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How probability for freak wave formation can be found

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Abstract. The statistics of arising of freak waves on a surface of an ideal heavy fluid is studied. The freak waves arise during the evolution of statistically homogeneous initial conditions with a Gaussian probability. Simple equation describing evolution of “almost” 1-D water waves is derived using important property of vanishing four-wave interaction.

1 Introduction

It is clear now that freak waves are “a native ingredient” of the surface wave dynamics on deep water. Freak waves appear inevitably as a result of nonlinear evolution of modulation instability not only for monochromatic Stokes waves [1,2], but also for stochastic spectra with a narrow enough spectral band. Understanding of this fact leads us to formulation of the fundamental problem: *what is a freak waves statistics?* Could a captain in the sea estimate a probability to face a freak wave during next one, two, five, ten hours? What information about the spectrum of the wind-driven sea must he (or she) have in possession? This is a problem of a great practical importance which can be solved by combined efforts of the satellite experts realizing monitoring of the ocean surface, and applied mathematicians performing massive numerical simulation of ocean waves. In this article we discuss only the second approach to the problem. We discuss different mathematical models which are used or could be used for analytical and numerical description of ocean waves in connection with their respectivity for study of the freak wave statistics.

No doubts that the most adequate approach to the problem is development of codes for solution of the exact full 3-dimensional Euler equations for incompressible fluid with a free surface. In the ideal case the flow beneath the surface should not be considered as potential one. Creation of this code would be a feat, and Dmitry Chalikov [3] is close to perform it. We cordially wish him a good luck, but we are cautious about ”industrial” perspectives of this approach.

A sophisticated exact code can describe formation and breaking of an individual freak wave, but even the best code supported by the most powerful computers in a closed future will not be able to reproduce scientifically the masterpiece of Hokusai. It does not mean that such a code is useless. It could really reproduce in reality fine details of formation and breaking (or disappearance) of an individual freak wave. But to find the probability distribution function, PDF, describing appearance of freak waves from a relatively smooth sea, one has to perform

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thousands numerical experiments. This ambitious problem can be solved at least at a current state of computer technique, only in the frameworks of more rough and approximate, but much less time consuming analytical models.

A model becomes much simpler if we assume that the flow is potential. This simplification is not that bad – in fact, the shear flow velocity is much less than phase and group velocities of energy containing waves. A full 3-dimensional exact potential model for description of surface waves was developed by some authors years ago [4,5]. However, even these codes are too complicated for using in the massive scale.

A realistic and popular model for massive modelling of freak waves is the nonlinear Shredinger equation (NLSE). However applicability of this equation is limited by very low values of average steepness (according our estimates $\mu = \sqrt{< \nabla\eta^2 >} < 0.05$). Real freak wave is more steep. Usually they appear from a more rough sea. Much better model is the Dysthe equation [6]. It is applicable for description of more rough sea, however, its derivation is still based on the "envelope" approach, presuming that a freak wave is a member of a group of waves of comparable amplitudes. However in a typical situation the freak wave is a single and very singular event. Thus, using any "weak modulation" model from the very beginning is a hopeless idea.

In this article we discuss two mathematical models used by our group to study of freak wave statistics. The first one is the exact Euler equations describing potential flow of ideal fluid with free surface for 2-dimensional geometry. To solve these equations we use the conformal mapping of an area filled with fluid to the lower half-plane. We have already discussed this model in [2,7,8]. The second model is new. In fact this is a refined and maximally simplified "Zakharov equation". In the presented form this equation is very suitable for study of nonlinear wave dynamics in "almost 2-dimensional" geometry, when dependence on the transverse coordinate is slow. Let us notice that similar situation is typical for a real sea. In this article we discuss only analytical aspects of the new model. A numerical code and results of simulations will be published separately.

2 On the statistic of freak waves

Here we present results on quantitative study of freak waves arising from spectrally narrow packages of gravitational waves (see [9]–[14]).

2.1 Exact basic equations

We solved numerically Euler's equation which describes ideal fluid with a free surface in 2D-geometry $0 < x < 2\pi$, $-\infty < y < \eta(x)$. Boundary conditions on the interval ends ($x = 0, 2\pi$) were assumed periodic.

Fluid flow is considered potential, and fluid is incompressible

$$v = \nabla\phi, \quad \operatorname{div} v = 0.$$

Therefore the potential satisfies the Laplace equation

$$\Delta\phi = 0.$$

We perform conformal mapping of the area occupied with a fluid to the bottom half-plane (with coordinates $w = u + iv$). This mapping is defined by the function $z = z(w)$, $z = x + iy$.

The dynamical equations are formulated for variable (see [7])

$$R = \frac{1}{z'_w} \quad V = i\frac{\partial\Phi}{\partial z}$$

and have following form

$$\begin{aligned}
R_t(u,t) &= i(UR_u - U_uR),\\
V_t(u,t) &= i(UV_u - B_uR) + g(R-1),\\
U &= P(VR^* + RV^*),\\
B &= P(VV^*),
\end{aligned} \tag{1}$$

where P is the projector operator generating a function which is analytical in the lower half-plane $P = \frac{1}{2}(1+iH)$, H is the Hilbert operator for periodic case

$$H[f](y) = \frac{1}{2\pi} v.p. \int_0^{2\pi} \frac{f(u')}{\tan(\frac{u'-u}{2})} du'.$$

The system (1) is widely used now. Rigorous mathematical results about resolvability of the system (1), and also the description of methods of its numerical simulation are obtained in the works [15–19].

2.2 Problem set-up

In our experiments initial conditions were defined as ensemble of waves running in one direction with characteristic value of wave number $K_0 = 25$.

We assumed that initial perturbation of a surface is set by the sum of harmonics with random phases

$$\eta_0(x) = \sum_{-\frac{1}{2}K_{max}}^{\frac{1}{2}K_{max}} \phi(k-k_0)\cos(kx - \xi_k). \tag{2}$$

Here K_{max} is full number of spectral modes; ξ_k is the random variate distributed on an interval $-\frac{1}{2}K_{max} < k < \frac{1}{2}K_{max}$.

Initial values of velocities in (2) were calculated according to the linear theory. Equations for conformal mapping and complex velocity were solved by means of the iterative algorithm offered [7] and described in details in [16].

The function $\phi(k)$ is defined by formula

$$\phi(k) = \begin{cases} \delta_k, & |k| > K_w; \\ \kappa \exp(-\alpha k^2) + \delta_k, & |k| \le K_w. \end{cases} \tag{3}$$

Here δ_k is independent random uniformly distributed parameters on the interval $-\frac{1}{2}K_{max} < k < \frac{1}{2}K_{max}$.

Number $1 \le K_w \le 10$ defines spectral width, κ, α are "internal" parameters of a spectrum are defined in a way to provide for "external" parameters preset values: average steepness μ

$$\mu^2 = \frac{1}{2\pi}\int_0^{2\pi} \eta_x^2 dx$$

and dispersion D

$$D = \left(\int_{-K_w}^{K_w} k^2 e^{-\alpha k^2} dk\right)\left(\int_{-K_w}^{K_w} e^{-\alpha k^2} dk\right)^{-1}.$$

Further, contribution of random noise to the total energy E and we is no more than three percent. 5000 individual experiments have been done. In each experiment time varied in the range of $0 < t < 200$ that corresponded approximately to 500 periods of waves. If breaking

Table

D	$\mu^2 = 1.54 \cdot 10^{-3}$	$\mu^2 = 2.06 \cdot 10^{-3}$	$\mu^2 = 2.56 \cdot 10^{-3}$	$\mu^2 = 3.08 \cdot 10^{-3}$
$D = 0.07$ $K_w = 1$	0.141	0.638	0.828	0.849
$D = 2$ $K_w = 1$	0.152	0.457	0.616	0.554
$D = 4$ $K_w = 2$	0.011	0.231	0.346	0.272
$D = 6$ $K_w = 3$	0.000	0.192	0.305	0.246
$D = 8$ $K_w = 4$	0.011	0.154	0.280	0.195
$D = 10$ $K_w = 5$	0.022	0.125	0.247	0.186
$D = 12$ $K_w = 6$	0.010	0.173	0.256	0.172
$D = 14$ $K_w = 7$	0.000	0.058	0.216	0.170
$D = 16$ $K_w = 8$	0.000	0.136	0.208	0.151
$D = 18$ $K_w = 9$	0.000	0.118	0.219	0.134
$D = 20$ $K_w = 10$	0.034	0.127	0.206	0.099

of waves takes place, the experiment stopped ahead of the time. In the simulations the full number of harmonics was $K_{max} = 2048$ or $K_{max} = 4096$ depending on a total energy which varied within $1.5 \cdot 10^{-4} \leq E \leq 4 \cdot 10^{-4}$.

Registration of freak waves was made as follows. For every individual experiment the value ν was estimated by the formula

$$\nu = \frac{\max \eta(x,t)}{\langle|\eta|\rangle}.$$

Here the maximum in numerator undertakes on coordinate and on time an interval $0 < t < T$,

$$\langle|\eta|\rangle = \frac{1}{T}\int_0^T \max_{x\in(0,2\pi)} |\eta(x,t)| dt.$$

The freak wave thought to be appeared if the parameter ν exceeded critical value $\nu = 1.8$. This definition quantitatively not essentially differs from standard when it is considered that freak waves twice exceed significant wave height. It was required also that the local steepness of a wave $|\eta_x|$ exceeded critical value $\max\limits_{0<x<2\pi} |\eta_x| \leq 0.3$. This requirement is caused by obvious physical reasons and is rather essential.

2.3 Numerical results

Results of experiments are presented in the table.

It follows from our data that even for waves of enough moderate steepness ($\mu^2 \simeq 2.06 \cdot 10^{-3}$, $\mu \simeq 0.045$) formation of an extreme wave occurs after short interval of time, 500 periods. (if wave period is 2 seconds, than freak wave appears after half an hours) It is rather probable event even if the spectral width on wave numbers is comparable with characteristic wave number. Actually, this experiment underlines that formation of extreme waves is the ordinary event. In the Fig. 1 the initial profile of a wave is shown.

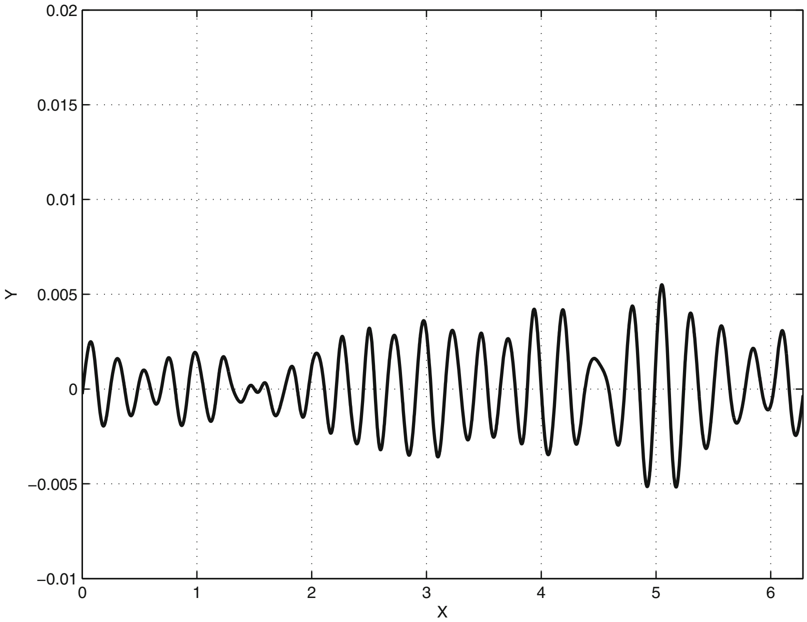

Fig. 1. Profile of an initial wave, average steepness: $\mu^2 = 2.56 \cdot 10^{-3}$, dispersion: $D = 4$.

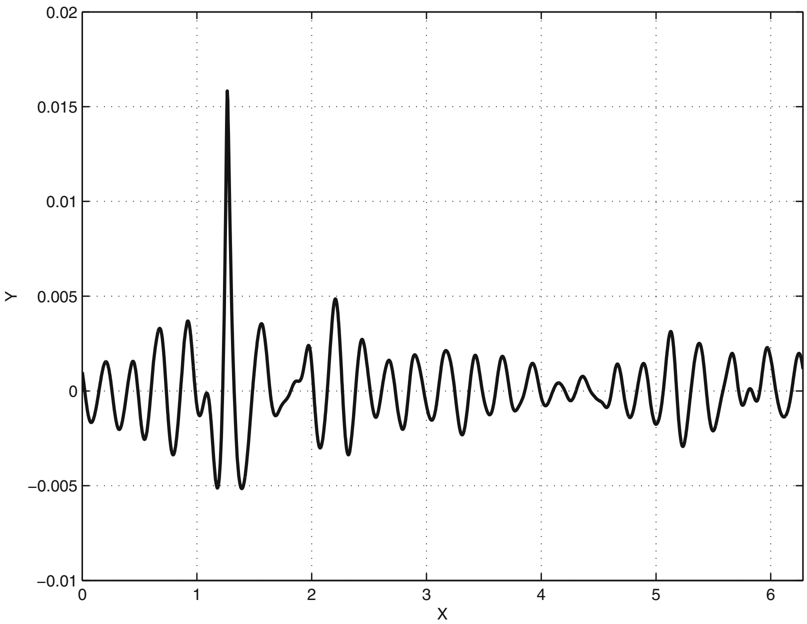

Fig. 2. Profile of freak wave, time: $t = 67.2$, parameter $\nu = 2.13$ maximal steepness: 0.558.

During the evolution there was a freak wave which profile is shown in the Fig. 2.

In the Fig. 3 the density of an impulse at the moment of formation of this freak wave is shown.

It is interesting that probability of occurrence of the extreme waves, considered as function from an average steepness has the maximum at rather moderate steepness ($\mu^2 = 2.0 \cdot 10^{-3}$) and then decreases while steepness increases. This fact shows the important role of competing effect – collapse of waves. The used scheme of the calculation allows to make experiment only up to the first collapse. We believe that the use of more perfect techniques, dependence of probability of occurrence of extreme waves on a steepness remains monotonous.

In Fig. 4 probabilities of occurrence of freak waves as a function of dispersion are shown.

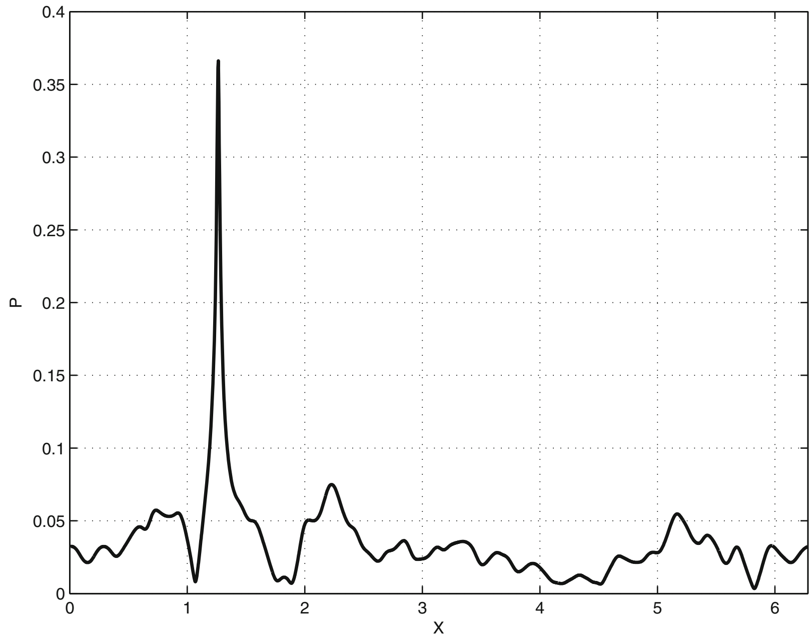

Fig. 3. Density of an impulse at the moment of formation of the freak wave presented at Fig. 2.

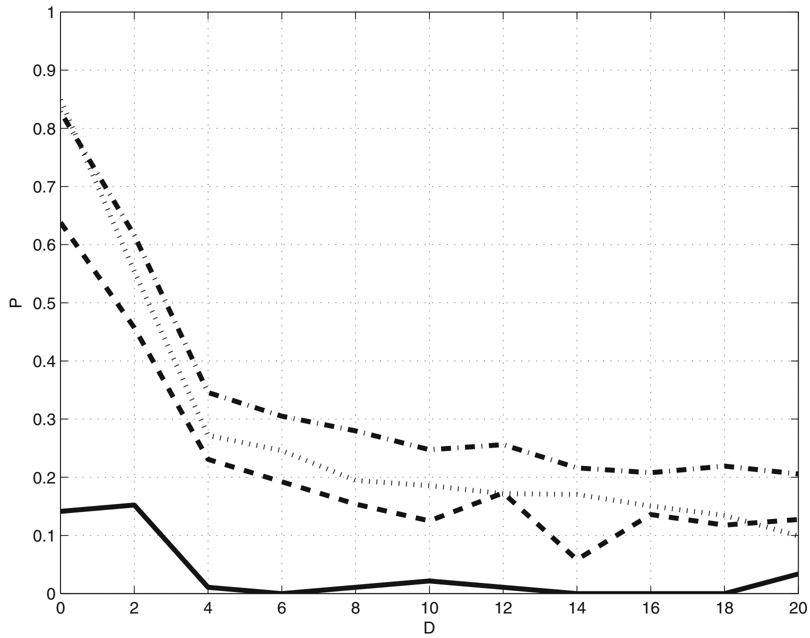

Fig. 4. Frequencies of probability of occurrence of freak waves depending on dispersion. $\mu^2 = 1.54 \cdot 10^{-3}$ – continuous line, $\mu^2 = 2.56 \cdot 10^{-3}$ – dash, $\mu^2 = 2.06 \cdot 10^{-3}$ – dot line, $\mu^2 = 3.08 \cdot 10^{-3}$ – dot-dash.

3 Compact model for deep water waves

In this section we study dynamics of water waves in "almost 2-dimensional" geometry, when dependence on the transverse coordinate is slow. In fact this is simplified "Zakharov equation".

3.1 Four-wave interaction

The main mechanism of of gravity wave interaction is four-wave scattering, satisfying the following resonant conditions

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}, \end{aligned} \tag{4}$$

here k_i are wave vectors of the waves, and $\omega_k = \sqrt{gk}$ is the dispersion law. The corresponding effective Hamiltonian is

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{2} \int \mathbf{T}_{kk_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots; \tag{5}$$

here b_k complex amplitudes of propagating waves [20,21].

The system (4) describes a certain two-dimensional manifold in four-dimensional space (k, k_1, k_2, k_3). This manifold has a trivial component

$$k_2 = k_1, \quad k_3 = k, \quad or \quad k_2 = k, \quad k_3 = k_1, \tag{6}$$

but it has also a nontrivial part. Let $k, k_1, k_3 > 0, k_2 < 0$. Now (4) describes the rational manifold, which can be parametrized in the following way:

$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1+\zeta+\zeta^2)^2; \end{aligned} \tag{7}$$

here $0 < \zeta < 1$ and $a > 0$. It is easy to see that these two manifolds, (6) and (7), represent the general solution for resonant interaction (except trivial permutations).

It was shown in [22] that the coefficient

$$\mathbf{T}_{kk_1}^{k_2 k_3} = 0$$

on the manifold (7). This remarkable property means that the system with Hamiltonian (5) is almost integrable.

3.2 Derivation of four-wave Hamiltonian

Let us start set of equation describing a two dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field:

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \qquad (\phi_z \to 0, z \to -\infty), \\ \eta_t + \eta_x \phi_x &= \phi_z \Big|_{z=\eta} \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= 0 \Big|_{z=\eta}; \end{aligned} \tag{8}$$

here $\eta(x,t)$ – is the shape of a surface, $\phi(x,z,t)$ – is a potential function of the flow and g – is a gravitational constant. As was shown in [20], the variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)|_{z=\eta}$ are canonically conjugated, and their Fourier transforms satisfy the equations

$$\frac{\partial \psi_k}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta_k^*} \qquad \frac{\partial \eta_k}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi_k^*}.$$

Here $\mathcal{H} = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz \qquad U = \frac{g}{2}\int \eta^2 dx.$$

A Hamiltonian can be expanded in an infinite series in powers of a characteristic wave steepness $k\eta_k \ll 1$ (see[20,21]). Our goal is to derive effective four wave Hamiltonian. Effective - means that three-wave interactions (which can not provide resonance interaction on deep water) must be excluded by some transformation. It is easier make derivation after conformal transformation of the 2-D domain (with the curved boundary) occupied by the fluid to the lower half-plane [7,23].

So, one can perform, as before, the conformal transformation to map the domain that is filled with fluid,

$$-\infty < x < \infty, \qquad -\infty < z < \eta(x,t)$$

in physical plane to the lower half-plane

$$-\infty < u < -\infty, \qquad -\infty < v < 0, \qquad w = u + iv$$

in conformal w-plane. Now, the shape of surface $\eta(x,t)$ is presented by parametric equations

$$z = z(u,t),\; x = x(u,t),$$

where $x(u,t)$ and $z(u,t)$ are related through Hilbert transformation

$$z = \hat{H}\left(x(u,t) - u\right), \qquad x(u,t) = u - \hat{H}z(u,t). \tag{9}$$

Here

$$\hat{H}(f(u)) = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u')\mathrm{d}u'}{u'-u}.$$

We introduce also complex velocity potential

$$\Phi(w,t) = \Psi(u,v,t) + i\Theta(u,v,t).$$

Dynamic equation (boundary conditions on the surface) now have the following form:

$$\begin{aligned} z_t x_u - x_t z_u &= -\hat{H}\Psi_u, \\ \Psi_t x_u - x_t\Psi_u + gzx_u &= \hat{H}\left(\Psi_t z_u - z_t\Psi_u + gzz_u\right) \end{aligned} \tag{10}$$

Hamiltonian

$$\mathcal{H} = -\int_{-\infty}^{\infty}\Psi\hat{H}\Psi_u du \;+\; \frac{g}{2}\int_{-\infty}^{\infty} z^2 x_u du$$

with the hamiltonian variables z and $\mathcal{P}$, where $\mathcal{P}$ and Ψ are related through

$$\Psi = \frac{\mathcal{P}x_u + \hat{H}(\mathcal{P}z_u)}{z_u^2 + x_u^2}$$

and the Hamiltonian of the system is

$$\mathcal{H} = -\int_{-\infty}^{\infty}\Psi\hat{H}\Psi_u du + \frac{g}{2}\int_{-\infty}^{\infty} z^2 x_u du. \tag{11}$$

The equations of motion can be written in the explicit Hamiltonian form (which includes Hilbert transformation):

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta z} \qquad \frac{\partial z}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathcal{P}}.$$

Hamiltonian is quadratic function of $\mathcal{P}$ and can be expanded in powers of z:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4^{2\Leftrightarrow 2} + \dots$$

Before doing that it is convenient to introduce conformal normal complex variables a_k[1]

$$z_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \mathcal{P}_k = -i\sqrt{\frac{2g}{\omega_k}}(a_k - a^*_{-k})$$

which satisfies the equation of motion

$$\frac{\partial a_k}{\partial t} + i\frac{\delta \mathcal{H}}{\delta a_k^*} = 0.$$

Quadratic, cubic and fourth order terms are:

$$\begin{aligned}
\mathcal{H}_2 &= \int \omega_k a_k a_k^* dk \\
\mathcal{H}_3 &= \frac{1}{2}\int V^{k_1}_{k_2k_3}\{a^*_{k_1}a_{k_2}a_{k_3} + a_{k_1}a^*_{k_2}a^*_{k_3}\}\delta_{k_1-k_2-k_3}dk_1dk_2dk_3 \\
&\quad + \frac{1}{6}\int U_{k_1k_2k_3}\{a_{k_1}a_{k_2}a_{k_3} + a^*_{k_1}a^*_{k_2}a^*_{k_3}\}\delta_{k_1+k_2+k_3}dk_1dk_2dk_3, \\
\mathcal{H}_4^{2\Leftrightarrow 2} &= \frac{1}{4}\int W^{kk_1}_{k_2k_3}a^*_k a^*_{k_1}a_{k_2}a_{k_3}\delta_{k+k_1-k_2-k_3}dkdk_1dk_2dk_3.
\end{aligned}$$

Expressions for $V^{k_1}_{k_2k_3}$, $U_{k_1k_2k_3}$ and $W^{kk_1}_{k_2k_3}$ can be found in Sect. 4.

Hamiltonian $\mathcal{H}$ in the normal variables a_k is not "canonical". It has cubic term $\mathcal{H}^3$ which can be excluded by appropriate transformation (see [24–26]):

$$\begin{aligned}
a_k &= b_k + \int \Gamma^k_{k_1k_2}b_{k_1}b_{k_2}\delta_{k-k_1-k_2} - 2\int \Gamma^{k_2}_{kk_1}b^*_{k_1}b_{k_2}\delta_{k+k_1-k_2} \\
&\quad + \int \Gamma_{kk_1k_2}b^*_{k_1}b^*_{k_2}\delta_{k+k_1+k_2} + \int B^{kk_1}_{k_2k_3}b^*_{k_1}b_{k_2}b_{k_3}\delta_{k+k_1-k_2-k_3} + \dots \\
B^{kk_1}_{k_2k_3} &= \Gamma^{k_1}_{k_2k_1-k_2}\Gamma^{k_3}_{kk_3-k} + \Gamma^{k_1}_{k_3k_1-k_3}\Gamma^{k_2}_{kk_2-k} \\
&\quad -\Gamma^{k}_{k_2k-k_2}\Gamma^{k_3}_{k_1k_3-k_1} - \Gamma^{k_1}_{k_3k_1-k_3}\Gamma^{k_2}_{k_1k_2-k_1} \\
&\quad -\Gamma^{k+k_1}_{kk_1}\Gamma^{k_2+k_3}_{k_2k_3} + \Gamma_{-k-k_1kk_1}\Gamma_{-k_2-k_3k_2k_3} + \tilde{B}^{kk_1}_{k_2k_3}.
\end{aligned} \tag{12}$$

Here

$$\Gamma^k_{k_1k_2} = -\frac{1}{2}\frac{V^k_{k_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \qquad \Gamma_{kk_1k_2} = -\frac{1}{2}\frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}.$$

Coefficient $\tilde{B}^{kk_1}_{k_2k_3}$ is an arbitrary function, satisfying the following symmetry conditions:

$$\tilde{B}^{kk_1}_{k_2k_3} = \tilde{B}^{k_1k}_{k_2k_3} = \tilde{B}^{kk_1}_{k_3k_2} = -(\tilde{B}^{k_2k_3}_{kk_1})^*.$$

The transformation (12) is canonical up to terms of the order of $|b_k|^4$.

Hamiltonian $\mathcal{H}$ in the new variables b_k has the simple, *effective* form (cubic term vanishes):

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{4}\int T^{kk_1}_{k_2k_3}b^*_k b^*_{k_1}b_{k_2}b_{k_3}\delta_{k+k_1-k_2-k_3}dkdk_1dk_2dk_3.$$

[1] The following Fourier transform is introduced: $f_k = \frac{1}{\sqrt{2\pi}}\int f(u)e^{-iku}du$, $f(u) = \frac{1}{\sqrt{2\pi}}\int f_k e^{iku}dk$.

3.3 Quasi-integrable 1-D equation

The form of $T^{kk_1}_{k_2k_3}$ depends on choice of the function $\tilde{B}^{kk_1}_{k_2k_3}$ in the transformation (12). The simplest choice is $\tilde{B}^{kk_1}_{k_2k_3} = 0$. Then (as it was shown in [22,27])

$$
\begin{aligned}
T^{kk_1}_{k_2k_3} = W^{kk_1}_{k_2k_3}
&-\frac{1}{2}V^{k}_{k_2k-k_2}V^{k_3}_{k_1k_3-k_1}\left[\frac{1}{\omega_{k_2}+\omega_{k-k_2}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_3-k_1}-\omega_{k_3}}\right]\\
&-\frac{1}{2}V^{k_1}_{k_2k_1-k_2}V^{k_3}_{kk_3-k}\left[\frac{1}{\omega_{k_2}+\omega_{k_1-k_2}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_3-k}-\omega_{k_3}}\right]\\
&-\frac{1}{2}V^{k}_{k_3k-k_3}V^{k_2}_{k_1k_2-k_1}\left[\frac{1}{\omega_{k_3}+\omega_{k-k_3}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_2-k_1}-\omega_{k_2}}\right]\\
&-\frac{1}{2}V^{k_1}_{k_3k_1-k_3}V^{k_2}_{kk_2-k}\left[\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_2-k}-\omega_{k_2}}\right]\\
&-\frac{1}{2}V^{k+k_1}_{kk_1}V^{k_2+k_3}_{k_2k_3}\left[\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right]\\
&-\frac{1}{2}U_{-k-k_1kk_1}U_{-k_2-k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right]. \qquad (13)
\end{aligned}
$$

Let us denote this particular choice of coefficient $T^{kk_1}_{k_2k_3}$ as $\hat{T}^{kk_1}_{k_2k_3}$. It

$$\hat{T}^{kk_1}_{k_2k_3} = 0$$

on the manifold (7) while it diagonal part $\hat{T}^{kk_1}_{kk_1}$ which we denote as T_{kk_1} has the simple form:

$$\hat{T}_{kk_1} = \frac{1}{4\pi^2}|k||k_1|min(|k|,|k_1|).$$

Using this diagonal part one can construct the following function (with tilde):

$$
\begin{aligned}
\tilde{T}^{kk_1}_{k_2k_3} &= \left[\frac{1}{2}(T_{kk_2}+T_{kk_3}+T_{k_1k_2}+T_{k_1k_3})-\frac{1}{4}(T_{kk}+T_{k_1k_2}+T_{k_2k_2}+T_{k_3k_3})\right]\theta(kk_1k_2k_3),\\
\theta(x) &= \begin{cases}1, & \text{if } x\geq 0,\\ 0, & \text{if } x<0.\end{cases} \qquad (14)
\end{aligned}
$$

It is obvious that both diagonal parts of $\tilde{T}^{kk_1}_{k_2k_3}$ and $\hat{T}^{kk_1}_{k_2k_3}$ are the same:

$$\tilde{T}^{kk_1}_{kk_1} = \hat{T}^{kk_1}_{kk_1} = T_{kk_1}.$$

$\tilde{T}^{kk_1}_{kk_1}$ coincides with original four-wave coefficient on the resonant manifold (7) up to some canonical transformation like (12) with $\tilde{B}^{kk_1}_{k_2k_3}$ as follow

$$\tilde{B}^{kk_1}_{k_2k_3} = \frac{\tilde{T}^{kk_1}_{k_2k_3}-\hat{T}^{kk_1}_{k_2k_3}}{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}}.$$

Corresponding hamiltonian

$$\mathcal{H} = \int \omega_k b_k b^*_k dk + \frac{1}{4}\int \tilde{T}^{kk_1}_{k_2k_3} b^*_k b^*_{k_1} b_{k_2} b_{k_3}\delta_{k+k_1-k_2-k_3}dk\, dk_1\, dk_2\, dk_3$$

describes "quasi-integrable" system in a sense that four-wave interaction are absent, and only five and higher order are in force [27,28].

3.4 "Almost" 1-D water waves

Dynamic equation for b_k is

$$i\frac{\partial b_k}{\partial t} = \omega_k b_k + b_k \int \tilde{T}^{kk_1}_{k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3}\, dk\, dk_1\, dk_2\, dk_3. \tag{15}$$

Now let us consider slow dependence of b_k on transverse coordinate y. The most important contribution to transverse dynamics comes from linear term, so that

$$\omega_k b_k \to \hat{\omega}_k(y) b_k(y),$$

while in the nonlinear frequency shift (which is much smaller then linear frequency) dependence on y appears only in b_k:

$$i\frac{\partial b_k(y)}{\partial t} = \hat{\omega}_k(y) b_k + b_k(y) \sum_{k_1=-\infty}^{\infty} T_{kk_1} |b_{k_1}(y)|^2. \tag{16}$$

Here $\hat{\omega}_k(y)$ in k-space has obvious form:

$$\hat{\omega}_k(y) -> (k^2 + k_y^2)^{\frac{1}{4}}.$$

Equation (16) can be solved on the computer much more effectively due to drastic simplicity $\tilde{T}^{kk_1}_{k_2k_3}$ in comparison with original $T^{kk_1}_{k_2k_3}$ from (13) for Zakharov's equation.

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4 Supplement

$$V^{k_1}_{k_2k_3} = \frac{g^{\frac{1}{4}}}{2\sqrt{4\pi}}(|k_1|+|k_2|+|k_3|)\left(|k_1k_2k_3|^{\frac{1}{4}} + \left(\frac{k_1}{k_2k_3}\right)^{\frac{1}{4}}|k_1| - \left(\frac{k_2}{k_1k_3}\right)^{\frac{1}{4}}|k_2| - \left(\frac{k_3}{k_1k_2}\right)^{\frac{1}{4}}|k_3|\right).$$

$$U_{k_1k_2k_3} = \frac{g^{\frac{1}{4}}}{2\sqrt{4\pi}}(|k_1|+|k_2|+|k_3|)\left(|k_1k_2k_3|^{\frac{1}{4}} + \left(\frac{k_1}{k_2k_3}\right)^{\frac{1}{4}}|k_1| + \left(\frac{k_2}{k_1k_3}\right)^{\frac{1}{4}}|k_2| + \left(\frac{k_3}{k_1k_2}\right)^{\frac{1}{4}}|k_3|\right).$$

$$\begin{aligned} W^{k_1k_2}_{k_3k_4} = \frac{-1}{4\pi}\Bigg(&|\frac{k_1k_2}{k_3k_4}|^{\frac{1}{4}} M^{-k_1-k_2}_{k_3k_4} - \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}} M^{-k_1k_3}_{-k_2k_4} - |\frac{k_1k_4}{k_2k_3}|^{\frac{1}{4}} M^{-k_1k_4}_{k_3-k_2} \\ &- \left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}} M^{-k_2k_3}_{-k_1k_4} - \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}} M^{-k_2k_4}_{-k_1k_3} + \left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}} M^{k_3k_4}_{-k_1-k_2}\Bigg). \end{aligned} \tag{17}$$

here

$$\begin{aligned} M^{k_1k_2}_{k_3k_4} = &\left(|k_3|+|k_4|\right)(k_1k_2+|k_1k_2|) + \tfrac{1}{4}L_{-k_1k2}(|k_1+k_3|+|k_1+k_4|+|k_2+k_3|+|k_2+k_4|) \\ &+ \tfrac{1}{2}(|k_1|k_2(2k_2+k_3+k_4) + k_1|k_2|(2k_1+k_3+k_4)) \quad (18) \\ &+ |k_1|k_2(|k_3|\text{sign}(k_2+k_4) + |k_4|\text{sign}(k_2+k_3)) \quad (19) \\ &+ k_1|k_2|(|k_3|\text{sign}(k_1+k_4) + |k_4|\text{sign}(k_1+k_3))) \quad (20) \end{aligned}$$

and

$$L_{k_1k_2} = k_1k_2 + |k_1k_2|.$$

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