-23/6}$, the integrand behaves like $k_1^{-7/6}$ and the integral diverges. For steeper KZ spectra, the divergence is stronger.

Let us estimate $\Gamma_k$ for the case of a ‘mature sea’, when the spectrum can be taken in the form

$$N_k \approx \frac{3}{2} \frac{E}{k^4} k_p^{1/2} \theta(k - k_p).$$  \hspace{1cm} (6.10)

Here $E$ is the total energy. By plugging (6.10) into (6.9), one gets the equation

$$\Gamma_{\omega} = 36\pi \omega \left( \frac{\omega}{\omega_p} \right)^3 \left( \frac{k_p}{k_4} \right)^4 \theta(k - k_p).$$  \hspace{1cm} (6.11)

which includes a huge enhancing factor: $36\pi \approx 113.04$. For a very modest value of steepness, $\mu_p \approx 0.05$, we get

$$\Gamma_{\omega} \approx 7.06 \times 10^{-4} \omega \left( \frac{\omega}{\omega_p} \right)^3 \cos^2 \theta.$$  \hspace{1cm} (6.12)

In the isotropic case, to find $\Gamma_k$ for $\omega/\omega_p \gg 1$ we need to perform a simple integration over angles that yields

$$\int_0^{2\pi} \int_0^{2\pi} r_{\theta_1, \theta_2} d\theta_1 d\theta_2 = \frac{5}{2}(2\pi)^2;$$

thus instead of equation (6.11) we get

$$\Gamma_k = 5\pi g^{3/2} k^2 \int_0^{\infty} k_1^{13/2} \tilde{N}(k_1)^2 \, dk_1.$$  \hspace{1cm} (6.13)

or

$$\Gamma_{\omega} = \frac{45\pi}{2} \pi g^{3/2} \omega \left( \frac{\omega}{\omega_p} \right)^3 \mu_p^4.$$  \hspace{1cm} (6.14)

Finally, assuming that

$$N_{kp} \approx \frac{3}{2} \frac{E}{k^4},$$

we get from equation (6.8) the following estimate for $\Gamma_p = \Gamma_{kp}$:

$$\Gamma_p \approx 9\pi \omega_p \mu_p^4.$$  \hspace{1cm} (6.15)

Even in this case, we have a pretty high enhancing factor: $9\pi \approx 28.26$. In fact, in all known models, $\Gamma_k$ surpasses $\gamma_k$ at least in order of magnitude even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \approx \frac{3}{2} \frac{E}{k^4} k_p^{1/2}.$$  \hspace{1cm} (6.16)

Here $\omega \approx 4\pi \omega_p/\delta \omega$ is the enhancing factor due to peakedness. If $\Lambda \mu_p^4 \approx 1$, then $\Gamma_p$ is associated with the maximal growth of modulational instability for a monochromatic wave: $\Gamma_p \approx \gamma_{\text{mod}} \approx \omega_p \mu_p^4$. If $\Lambda \sim 1/\mu_p^4$, the nonlinearity becomes so strong that the weak-turbulent statistical approach is not applicable. This is quite a realistic possibility. Suppose that $\mu_p \approx 0.11$ and $\omega_p/\delta \omega \approx 5$. Then $\Lambda \mu_p^4 \approx 0.76$ and the weak-turbulent description is hardly correct. In the case of strong nonlinearity the wind-driven sea generates freak waves (see [24, 25]). The very fact of their existence as a common phenomenon is implicit proof of $S_{nl}$ domination in the energy balance.

Note that $\Gamma_k$ diverges for KZ spectra. However, it does not hurt the spectra existence, because in the full kinetic equation the divergence of $\Gamma_k$ is canceled by divergence of $F_k$. Indeed, if we consider the contribution of small wave numbers in integral (2.5), we end up with the following expression:

$$F_k = 2\pi g^2 N_k \int |T_{kk},k,k| \delta(\omega_k - \omega_k) N_k N_k \, dk, \, dk_3 \, dk_3 \simeq N_k \Gamma_k.$$  \hspace{1cm} (6.17)

Neglecting $\gamma_k$, equation (4.1) is satisfied automatically.

The results obtained in this section show that the four-wave nonlinear interaction has a very strong effect. The strong turbulence of the near-surface air boundary layer makes the development of a reliable theory for air–water interaction, including a well-justified analytical calculation of $\gamma_k$, an extremely difficult task. Making field and laboratory measurements of $\gamma_k$ is also difficult, and the scatter in the determination of $\gamma_k$ is itself the order of $\gamma_k$. Anyway, a comparison of the above calculated $\Gamma_k$ with experimental
The energy spectrum is described by the solution of equation (6.11), which has a rich family of solutions. In particular, this equation describes the angular spreading. As a result, we can conclude that $S_{nl}$ is the leading term in the balance equation (1.11) and that the rear face of the spectrum is described by the solution of equation (4.1), which has a rich family of solutions. In particular, this equation describes the angular spreading.

In figure 4, we demonstrate that for the nonlinear interaction term $S_{nl} = F_k - \Gamma_k N_k$, the magnitudes of constituents $F_k$ and $\Gamma_k N_k$ essentially exceed their difference. They are one order higher than the magnitude of $S_{nl}$.

The dominance of $S_{nl}$ was not apparent until now for two reasons. Firstly, it is not correct to compare $S_{nl}$ and $S_{dis}$, instead, one should compare $\Gamma_k$ and $\gamma_k$. Secondly, the widely accepted models for $S_{nl}$ essentially overestimate the dissipation due to white capping. As a result, the dominance of $S_{nl}$ is masked. We offer an alternative model for $S_{nl}$, which will be published in a forthcoming article [27]. Preliminary results obtained in this direction are given in [28].

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