A dynamic equation for water waves in one horizontal dimension

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**ABSTRACT**

We apply a canonical transformation to a water wave equation to remove cubic nonlinear terms and to drastically simplify fourth-order terms in the Hamiltonian. This transformation explicitly uses the vanishing exact four-wave interaction for water gravity waves for a 2D potential fluid. After transformation, the well-known but cumbersome Zakharov equation is drastically simplified and can be written in $X$-space in a compact form. This new equation is very suitable for analytical studies and numerical simulations.

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1. Introduction

The work described here was motivated by two remarkable facts regarding the hydrodynamics of a one-dimensional free surface:

- We previously showed that the four-wave interaction coefficient vanishes on the resonant manifold [1]
  \[ k + k_1 = k_2 + k_3, \]
  \[ \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \]

- We have also demonstrated that a giant breather that is highly nonlinear exists on the fluid surface in the absence of radiation [2,3]. Moreover, the space–time spectrum of the breather consists of waves propagating in the same direction.

These two facts indicate that fourth-order wave interactions can be drastically simplified by some canonical transformation of the Hamiltonian. Below we show the form of this transformation. The dynamic equation derived using this transformation is very elegant and simple and can easily be generalized for “almost one-dimensional waves.”

2. Zakharov’s equation

The one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field is described by the following set of equations:

\[ \phi_{xx} + \phi_{zz} = 0 \quad (\phi \rightarrow 0, z \rightarrow -\infty), \]
\[ \eta_t + \eta \phi_x = \phi_z |_{z=\eta}, \]
\[ \phi_t + \gamma (\phi_x^2 + \phi_z^2) + g \eta = 0 |_{z=\eta}, \]

where $\eta(x, t)$ is the shape of the surface, $\phi(x, z, t)$ is the potential function for the flow and $g$ is gravitational acceleration. As previously shown [4], the variables $\eta(x, t)$ and $\psi(x, t) = \phi(x, z, t)|_{z=\eta}$ are canonically conjugated and satisfy the equations

\[ \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \]

where $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

\[ K = \frac{1}{2} \int dx \int_{-\infty}^{\eta} v^2 dz \quad U = \frac{g}{2} \int \eta^2 dx. \]

It is convenient to introduce a normal complex variable $a_k$:

\[ \eta_k = \sqrt{\frac{g}{2\omega_k}} (a_k + a_k^*), \quad \psi_k = -i \sqrt{\frac{g}{2\omega_k}} (a_k - a_k^*). \]
where \( \omega_k = \sqrt{gk} \) is the dispersion law for the gravity waves, and the Fourier transformations \( \psi(x) \rightarrow \psi_k \) and \( \eta(x) \rightarrow \eta_k \) are defined as follows:

\[
f_k = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} \, dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int f_k e^{ikx} \, dk.
\]

The Hamiltonian can be expanded in an infinite series in powers of \( a_k \) [4,5]:

\[ H = H_2 + H_3 + H_4 + \cdots \]

This variable \( a_k \) satisfies the equation

\[
\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a^*_k} = 0,
\]

where

\[
H_2 = \int \omega_k a_k a_k^* \, dk,
\]

\[
H_3 = \int \left[ \sum_{k_1,k_2} \left( a_k a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^* \right) \delta_{k_1+k_2-k} \right] \, dk_1 dk_2 dk_3 dk_4.
\]

The fourth-order part of Hamiltonian is:

\[
H_4 = \frac{1}{2} \int \left[ \sum_{k_1,k_2,k_3,k_4} \left( a_k a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^* \right) \delta_{k_1+k_2+k_3+k_4-k} \right] \, dk_1 dk_2 dk_3 dk_4.
\]

Now we can apply canonical transformation from variables \( a_k \) to \( b_k \) to exclude nonresonant cubic terms and nonresonant fourth-order terms with coefficients \( C_{k_1,k_2,k_3} \) and \( R_{k_1,k_2,k_3,k_4} \). This transformation up to accuracy \( O(b^5) \) has the form [4,6,7]:

\[
a_k = b_k + \int \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_k \, dk_1 dk_2 - 2 \int \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} \, dk_1 dk_2 + \int \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} \, dk_1 \, dk_2 + \int \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} \, dk_1 \, dk_2 + \int \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} \, dk_1 \, dk_2,
\]

where

\[
b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} = \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} + \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} + \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2} + \Gamma_{k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_1} b_{k_2}.
\]

and \( \tilde{b}_{k_1} \) is an arbitrary function satisfying the following symmetry conditions:

\[
\tilde{b}_{k_1}^* = -\tilde{b}_{k_1}^* = -\tilde{b}_{k_1}^*.
\]

Coefficients \( C_{k_1,k_2,k_3} \) and \( S_{k_1,k_2,k_3,k_4} \) provide vanishing corresponding fourth-order terms in the new Hamiltonian.

After transformation of (2.1) the Hamiltonian takes the following form:

\[
H = \int \omega_k b_k b_k^* \, dk + \frac{1}{2} \int \left( T_{k_1,k_2,k_3} - (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tilde{b}_{k_1}^* b_{k_1} b_{k_1}^* b_{k_1} \right) \, dk_1 \, dk_2 \, dk_3 \, dk_4.
\]

If \( \tilde{b}_{k_1}^* = 0 \), Eq. (2.2) is known as Zakharchov’s equation. Here \( T_{k_1,k_2,k_3} \) satisfies the symmetry conditions:

\[
T_{k_1,k_2,k_3} = T_{k_1,k_2,k_3}, \quad T_{k_1,k_2,k_3} = T_{k_1,k_2,k_3}, \quad T_{k_1,k_2,k_3} = T_{k_1,k_2,k_3},
\]

and has the form:

\[
T_{k_1,k_2,k_3} = \left[ W_{k_1,k_2,k_3,k_4} - V_{k_1,k_2,k_3,k_4} \right] \left[ \frac{1}{\omega_k} + \frac{1}{\omega_{k_2} + \omega_{k_1} - \omega_k} \right] - V_{k_1,k_2,k_3,k_4} \left[ \frac{1}{\omega_k} + \frac{1}{\omega_{k_2} + \omega_{k_1} - \omega_k} \right] \times \left( \frac{1}{\omega_k} + \frac{1}{\omega_{k_2} + \omega_{k_1} - \omega_k} \right).
\]
of kcorrespondstotrivialfour-wavescattering: Tthe Hamiltonian. We can remove the cumbersome expression T
\[ T_{k_i k_j} = \frac{1}{\omega_{k_i + k_j} - \omega_{k_i} - \omega_{k_j}} \]

Simple calculations yield
\[ T_{k_i k_j} = \frac{1}{\omega_{k_i + k_j} - \omega_{k_i} - \omega_{k_j}} - U_{k_i k_j} U_{k_i k_j} \]

Now we address the key point of this article: appropriate choice of $B_{k_i k_j}$ in (2.2) is based on two items:

1. The coefficient $T_{k_i k_j}$ is identically zero for the resonant manifold [1]:
\[ k + k_i = k + k_j, \]
\[ \omega_k + \omega_{k_i} = \omega_k + \omega_{k_j}, \]
with nontrivial solution:
\[ k = a(1 + \zeta)^2, \]
\[ k_i = a(1 + \zeta)^2 \zeta^2, \]
\[ k_j = -a \zeta^2, \]
\[ k_3 = a(1 + \zeta + \zeta^2)^2, \]
where $0 < \zeta < 1$ and $a > 0$. This is the only nontrivial solution for (2.4).

2. We also consider waves moving in the same direction, which allows us to consider only positive wavenumbers $k_i$. This assumption came from numerical simulations [2,3].

This fact and observation allow a drastic simplification of the Hamiltonian. We can remove the cumbersome expression for $T_{k_i k_j}$ in (2.2) and retain only its diagonal part, which corresponds to trivial four-wave scattering:
\[ k_2 = k_1, \quad k_3 = k, \quad k_2 = k, \quad k_3 = k. \]

This is equal to
\[ T_{k_1 k_2} = \frac{1}{4\pi} |k||k_1| (|k + k_1| - |k - k_1|). \]

Using this diagonal part, we can construct the following function (with tilde):
\[ \tilde{T}_{k_1 k_2} = \left[ \frac{1}{2} (T_{k_1 k_2} + T_{k_2 k_1} + T_{k_1 k_2} + T_{k_1 k_2}) \right] \theta(k) \theta(k_1) \theta(k_2) \theta(k_3). \]

\[ \theta(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x > 0. \end{cases} \]

Consider the case in which all the waves move in the same direction. This means that all $k$ values have the same sign. Thus, let $k_i > 0$, and
\[ b = e^{i(kx - \omega t)}. \]

Then $\tilde{T}_{k_1 k_2}$ can be significantly simplified, and the modulus for $|k + k_1|, |k|$, and $|k_1|$ can be dropped. Now
\[ T_{k_1 k_2} = \frac{1}{4\pi} \Re \left( k(k_1 + k - |k - k_1|) \right). \]

Simple calculations yield
\[ \tilde{T}_{k_1 k_2} = \left[ -\frac{1}{8\pi} (kk_2 |k - k_2| + kk_3 |k - k_3| + k_1 k_2 |k - k_2| + k_1 k_3 |k - k_3|) + \frac{1}{8\pi} \Re \left( k(k_1 + k - |k - k_1|) + k_2 k_3 (k_2 + k_3) \right) \right] \times \theta(k) \theta(k_1) \theta(k_2) \theta(k_3). \]

\[ \text{3. Compact equation} \]

We choose $B_{k_i k_j}$ as follows:
\[ B_{k_i k_j} = \frac{\tilde{T}_{k_i k_j}}{k_{k_i k_j} - k_{k_i k_j}}. \]

This makes the four-wave coefficient in (2.2) equal to $\tilde{T}_{k_1 k_2}$. Note that Eq. (3.11) for $\tilde{T}_{k_1 k_2}$ has no singularity for the resonance manifold (2.4) because $T_{k_i k_j}$ vanishes on that manifold. Using the following relations for $\hat{k}$ and the space derivative,
\[ \hat{k} \theta \Leftrightarrow \frac{\partial}{\partial x} b(x), \]
\[ \hat{k} \omega \Leftrightarrow -i \frac{\partial}{\partial x} b(x), \]
\[ |k_1 - k_2 B_{k_2 k_3} \Leftrightarrow \hat{k} ((b(x))^2), \]

the Hamiltonian can easily be written in $X$-space:
\[ \mathcal{H} = \int b^* g(\hat{k}) b dx + \frac{i}{16} \int \left[ b^2 \frac{\partial}{\partial x} (b^2) - b^2 \frac{\partial}{\partial x} (b^* b^2) \right] dx \]
\[ - \frac{1}{4} \int \left| b^2 \cdot \hat{k} ((b^2)^2) \right| dx. \]

The corresponding equation of motion is:
\[ \frac{\partial}{\partial t} = (g(\hat{k}))^2 b + \frac{i}{8} \left[ \left( b \frac{\partial}{\partial x} (b^2) - \frac{\partial}{\partial x} (b^* b^2) \right) \right] \]
\[ - \frac{1}{4} \left[ b \cdot \hat{k} ((b^2)^2) - \frac{\partial}{\partial x} (b^2 \hat{k} ((b^2)^2)) \right]. \]

\[ \text{4. Some solutions} \]

\[ \text{4.1. Monochromatic wave} \]

A monochromatic wave
\[ b(x) = B_0 e^{i(k_0 x - \omega_0 t)} \]

is the simplest solution of (3.17). Indeed, substituting (4.18) into Eq. (3.17) yields the following relation:
\[ \omega_0 = \omega_k + \frac{1}{2} k_0^2 |B_0|^2. \]

Recalling the transformation from $a_k$ to $b_k$, we can see that for waves with small amplitude ($a_k \simeq b_k$)
\[ |B_0|^2 = \frac{\omega b_0}{k_0^2} \eta_0^2 \]

and relation (4.19) coincides with the well-known Stokes correction to the frequency due to finite wave amplitude.
\[ \omega_0 = \omega_b \left( 1 + \frac{1}{2} k_0^2 |\eta_0|^2 \right). \]
4.2. Modulational instability of a monochromatic wave

We consider a perturbation to the solution

\[ b = B_0 e^{i(k_0 x - \omega_0 t)}, \]

where

\[ B_0 = \frac{1}{\sqrt{2 \pi}} \int b_{k_0} e^{i(k_0 x - k_1 t)} \, dx \]

and

\[ \omega_0 = \omega_{k_0} + \frac{1}{2} |B_0|^2 k_0^3. \]

The perturbed solution has the following form:

\[ b \Rightarrow (b_{k_0} + \delta b_{k_0+k} e^{-i\omega_0 t} + \delta b_{k_0-k} e^{-i\omega_0 t}) e^{-ik_0 t}. \]

If we introduce the steepness of the carrier wave \( \omega_{k_0} \mu^2 = |B_0|^2 k_0^2 \) and approximate \( d(k) \) as

\[ d(k) \approx -\frac{1}{8} \omega_{k_0}^2 k^2 = -\frac{1}{8} \omega_{k_0}^2 \frac{k^2}{k_0^2}, \]

then the growth rate is

\[ \gamma_k^2 = \frac{1}{8} \omega_{k_0}^2 (1 - 6 \mu^2) k^2 \left( \mu^2 \left( k_0 - \frac{|k|}{2} \right)^2 - \frac{k^2}{4} \right). \]

The difference between this formula and the well-known expression derived from the nonlinear Schrödinger equation is highlighted by the two terms in bold.

4.3. Breathers

An equation for the amplitude of the wave train can easily be derived from (3.17). We introduce envelope \( B(x, t) \) so that

\[ b(x, t) = B(x, t) e^{i(k_0 x - \omega_0 t)}. \]

Then the Hamiltonian can be written as:

\[ H = \int B^* (\hat{\omega}_{k_0+k} - \omega_{k_0}) B dx + \frac{1}{4} \int |B'|^2 \]

\[ \times \left[ \frac{i}{2} (B(B^* - ik_0 B') - B^*(B' + ik_0 B)) - \hat{K}|B|^2 \right] dx, \]

where the operator \( \hat{\omega}_{k_0+k} \) acts in \( k \)-space as \( \sqrt{g(k_0 + k)} \).

It is convenient to introduce the following operator \( \hat{S} \):

\[ \hat{S}B = \frac{\partial}{\partial x} B + ik_0 B. \]

Then the Hamiltonian can be written as:

\[ H = \int B^* (\hat{\omega}_{k_0+k} - \omega_{k_0}) B dx + \frac{1}{4} \int |\hat{S}B|^2 \]

\[ \times \left[ \frac{i}{2} (B(\hat{S}B)^* - B^* \hat{S}B) - \hat{K}|B|^2 \right] dx. \]

A variation of the Hamiltonian (4.26) dynamic equation for \( B(x, t) \) is:

\[ iB = (\hat{\omega}_{k_0+k} - \omega_{k_0}) B - \frac{1}{4} \hat{S} \left[ \frac{i}{2} (B(\hat{S}B)^* - B^* \hat{S}B) - \hat{K}|B|^2 \right] \hat{S}B \]

\[ - \frac{i}{8} (\hat{S}(|\hat{S}B|^2 B) + |\hat{S}B|^2 \hat{S}B) - \frac{1}{4} B \hat{K} |\hat{S}B|^2. \]

The breather is the solution of (4.27) in the following form:

\[ B(x, t) = B(x - Vt) e^{-i\Omega t}. \]

It satisfies the following equation:

\[ -i VB + \Omega B = (\hat{\omega}_{k_0+k} - \omega_{k_0}) B \]

\[ - \frac{1}{4} \hat{S} \left[ \frac{i}{2} (B(\hat{S}B)^* - B^* \hat{S}B) - \hat{K}|B|^2 \right] \hat{S}B \]

\[ - \frac{i}{8} (\hat{S}(|\hat{S}B|^2 B) + |\hat{S}B|^2 \hat{S}B) - \frac{1}{4} B \hat{K} |\hat{S}B|^2. \]

\[ V \text{ is close to the linear group velocity.} \]

The solution to (4.29) can be found numerically and is a generalization of the well-known soliton solution for the nonlinear Schrödinger equation. Indeed, for very small steepness we can neglect the derivative in the nonlinear terms, set \( V = \omega_{k_0}' \) and obtain a stationary NLS equation:

\[ \Omega B = -\omega_{k_0}' B'' + \frac{1}{2} |B|^2 B. \]

Taking into account first derivatives and the operator \( \hat{K} \) in the nonlinear term, we can easily obtain the Dysthe equation.
5. Conclusion

A simple equation describing the evolution of 1D water waves is derived based on the important property of a vanishing four-wave interaction for gravity water waves. This property allows drastic simplification of the well-known Zakharov equation for water waves, which is very cumbersome. When written in \(X\)-space instead of \(K\)-space, the equation allows further analytical and numerical study. The simple Hamiltonian obtained after canonical transformation raises a question about the integrability of equations for the potential flow of a fluid in a gravity field. This question remains open.

This new equation can be generalized for “almost 2D waves” or “almost 3D fluid”. Waves that are slightly inhomogeneous in the transverse direction can be considered in the spirit of the Kadomtsev–Petviashvili equation for the Korteweg–de-Vries equation: the frequency \(\omega_k\) can be treated as 2D, \(\omega_{k_x, k_y}\), while leaving the coefficient \(\tilde{T}_{kk}^{(1)}\) in (2.10) not dependent on \(y\). \(b\) now depends on both \(x\) and \(y\):

\[
\mathcal{H} = \int b^* \hat{\omega}_{k_x,k_y} b dx dy + \frac{1}{4} \int |b^*_y|^2 \]
\[
\times \left[ \frac{i}{2} (bb^*_y - b^*b_y^*) - \tilde{K}_y |b|^2 \right] dx dy.
\]  
(5.31)

For the pure 1D case, this is much more applicable than the nonlinear Schrödinger equation or the Dysthe equation. Note that it does not contain multiple integrations in Fourier space and is written in coordinate space.

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