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On Energy Balance in Wind-Driven Seas[1]

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Abstract—We show that in the energy balance in the wind-driven sea the process of four-wave nonlinear interaction plays the leading role. This process surpasses competing mechanisms—input energy from wind and dissipation of energy due to white capping at least in order of magnitude. This result is supported by analytical calculations and numerical simulations.

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Since pioneering works of O. Phillips [1] we know that the main nonlinear process in the deep sea is four-wave resonant interaction. Four interacting waves form a quadruplet with wave vectors $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ obeying resonance conditions

$$\begin{aligned}\mathbf{k} + \mathbf{k}_1 &= \mathbf{k}_2 + \mathbf{k}_3,\\ \omega + \omega_1 &= \omega_2 + \omega_3.\end{aligned} \tag{1}$$

Here $\omega_k = \sqrt{g|\mathbf{k}|}$ —dispersion relation for deep water waves (g—gravity acceleration).

Statistical theory of four-wave interactions was developed by Klauss Hasselmann in 1962–1963 [2]. He showed that spatial spectrum of wave action $N_{\mathbf{k}}(t)$ obeys the kinetic (Hasselmann) equation

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}}\omega_{\mathbf{k}}\nabla_{\mathbf{r}}N_{\mathbf{k}} = S_{nl} + S_{in} + S_{diss}. \tag{2}$$

Subscripts $\mathbf{k}$, $\mathbf{r}$ for ∇ are used for gradients in wavevector $\mathbf{k}$ and coordinate $\mathbf{r}$ spaces correspondingly. For $N_{\mathbf{k}}(t)$ and $\omega_{\mathbf{k}}$ the subscript $\mathbf{k}$ means dependence on wavevector $\mathbf{k}$. In (2) the term S_{nl} is responsible for four-wave interactions. Terms S_{in} and S_{diss} describe correspondingly input of wave action from wind and its dissipation. Exact analytical expressions for S_{in}, S_{diss} are not known. Different authors offer different phenomenological forms for these terms. In particular, in the most updated operational models for wind wave forecasting the source functions of wave input and dissipation use quasi-linear dependencies on spectral density as follows [3]

$$S_{in} = \gamma_{in}N_{\mathbf{k}}, \tag{3}$$

$$S_{diss} = \gamma_{diss}N_{\mathbf{k}}. \tag{4}$$

The corresponding increment γ_{in} (as well as decrement γ_{diss}), generally, depends on a number of arguments, first of all, on wave frequency as a key wave characteristic and wind speed as a key parameter of wind-sea interaction. These arguments are introduced in conventional non-dimensional form following the Kitaigorodskii approach [4]. These are: ratio of wave frequency to frequency of spectral peak $\frac{\omega}{\omega_p}$ (or mean over spectrum frequency ω_m) and ratio of wave phase speed C_p to a characteristic wind speed U_h (wind speed at height h or friction velocity $u_* = \sqrt{\langle u'w'\rangle}$ with u', w' being velocities of turbulent pulsations)—the so-called wave age $\frac{C_p}{U_h}$.

An additional important non-dimensional parameter—wave steepness

$$\mu_p^2 = \frac{E\omega_p^4}{g^2}, \tag{5}$$

defined in terms of total energy

$$E = \int E(\mathbf{k})d\mathbf{k}, \tag{6}$$

is used quite often to describe a feedback of wind sea state on features of wind-sea interaction and wave dissipation. Evidently, dependence on this parameter implies non-linearity of wave input and dissipation. The dependencies on the three arguments $\frac{\omega}{\omega_p}$, $\frac{C_p}{U_h}$, μ_p are used as work-pieces for parameterizing wave input and dissipation in all the modern wave forecasting models (see for review [3])

$$\gamma_{in} = \omega G_{in}\left(\frac{\omega}{\omega_p}, \frac{C_p}{U_h}, \mu_p, \ldots\right), \tag{7}$$

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$$\gamma_{diss} = \omega G_{diss}\left(\frac{\omega}{\omega_p}, \frac{C_p}{U_h}, \mu_p, \ldots\right). \tag{8}$$

In contrast to the phenomenological parameterizations (7), (8) the nonlinear interaction term S_{nl} can be derived "from the first principles". According to [5] it is written as

$$S_{nl} = \pi g^2 \int |T_{0123}|^2 (N_1N_2N_3 + N_0N_2N_3 - N_0N_1N_2 - N_0N_1N_3)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \tag{9}$$

Here subscripts of N, T, ω are used as abbreviations of arguments $\mathbf{k}_i$. In (9) the kernel $T(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = T_{0123} = T_{1023} = T_{0132} = T_{2301}$ is a homogeneous function of order 3, invariant with respect to rotation. Collection of its explicit (and very complicated) expressions can be found in [6].

On the "rear face" of the energy (or action) spectrum time derivative $\frac{\partial N}{\partial t}$ and advection terms are not essential [3]. In this equilibrium region equation (2) can be reduced to the stationary equation

$$S_{nl} + S_{in} + S_{diss} = 0. \tag{10}$$

Since the well-known article by G. J. Komen, S. Hasselmann and K. Hasselmann [7] many in the oceanographic community believe that in the stationary equation (10) all three terms are of the same order of magnitude. Some authors think that the wave input term S_{in} and dissipation term S_{diss} almost compensate each other, while the S_{nl} term is of the secondary importance. In our articles [6, 8] we expressed completely opposite opinion and claimed that in the balance equation (10) S_{nl} is the leading term, thus, essential information can be acquired from the conservative stationary equation

$$S_{nl} = 0. \tag{11}$$

It was shown [9, 10] that this equation has a rich family of exact solutions, including KZ (Kolmogorov-Zakharov) isotropic spectra

$$N^{(1)}(\mathbf{k}) = C_p P^{1/3} g^{-2/3} \mathbf{k}^{-4}, \tag{12}$$

$$N^{(2)}(\mathbf{k}) = C_q Q^{1/3} g^{-1/2} \mathbf{k}^{-23/6}. \tag{13}$$

Here P is the energy flux to high wave numbers, Q is action flux to low wave numbers. Solutions (12), (13) successfully describe asymptotics of energy spectra behind the spectral peak [6]. Moreover, non-stationary and spatially non-homogeneous spectra of growing wind sea can be described quite well by self-similar solutions of an asymptotic model where S_{in} and S_{diss} are assumed to be formally small. The conservative Hasselmann equation gives the lowest-order approximation of the model and the corresponding self-similar solutions while the next order approximation provides explicit relationships for total energy, characteristic frequency and total net input $\langle S_{in} + S_{diss}\rangle$ [6, 8].

In this article we display resolved arguments in support of our viewpoint and explain why the analysis made in article [7] cited above and in later papers was erroneous.

In fact, S_{nl} is a complicated nonlinear operator which can be split into two parts

$$S_{nl} = F_{\mathbf{k}} - \Gamma_{\mathbf{k}} N_{\mathbf{k}}, \tag{14}$$

where

$$F_{\mathbf{k}} = \pi g^2 \int |T_{0123}|^2 N_1N_2N_3 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \tag{15}$$

$$\Gamma_{\mathbf{k}} = \pi g^2 \int |T_{0123}|^2 (N_1N_2 + N_1N_3 - N_2N_3) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \tag{16}$$

Nonlinear forcing $F_{\mathbf{k}}$ (15), evidently, is definitely positive value while $\Gamma_{\mathbf{k}}$ (16) can be negative in some frequency intervals. Nevertheless, we refer to $\Gamma_{\mathbf{k}}N_{\mathbf{k}}$ as to nonlinear damping, emphasizing that magnitudes of every constituent $F_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}}N_{\mathbf{k}}$ are much higher than S_{nl} itself. Two constituents of S_{nl} are almost annihilating each other that means very strong relaxation in (2) due to four-wave resonant processes. External forcing terms (S_{in}, S_{diss}) do not affect this relaxation essentially. We show that inequalities $F_{\mathbf{k}} \gg |S_{nl}|$ and $|\Gamma_{\mathbf{k}}N_{\mathbf{k}}| \gg |S_{nl}|$ are valid in a wide range of sea states. Hence, physically correct way to estimate importance of S_{nl} is a comparison wave input and dissipation S_{in}, S_{diss} with its constituents (15), (16) rather than with S_{nl} itself. In particular, "net increment" $\gamma_{\mathbf{k}} = \gamma_{in} - \gamma_{diss}$ should be compared to nonlinear damping rate $\Gamma_{\mathbf{k}}$ due to four-wave resonances. Simple theoretical and numerical estimates show that generally $\Gamma_{\mathbf{k}}$ is one order or more higher than $\gamma_{\mathbf{k}}$. Key arguments are given below while details can be found in [5].

A "naive" dimensional consideration of nonlinear damping decrement (16) gives

$$\Gamma(\mathbf{k}) \simeq \frac{4\pi g^2}{\omega_{\mathbf{k}}}|\mathbf{k}|^{10} N_{\mathbf{k}}^2. \tag{17}$$

However, this estimate works only near spectral peak, i.e. $|\mathbf{k}| \approx |\mathbf{k}_p|$ ($\mathbf{k}_p$ is the wave number of the spectral maximum).

Let $|\mathbf{k}| \gg |\mathbf{k}_p|$. Now for $\Gamma_{\mathbf{k}}$ one gets

$$\Gamma(\mathbf{k}) \approx 2\pi g^2 \int |T_{0103}|^2 \delta(\omega - \omega_3) N_1 N_3 d\mathbf{k}_1 d\mathbf{k}_3. \tag{18}$$

The main source of $\Gamma(\mathbf{k})$ is the interaction of long and short waves as it is shown below. To estimate integral (18) more accurately, we assume that the spectrum of long waves is narrow in angle, i.e. $N(|\mathbf{k}_1|, \Theta) = \tilde{N}(|\mathbf{k}_1|)\delta(\Theta)$. Long waves propagate along the axis x and $\mathbf{k}_1$ is the wave vector of short wave with direction Θ. For the coupling coefficient one has $T_{0101} \approx T_{0103} 2|\mathbf{k}_1|^2|\mathbf{k}_0|\cos\Theta$ and

$$\Gamma(\mathbf{k}) = 8\pi g^{3/2}|\mathbf{k}|^2\cos^2\Theta\int_0^\infty|\mathbf{k}_1|^{13/2}\tilde{N}(\mathbf{k}_1)^2 d|\mathbf{k}_1|. \quad (19)$$

Even for the most mildly decaying Kolmogorov-Zakharov (KZ) spectrum, $N(\mathbf{k}) \sim |\mathbf{k}|^{23/6}$ and the integral in (19) diverges in small wavenumbers. For steeper KZ spectra the divergence is stronger. Let us estimate $\Gamma(\mathbf{k})$ for the case of "mature sea", taking the spectrum in the form

$$N(\mathbf{k}) = \frac{3}{2}\frac{E}{\sqrt{g}}\frac{|\mathbf{k}_p|^{3/2}}{|\mathbf{k}|^4}H(\mathbf{k}-\mathbf{k}_p). \quad (20)$$

Here E is the total energy, $H(x)$ is the Heaviside step function. By plugging (20) to (19) one gets the equation

$$\Gamma(\mathbf{k}) = 36\pi\omega\left(\frac{\omega}{\omega_p}\right)^3\mu_p^4\cos^2\Theta, \quad (21)$$

that includes a huge "enhancing factor" $36\pi \approx 113.1$ at formally small steepness parameter μ_p. In the isotropic case, we need to perform one more integration over angle. It yields expression for $\Gamma(\mathbf{k})$ with smaller enhancing factor (cf. 21):

$$\Gamma(\mathbf{k}) = 22.5\pi\omega\left(\frac{\omega}{\omega_p}\right)^3\mu_p^4. \quad (22)$$

A representative estimate by Plant [11] gives increment in source function (3)

$$\gamma_{in} = 5.1\cdot 10^{-5}\omega\left(\frac{U_{10}\omega_p}{g}\right)^2\left(\frac{\omega}{\omega_p}\right)^2, \quad (23)$$

where U_{10} is wind speed at standard height 10 meters above the sea surface. Two independent parameters—steepness μ_p and wind speed U_{10} determine the answer on relative balance of wave generation and nonlinear transfer. For the ratio of the linear input (23) and nonlinear damping one gets from (21)

$$\frac{\Gamma_{\mathbf{k}}}{\gamma_{in}} \approx 2.26\cdot 10^5\left(\frac{\omega}{\omega_p}\right)\mu_p^4\left(\frac{U_{10}\omega_p}{g}\right)^{-2}. \quad (24)$$

Let for estimates in (24) the formally minimal value $\frac{\omega}{\omega_p} = 1$ и $\frac{U_{10}\omega_p}{g} = 2$. Even for the most aggressive wave input by Plant [11] and rather young wind sea the nonlinear damping appears to be stronger than wind input for rather quiet sea ($\mu_p > 0.0365$). More accurate comparison is given below (see Fig. 3).

For illustration of the above theoretical speculations we use the case of the cited milestone paper by Komen et al. [7]. For a quarter of a century the majority of the wind-wave community follows the stereotypes of the paper in his treatment of wind sea balance. The fully developed (mature) sea is considered as a result of competition of wind input and wave dissipation rather than a relaxation of wind sea to an inherent state due to effect of four-wave nonlinear resonant interactions.

For the simulation we used a novel numerical approach developed by N. N. Ivenskikh based on WRT-algorithm first proposed by Webb [12] and implemented by Tracy & Resio [13]. Note, that a number of numerical approaches for the Hasselmann equation (including the authentic one by Tracy & Resio code) hides the physically relevant decomposition of S_{nl} into nonlinear forcing $F_{\mathbf{k}}$ and nonlinear damping $\Gamma_{\mathbf{k}}N_{\mathbf{k}}$.

In our Fig. 1 we reproduce literally the case of Fig. 1 in [7] with an analogue of the Pierson-Moskowitz spectral distribution [14]

$$E_{PM}(f) = \frac{\alpha g^2}{(2\pi)^4 f^5}\exp\left[-\frac{5}{4}\left(\frac{f_{PM}}{f}\right)^4\right], \quad (25)$$

where Phillips' constant $\alpha = 0.0081$. Peak frequency

$$f_{PM} = \frac{0.13g}{U_{10}} \quad (26)$$

is determined by characteristic wind speed U_{10}. We skip definitions of other parameters that can be found in [7]. Figure 1 shows perfect agreement of our results with ones of [7]. Magnitudes of all the constituents of wind-wave balance S_{nl}, S_{in}, S_{diss} appear to be very close to each other. Note, that S_{in} was calculated basing on the well-known parameterization by Snyder et al. [3] while the dissipation term S_{diss} was estimated as a residual$-(S_{in} + S_{nl})$ to provide full balance in (10). Positiveness of dissipation term S_{diss} in Fig. 1d) shows inconsistency of the chosen setup [7]. Playing with Phillips' parameter α and a parameter in Snyder's formula authors of [7] reach reasonable agreement of S_{diss} with the white-capping model by Hasselmann [15] (dashed curve in Fig. 1d).

Decomposition of the collision integral S_{nl} into two parts in accordance with (14) gives an impressive result: the constituents of S_{nl} appear to be one order higher in magnitude as S_{nl} almost in the whole domain of simulation except rather narrow vicinity of spectral peak (see Fig. 2).

In terms of decrement of nonlinear damping $\Gamma_{\mathbf{k}}$ one has an extremely resolved result: $\Gamma_{\mathbf{k}}$ exceeds significantly all the conventional increments of wave growth in a wide range of non-dimensional frequencies. Figure 3 shows results of the comparison fairly well for down-wind direction ($\Theta = 0$). Theoretical dependence (21) gives a remarkable reference for estimates of decrement $\Gamma_{\mathbf{k}}$. Slight divergence of numerical results and the theoretical dependence is explained naturally by high-frequency tail ($E(\omega) \sim \omega^{-5}$) in the spectral distribution [7] used in our simulations.

We presented analytical and numerical arguments that support the fact of leading role of nonlinear transfer in balance of wind-driven seas. Our argumentation is based on a milestone case of fully developed sea introduced by Komen et al. (1984). Even in this

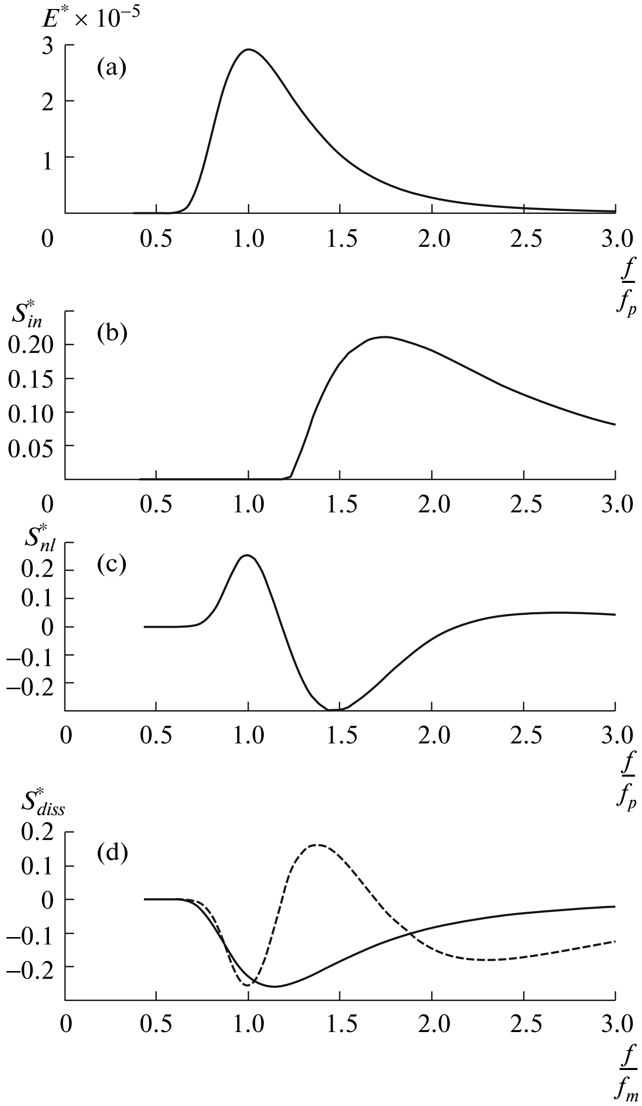

Fig. 1. Reproduction of Fig. 1 by Komen et al. [7]. Dependencies on non-dimensional frequencies for (a)—energy $E(f)$; (b)—wave input term S_{in} integrated over direction; (c)—collision integral S_{nl} integrated over direction; (d)—dissipation term S_{diss} derived from condition of full balance (10) (dashed line) and from white-capping parameterization [15]. All values are nondimensionalized in terms of u^* and g.

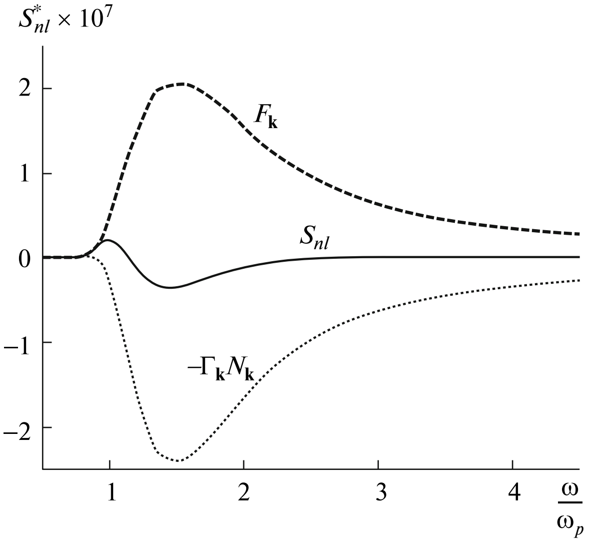

Fig. 2. Decomposition of the collision integral S_{nl} in Fig. 1c (solid line) for the case by Komen et al. [7] into nonlinear forcing (dashed) and damping (dotted) terms (see (13)–(15)).

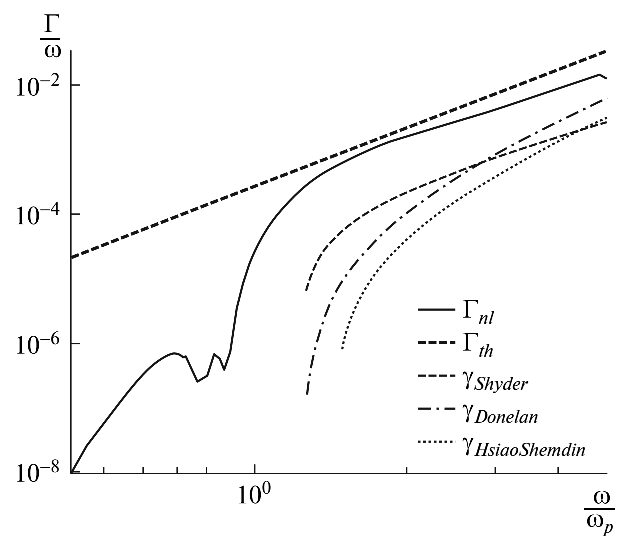

Fig. 3. Nonlinear damping coefficient $\Gamma_{\mathbf{k}}$ given by theoretical estimate (21) and by the numerical simulation (dashed and solid bold curves, correspondingly). Conventional dependencies of wind growth increments [3] are shown by thin curves with authors' names in legend.

extreme case when competition of wind input and wave dissipation is, evidently, responsible for appearance of a stationary state the constituents of S_{nl} exceed dramatically S_{in} and S_{diss}. One can suppose and then to show numerically that the leadership of nonlinear transfer will be more definite at earlier stages of wave development when waves are lower but essentially steeper.

Our hypothesis on dominant role of wave quartets composed by pairs of long and short waves in relaxation of wave spectra found its impressive confirmation: results of numerical simulation in Fig. 3 follow remarkably well the asymptotic dependence (21).

Our key result on dominating effect of nonlinear transfer on wave spectra evolution should not be interpreted as a call to ignore effects of wind input and wave dissipation. As we have formulated earlier in [8] the leadership of S_{nl} "does not mean that we disregard wind input and dissipation, we just put them into their proper place." The strong nonlinear forcing and damping that compose the conservative term S_{nl} determine strong relaxation and a universality of spectral shaping due to "inherent" wave dynamics while S_{in}, S_{diss} are responsible for growth of total energy.

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