

Compact Equation for Gravity Waves on Deep Water[†]

A. I. Dyachenko^{a, b,*} and V. E. Zakharov^{a-d}

^a Novosibirsk State University, Novosibirsk, 630090 Russia

^b Landau Institute for Theoretical Physics, Chernogolovka, Moscow region, 142432 Russia

* e-mail: alexd@itp.ac.ru

^c Department of Mathematics, University of Arizona, 857201 Tucson, AZ, USA

^d Lebedev Physical Institute, Russian Academy of Sciences, Moscow, 119991 Russia

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Using the fact of vanishing four waves interaction for water gravity waves for 2D potential fluid, we have significantly simplified the well-known but cumbersome Zakharov equation. The Hamiltonian of the obtained equation is very simple and includes only the fourth order nonlinear term. It raises the question of the integrability of free surface hydrodynamics. This new equation is very suitable both for analytic study and for numerical simulation.

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1. ZAKHAROV EQUATION

A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations:

$$\phi_{xx} + \phi_{zz} = 0 \quad (\phi_z \rightarrow 0, z \rightarrow -\infty),$$

$$\eta_t + \eta_x \phi_x = \phi_x|_{z=\eta},$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0|_{z=\eta},$$

where $\eta(x, t)$ is the shape of a surface, $\phi(x, z, t)$ is the potential function of the flow, and g is a gravitational acceleration. As was shown in [4], the variables $\eta(x, t)$ and $\psi(x, t) = \phi(x, z, t)|_{z=\eta}$ are canonically conjugated, and satisfy the equations

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$K = \frac{1}{2} \int dx \int_{-\infty}^{\eta} v^2 dz, \quad U = \frac{g}{2} \int \eta^2 dx.$$

It is convenient to introduce normal complex variable a_k :

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a_{-k}^*), \quad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a_{-k}^*),$$

where $\omega_k = \sqrt{gk}$ is the dispersion law for the gravity waves, and Fourier transformations $\psi(x) \rightarrow \psi_k$ and $\eta(x) \rightarrow \eta_k$ are defined as follows:

$$f_k = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int f_k e^{+ikx} dk.$$

The Hamiltonian can be expanded in an infinite series in powers of a_k (see [4, 5])

$$H = H_2 + H_3 + H_4 + \dots$$

This variable a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0,$$

where

$$H_2 = \int \omega_k a_k a_k^* dk,$$

$$H_3 = \int V_{k_1 k_2}^k \{ a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^* \} \delta_{k-k_1-k_2} dk dk_1 dk_2$$

$$+ \frac{1}{3} \int U_{kk_1 k_2} \{ a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^* \} \delta_{k+k_1+k_2} dk dk_1 dk_2,$$

$$V_{k_1 k_2}^k = \frac{g^{\frac{1}{4}}}{8\sqrt{\pi}} \left(|k|^{\frac{1}{4}} L_{k_1 k_2} - |k_2|^{\frac{1}{4}} L_{-kk_1} - |k_1|^{\frac{1}{4}} L_{-kk_2} \right),$$

$$U_{kk_1 k_2} = \frac{g^{\frac{1}{4}}}{8\sqrt{\pi}} \left(|k|^{\frac{1}{4}} L_{k_1 k_2} + |k_2|^{\frac{1}{4}} L_{kk_1} + |k_1|^{\frac{1}{4}} L_{kk_2} \right),$$

[†]The article is published in the original.

$$L_{kk_1} = \frac{1}{|k_1|^{\frac{1}{4}}} (k k_1 + |k| |k_1|).$$

The fourth order part of the Hamiltonian has the form

$$\begin{aligned} H_4 &= \frac{1}{2} \int W_{k_1 k_2}^{k_3 k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1 + k_2 - k_3 - k_4} dk_1 dk_2 dk_3 dk_4 \\ &\quad + \frac{1}{3} \int G_{k_1 k_2 k_3}^{k_4} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4} + \text{c.c.}) \\ &\quad \times \delta_{k_1 + k_2 + k_3 - k_4} dk_1 dk_2 dk_3 dk_4 \\ &\quad + \frac{1}{12} \int R_{k_1 k_2 k_3 k_4} (a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^* + \text{c.c.}) \\ &\quad \times \delta_{k_1 + k_2 + k_3 + k_4} dk_1 dk_2 dk_3 dk_4. \end{aligned}$$

Here,

$$\begin{aligned} W_{k_1 k_2}^{k_3 k_4} &= -\frac{1}{32\pi} [M_{-k_3 - k_4}^{k_1 k_2} + M_{k_1 k_2}^{-k_3 - k_4} - M_{k_2 - k_4}^{k_1 - k_3} \\ &\quad - M_{k_1 - k_4}^{k_2 - k_3} - M_{k_2 - k_3}^{k_1 - k_4} - M_{k_1 - k_3}^{k_2 - k_4}] \\ G_{k_1 k_2 k_3}^{k_4} &= -\frac{1}{32\pi} [M_{k_1 k_2}^{k_3 - k_4} + M_{k_1 k_2}^{k_2 - k_4} + M_{k_2 k_3}^{k_1 - k_4} \\ &\quad - M_{k_3 - k_4}^{k_1 k_2} - M_{k_2 - k_4}^{k_1 k_3} - M_{k_1 - k_4}^{k_2 k_3}] \\ R_{k_1 k_2 k_3 k_4} &= -\frac{1}{32\pi} [M_{k_1 k_2}^{k_3 k_4} + M_{k_1 k_3}^{k_2 k_4} + M_{k_1 k_4}^{k_2 k_3} \\ &\quad + M_{k_2 k_3}^{k_1 k_4} + M_{k_2 k_4}^{k_1 k_3} + M_{k_3 k_4}^{k_1 k_2}]. \end{aligned}$$

Here,

$$\begin{aligned} M_{k_1 k_2}^{k_3 k_4} &= |k_1 k_2|^{\frac{3}{4}} |k_3 k_4|^{\frac{1}{4}} (|k_1 + k_3| + |k_1 + k_4| \\ &\quad + |k_2 + k_3| + |k_2 + k_4| - 2|k_1| - 2|k_2|). \end{aligned}$$

Now one can apply canonical transformation from variables a_k to b_k to exclude non resonant cubic terms along with non resonant fourth order terms with coefficients $G_{k_1 k_2 k_3 k_4}^{k_4}$ and $R_{k_1 k_2 k_3 k_4}$. This transformation up to the accuracy $O(b^5)$ has the form [4, 6, 7]

$$\begin{aligned} a_k &= b_k + \int \Gamma_{k_1 k_2}^{k_3 k_4} b_{k_1} b_{k_2} \delta_{k_1 + k_2 - k_3 - k_4} dk_1 dk_2 \\ &\quad - 2 \int \Gamma_{k_1 k_2}^{k_2 k_3} b_{k_1}^* b_{k_2} \delta_{k_1 + k_2 - k_3} dk_1 dk_2 \\ &\quad + \int \Gamma_{k_1 k_2}^{k_3 k_4} b_{k_1}^* b_{k_2}^* \delta_{k_1 + k_2 + k_3} dk_1 dk_2 \\ &+ \int B_{k_1 k_2}^{k_3 k_4} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k_1 + k_2 - k_3} dk_1 dk_2 dk_3 \\ &+ \int C_{k_1 k_2}^{k_3 k_4} b_{k_1}^* b_{k_2}^* b_{k_3}^* \delta_{k_1 + k_2 + k_3} dk_1 dk_2 dk_3 \end{aligned} \tag{1}$$

$$+ \int S_{k_1 k_2 k_3} b_{k_1}^* b_{k_2}^* b_{k_3}^* \delta_{k_1 + k_2 + k_3} dk_1 dk_2 dk_3.$$

Here,

$$\begin{aligned} B_{k_1 k_2}^{k_3 k_4} &= \Gamma_{k_1 - k_2}^{k_3} \Gamma_{k_3 - k}^{k_4} + \Gamma_{k_3 k_1 - k_3}^{k_1} \Gamma_{k_2 - k}^{k_2} \\ &\quad - \Gamma_{k_2 k - k_2}^{k_1} \Gamma_{k_1 k_3 - k_1}^{k_3} - \Gamma_{k_3 k_1 - k_3}^{k_1} \Gamma_{k_1 k_2 - k_1}^{k_2} \\ &\quad - \Gamma_{k_1 k_1}^{k+k} \Gamma_{k_2 k_3}^{k_2+k_3} + \Gamma_{-k-k_1 k_1} \Gamma_{-k_2 - k_3 k_2} + \tilde{B}_{k_2 k_3}^{k k_1}, \end{aligned}$$

$$\Gamma_{k_1 k_2}^k = -\frac{V_{k_1 k_2}^k}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \quad \Gamma_{k_1 k_2} = -\frac{U_{k_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}},$$

$\Gamma_{k_1 k_2}^k$ and $\Gamma_{k_1 k_2}$ provides vanishing of cubic terms in the new Hamiltonian and $\tilde{B}_{k_2 k_3}^{k k_1}$ is an arbitrary function satisfying the following symmetry conditions:

$$\tilde{B}_{k_2 k_3}^{k k_1} = \tilde{B}_{k_3 k_2}^{k_1 k} = \tilde{B}_{k_3 k_2}^{k k_1} = -(\tilde{B}_{k_1 k_1}^{k_2 k_3})^*.$$

$\tilde{B}_{k_3 k_2}^{k k_1}$ manages 2 ↔ 2 coefficient in the new Hamiltonian. Coefficients $C_{k_1 k_2}^{k_3 k_4}$ and $S_{k_1 k_2 k_3}^{k_4}$ provide vanishing 3 ↔ 1 and 4 ↔ 0 terms in the Hamiltonian.

After transformation (1) the Hamiltonian acquires the following form:

$$\begin{aligned} H &= \int \omega_k b_k b_k^* dk + \frac{1}{2} \int [T_{k k_1}^{k_2 k_3} - (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \\ &\quad \times \tilde{B}_{k k_1}^{k_2 k_3}] b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots \end{aligned} \tag{2}$$

If $\tilde{B}_{k k_1}^{k_2 k_3} = 0$, equation (2) is known as the Zakharov equation. Here, $T_{k k_1, k_2 k_3}^{k_2 k_3}$ satisfies the symmetry conditions

$$T_{k k_1}^{k_2 k_3} = T_{k_1 k}^{k_2 k_3} = T_{k k_1}^{k_3 k_2} = T_{k_2 k_3}^{k k_1}$$

and has the form

$$\begin{aligned} T_{k k_1}^{k_2 k_3} &= W_{k_1 k}^{k_2 k_3} \\ &\quad - \left[\frac{V_{k k_2 k - k_2} V_{k_3 k_1 k_3 - k_1}}{\omega_{k_2} + \omega_{k - k_2} - \omega_k} + \frac{V_{k k_2 k - k_2} V_{k_3 k_1 k_3 - k_1}}{\omega_{k_1} + \omega_{k_3 - k_1} - \omega_{k_3}} \right] \\ &\quad - \left[\frac{V_{k_1 k_2 k_1 - k_2} V_{k_3 k k_3 - k}}{\omega_{k_2} + \omega_{k_1 - k_2} - \omega_{k_1}} + \frac{V_{k_1 k_2 k_1 - k_2} V_{k_3 k k_3 - k}}{\omega_k + \omega_{k_3 - k} - \omega_{k_3}} \right] \\ &\quad - \left[\frac{V_{k k_3 k - k_3} V_{k_2 k_1 k_2 - k_1}}{\omega_{k_3} + \omega_{k - k_3} - \omega_k} + \frac{V_{k k_3 k - k_3} V_{k_2 k_1 k_2 - k_1}}{\omega_{k_1} + \omega_{k_2 - k_1} - \omega_{k_2}} \right] \\ &\quad - \left[\frac{V_{k_1 k_3 k_1 - k_3} V_{k_2 k_2 k_2 - k}}{\omega_{k_3} + \omega_{k_1 - k_3} - \omega_{k_1}} + \frac{V_{k_1 k_3 k_1 - k_3} V_{k_2 k_2 k_2 - k}}{\omega_k + \omega_{k_2 - k} - \omega_{k_2}} \right] \end{aligned} \tag{3}$$

$$-\left[\frac{V_{k+k_1 k k_1} V_{k_2+k_3 k_2 k_3}}{\omega_{k+k_1} + \omega_k - \omega_{k_1}} + \frac{V_{k+k_1 k k_1} V_{k_2+k_3 k_2 k_3}}{\omega_{k_2+k_3} - \omega_{k_2} - \omega_{k_3}} \right] \\ - \left[\frac{U_{-k-k_1 k k_1} U_{-k_2-k_3 k_2 k_3}}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} + \frac{U_{-k-k_1 k k_1} U_{-k_2-k_3 k_2 k_3}}{\omega_{k_2+k_3} + \omega_{k_2} + \omega_{k_3}} \right].$$

At this moment we approach to the key point of this article. Namely, appropriate choice of $\tilde{B}_{k_2 k_3}^{k k_1}$ in (2) is based on two things:

1. The coefficient $T_{k k_1}^{k_2 k_3}$ is identically equal to zero on the resonant manifold [1]:

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}, \end{aligned} \quad (4)$$

with nontrivial solution:

$$\begin{aligned} k &= a(1 + \zeta)^2, \\ k_1 &= a(1 + \zeta)^2 \zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1 + \zeta + \zeta^2)^2; \end{aligned} \quad (5)$$

where $0 < \zeta < 1$ and $a > 0$

2. We consider waves moving in the same direction; this allows us to consider only positive wave-numbers k . This assumption came from numerical simulations [2, 3].

This fact and that observation allow drastically simplify Hamiltonian. One can vanish cumbersome expression for $T_{k k_1}^{k_2 k_3}$ in (2) keeping only its diagonal part. This diagonal part corresponds to trivial four-wave scattering

$$k_2 = k_1, \quad k_3 = k, \quad \text{or} \quad k_2 = k, \quad k_3 = k_1. \quad (6)$$

It is equal to

$$T_{k k_1} = T_{k k_1}^{k k_1} = \frac{1}{4\pi} |k| |k_1| (|k + k_1| - |k - k_1|). \quad (7)$$

Using this diagonal part, one can construct the following function (with tilde):

$$\tilde{T}_{k_2 k_3}^{k k_1} = \left[\frac{1}{2} (T_{k k_2} + T_{k k_3} + T_{k_1 k_2} + T_{k_1 k_3}) \right. \\ \left. - \frac{1}{4} (T_{k k} + T_{k_1 k_1} + T_{k_2 k_2} + T_{k_3 k_3}) \right] \theta(k k_1 k_2 k_3), \quad (8)$$

$$\theta(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x > 0. \end{cases}$$

This function was introduced in our work [8], but only now we discovered possibility of essential simplification of the Zakharov equation and its Hamiltonian.

Let us consider the case when all waves move in the same direction. It means that all $\tilde{T}_{k_2 k_3}^{k k_1}$ have the same sing. Thus, let $k_i > 0$, and

$$b = e^{i(kx - \omega t)}.$$

Then $\tilde{T}_{k_2 k_3}^{k k_1}$ can be significantly simplified, *modulus for $|k + k_1|$ along with $|k|$ and $|k_1|$ can be dropped*. Now

$$\hat{T}_{k k_1} = \frac{1}{4\pi} k k_1 (k + k_1 - |k - k_1|).$$

Simple calculations end up with

$$\begin{aligned} \tilde{T}_{k_2 k_3}^{k k_1} &= \frac{1}{8\pi} [k k_1 (k + k_1) + k_2 k_3 (k_2 + k_3)] \\ &\quad - \frac{1}{8\pi} (k k_2 |k - k_2| + k k_3 |k - k_3| \\ &\quad + k_1 k_2 |k_1 - k_2| + k_1 k_3 |k_1 - k_3|). \end{aligned} \quad (9)$$

2. COMPACT EQUATION

Let us make choice for $\tilde{B}_{k_2 k_3}^{k k_1}$ as follows:

$$\tilde{B}_{k_2 k_3}^{k k_1} = \frac{\tilde{T}_{k_2 k_3}^{k k_1} - \tilde{T}_{k_2 k_3}^{k k_1} \theta(k k_1 k_2 k_3)}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}. \quad (10)$$

It makes four-wave coefficient in (2) equal to $\tilde{T}_{k_2 k_3}^{k k_1}$. Note that expression (10) for $\tilde{B}_{k_2 k_3}^{k k_1}$ has no singularity on resonance manifold. Using following relations for \hat{K} and space derivative

$$k b_k^* \Leftrightarrow i \frac{\partial}{\partial x} b^*(x), \quad (11)$$

$$k b_k \Leftrightarrow -i \frac{\partial}{\partial x} b(x), \quad (12)$$

$$|k - k_2| b_k^* b_{k_2} \Leftrightarrow \hat{K}(|b(x)|^2). \quad (13)$$

Hamiltonian can be easily written in X -space:

$$\begin{aligned} \mathcal{H} &= \int b^* \hat{\omega} b dx + \frac{i}{16} \int \left[b^{*2} \frac{\partial}{\partial x} (b^2) - b^2 \frac{\partial}{\partial x} (b^{*2}) \right] dx \\ &\quad - \frac{1}{4} \int |b|^2 \hat{K}(|b'|^2) dx. \end{aligned} \quad (14)$$

After integrating by parts Hamiltonian acquires very nice form:

$$\begin{aligned} \mathcal{H} &= \int b^* \hat{\omega} b dx \\ &+ \frac{1}{4} \int |b'|^2 \left[\frac{i}{2} (bb'^* - b^* b') - \hat{K}|b|^2 \right] dx. \end{aligned} \quad (15)$$

This is the main result of the article. Corresponding equation of motion is the following:

$$\begin{aligned} i \frac{\partial b}{\partial t} &= \hat{\omega} b + \frac{i}{8} \left[b^* \frac{\partial}{\partial x} \left(b'^2 \right) - \frac{\partial}{\partial x} \left(b^{*'} \frac{\partial}{\partial x} b^2 \right) \right] \\ &- \frac{1}{4} \left[b \hat{K}(|b|^2) - \frac{\partial}{\partial x} [b' \hat{K}(|b|^2)] \right]. \end{aligned} \quad (16)$$

Along with usual quantities such as energy and both momenta equation (16) conserves action or number of waves:

$$N = \int |b|^2 dx.$$

3. SOME SOLUTIONS

3.1. Monochromatic Wave

Monochromatic wave with arbitrary amplitude B_0

$$b(x) = B_0 e^{i(k_0 x - \omega_0 t)} \quad (17)$$

is the simplest solution of (16). Indeed, substituting Eq. (17) into Eq. (16), one can get the relation

$$\omega_0 = \omega_{k_0} + \frac{1}{2} k_0^3 |B_0|^2. \quad (18)$$

Recalling transformation from a_k to b_k one can see that for waves with small amplitude ($a_k \approx b_k$)

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0} \eta_0^2,$$

and relation (18) coincides with well known Stokes correction to the frequency due to finite wave amplitude

$$\omega_0 = \omega_{k_0} \left(1 + \frac{1}{2} k_0^2 |\eta_0|^2 \right). \quad (19)$$

3.2. Modulational Instability of Monochromatic Wave

We consider perturbation to the solution

$$b = B_0 e^{i(k_0 x - \omega_0 t)},$$

where

$$B_0 = \frac{1}{\sqrt{2\pi}} \int b_{k_0} e^{i(k_0 x - kx)} dx$$

and

$$\omega_0 = \omega_{k_0} + \frac{1}{2} |B_0|^2 k_0^3, \quad \frac{1}{4\pi} |b_{k_0}|^2 k_0^3 = \frac{1}{2} T_{k_0 k_0}^{|b_{k_0}|^2}.$$

Perturbed solution has the form

$$b \Rightarrow (b_{k_0} + \delta b_{k_0+k} e^{-i\Omega_k t} + \delta b_{k_0-k} e^{-i\Omega_{-k} t}) e^{-i\omega_0 t} \quad (20)$$

where

$$\Omega_k = -\Omega_{-k}.$$

Suppose δb_{k_0+k} grows as

$$\delta b_{k_0+k} \Rightarrow \delta b_{k_0+k} e^{\gamma_k t}$$

one can easily obtain the following formula for γ_k :

$$\begin{aligned} \gamma_k^2 &= \left[-d(k) - \frac{3|B_0|^2}{4} k_0 k^2 \right] \\ &\times \left[d(k) + |B_0|^2 k_0 \left(k_0 - \frac{|k|}{2} \right)^2 \right]. \end{aligned} \quad (21)$$

If we introduce steepness of the carrier wave $\omega_{k_0} \mu^2 = |B_0|^2 k_0^2$ and approximate $d(k)$ as

$$d(k) \approx -\frac{1}{8} \omega_{k_0}'' k^2 = -\frac{1}{8} \omega_{k_0} \frac{k^2}{k_0^2},$$

then growth rate is equal to:

$$\gamma_k^2 = \frac{1}{8} \frac{\omega_{k_0}^2}{k_0^4} (1 - 6\mu^2) k^2 \left[\mu^2 \left(k_0 - \frac{|k|}{2} \right)^2 - \frac{k^2}{8} \right].$$

This expression for growth rate is more accurate than usually derived from nonlinear Schrödinger equation. The difference is seen from two terms marked with boldface.

3.3. Breathers

Equation for amplitude of the wave train can be easily derived from (16). Let us introduce envelope $B(x, t)$ so that

$$b(x, t) = B(x, t) e^{i(k_0 x - \omega_0 t)}. \quad (22)$$

$B(x, t)$ is also normal Hamiltonian variable and Hamiltonian for it is the following:

$$\begin{aligned} \mathcal{H} &= \int B^* (\hat{\omega}_{k_0+k} - \omega_{k_0}) B dx \\ &+ \frac{1}{4} \int |B' + ik_0 B|^2 \end{aligned} \quad (23)$$

$$\times \left[\frac{i}{2} (B(B'^* - ik_0 B^*) - B^*(B' + ik_0 B)) - \hat{K}|B|^2 \right] dx.$$

Breather is the solution equation with Hamiltonian (23) in the following form:

$$B(x, t) = B(x - Vt)e^{-i\Omega t}. \quad (24)$$

V is close to linear group velocity.

4. CONCLUSIONS

Simple equation describing evolution of 1D water waves is derived. Derivation of this equation is based on the important property of vanishing four-wave interaction for gravity water waves. This property allows us to simplify drastically well-known Zakharov equation for water waves, which is very cumbersome. Written in X -space instead of K -space, it allows further analytical and numerical study. Simple Hamiltonian which is obtained after canonical transformation raises the question about integrability of the equations for potential flow of fluid in the gravity field. It remains still open.

This new equation can be generalized for the “almost” 2D waves, or “almost” 3D fluid. When considering waves slightly inhomogeneous in transverse direction, one can think in the spirit of Kadomtsev–Petviashvili equation for Korteweg-de-Vries equation, namely one can treat now frequency ω_k as two dimensional, ω_{k_x, k_y} , while leaving coefficient $\tilde{T}_{k_2 k_3}^{kk_1}$ in (9) not dependent on y . Now b depends on both x and y :

$$\begin{aligned} \mathcal{H} = & \int b^* \hat{\omega}_{k_x, k_y} b dx dy \\ & + \frac{1}{4} \int |b'_x|^2 \left[\frac{i}{2} (bb'^*_x - b^*b'_x) - \hat{K}_x |b|^2 \right] dx dy. \end{aligned} \quad (25)$$

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