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# Statistics of Rogue Waves in Computer Experiments

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Rogue waves have been studied in the exact simulation of complete hydrodynamic equations for an ideal fluid with a free surface. The statistical characteristics of rogue waves such as the occurrence intensity, average existence time, and maximum energy dissipation at collapse have been obtained in computer experiments.

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Surface waves of anomalously large amplitudes, so-called rogue waves, are actively studied in oceanology, theoretical physics, and mathematics (see, e.g., [1–9]) mainly as a nonlinear effect in the hydrodynamics of an ideal fluid with a free surface. The probabilities of the occurrence of rogue waves for various parameters of the initial waves were estimated in our preceding computer experiments [10] on the simulation of the dynamics of surface waves with exact equations describing the hydrodynamics of an ideal potential fluid with a free surface. In that work, we used the equations free of dissipation and pumping. Collapses of waves during the experiments distorted the statistics of rogue waves because an experiment in this case was terminated prematurely.

The equations used in this work to describe the surface waves of an ideal fluid include both energy dissipation and pumping. These equations make it possible to maintain a given energy level in the experiments. Dissipation corresponds to energy loss because of the collapse of waves; for this reason, the experiments can be continued for any given time. Energy pumping is chosen such that waves travel in one direction.

We consider the dynamics of an ideal fluid with a free surface in two-dimensional geometry with an infinite bottom: $0 < x < 2\pi$, $-\infty < y < \eta(x, t)$. The boundary conditions at the ends of the interval $x = [0, 2\pi]$ are periodic.

The flow is assumed to be potential and the fluid is incompressible:

$$v = \nabla\phi, \quad \operatorname{div} v = 0.$$

Thus, the potential satisfies the Laplace equation

$$\Delta\phi = 0.$$

We perform the conformal mapping of the region occupied by the fluid onto the lower half-plane where the coordinates are specified as $w = u + iv$. This mapping is given by the function $z = z(w)$, $z = x + iy$.

The dynamic equations formulated in terms of the Dyachenko variables

$$R = 1/z'_w, \quad V = i\partial\Phi/\partial z$$

have the form

$$\begin{gathered} \dot{R}(u,t) = i(UR' - U'R) + F_R[R, V] - \alpha R'''', \\ \dot{V}(u,t) = i(UV' - B'R) + g(R-1) + F_V[R, V] - \alpha V'''', \\ U = P(VR^* + RV^*), \\ B = P(VV^*). \end{gathered} \tag{1}$$

Here, $P = (1 + iH)/2$ is the projection operator on the lower half-space, where $H$ is an analog of the Hilbert operator for the periodic case, which is specified as

$$H[f](u) = \frac{1}{2\pi} vp \int_0^{2\pi} \frac{f(u')}{\tan[(u'-u)/2]} du'.$$

The linear integral operators $F_R$ and $F_V$ correspond to pumping and have the following form in terms of Fourier transforms:

$$F_R[R, V]_k = -\beta_k (r_k - i\sqrt{k/g}\, v_k),$$

$$F_V[R, V]_k = \beta_k (r_k - i\sqrt{k/g}\, v_k).$$

Here,

$$\beta_k = \begin{cases} \dfrac{k-(K_0-D_F)}{D_F}, & K_0 - D_F < k \le K_0; \\ \dfrac{(K_0+D_F)-k}{D_F}, & K_0 < k < K_0 + D_F, \end{cases}$$

where $K_0$ is the wavenumber and $D_F$ is the spectral width of the pumping. The fourth derivatives in Eq. (1) correspond to dissipation. The pumping is switched on when the energy is below a given level and is switched off when the energy is above the given level.

The physical meaning of the dissipative terms and pumping operators in Eq. (1) is as follows. The collapse of waves is the main reason for the premature termination of a computer experiment. To avoid the termination, we use only dissipative terms suppressing high harmonics, which can be interpreted as collapse-induced dissipation. Pumping operators increase the energy of waves traveling in one direction, which corresponds to the chosen initial data. The pumping to the spectral peak makes it possible to increase the energy of the main waves considered in the experiment.

The system of equations (1) has become popular recently because of its properties very appropriate for theoretical and numerical analyses. The mathematical results concerning the solvability of this system, as well as the methods for its numerical solution, were reported in [11–14].

Our computer experiments basically correspond to the experiments described in [10]. An ensemble of waves traveling in one direction is initially specified. The average wavenumber is $K_0 = 25$.

The initial perturbation of the surface is specified by the sum of harmonics with random phases:

$$\eta_0(x) = \sum_{-\frac{1}{2}K_{\max}}^{\frac{1}{2}K_{\max}} \phi(k-K_0)\cos(kx-\xi_k). \quad (2)$$

Here, $K_{\max}$ is the total number of spectral modes and $\xi_k$ is a random variable uniformly distributed in the range $-K_{\max}/2 < k < K_{\max}/2$.

It is assumed that the initial potentials of velocities are related to Eq. (2) according to the linear theory.

The function $\phi(k)$ has the form

$$\phi(k) = \begin{cases} \delta_k, & |k| > K_w; \\ \kappa\exp(-\alpha k^2) + \delta_k, & |k| \le K_w. \end{cases} \quad (3)$$

Here, $\delta_k$ are independent random parameters uniformly distributed in the range $-K_{\max}/2 < k < K_{\max}/2$.

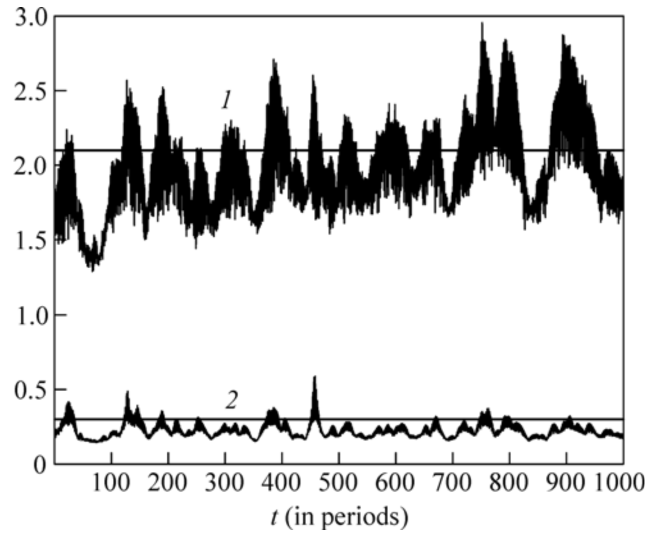


**Fig. 1.** Functions (*1*) $\nu(t)$ and (*2*) $\mu(t)$.

The spectral parameters $1 \le K_w \le 10$, $\kappa$, and $\alpha$ are specified so as to ensure given values of the average slope $\mu$ whose square is

$$\mu^2 = \frac{1}{2\pi}\int_0^{2\pi}\eta_x^2 dx,$$

and variance

$$D = \left(\int_{-K_w}^{K_w} k^2 e^{-\alpha k^2} dk\right)\left(\int_{-K_w}^{K_w} e^{-\alpha k^2} dk\right)^{-1}.$$

The contribution of random noise to the energy is less than three percent.

Each individual experiment is performed at $0 \le t \le 400$, which approximately corresponds to 1000 wave periods. The total number of harmonics in the calculations is $K_{\max} = 2048$. The spectral width of the pumping is $D_F = 15$ and the dissipation coefficient is $\alpha = 10^{-9}$.

Rogue waves were detected according to the standard amplitude criterion, which means that the following conditions are simultaneously satisfied at a certain value $t = t^*$:

$$\nu(t^*) = \frac{H_{\max}(t^*)}{H_s(t)} \ge 2.1,$$

$$\mu(t^*) = \max_{0<x<2\pi}|\eta_x(x, t^*)| \ge 0.3,$$

where $H_s(t)$ is the significant height of waves and $H_{\max}(t)$ is the maximum amplitude of a wave at time $t$.

The results of a typical computer experiment are shown in Fig. 1, where the functions $\nu(t)$ and $\mu(t)$ are plotted and the critical lines $\nu = 2.1$ and $\mu = 0.3$.

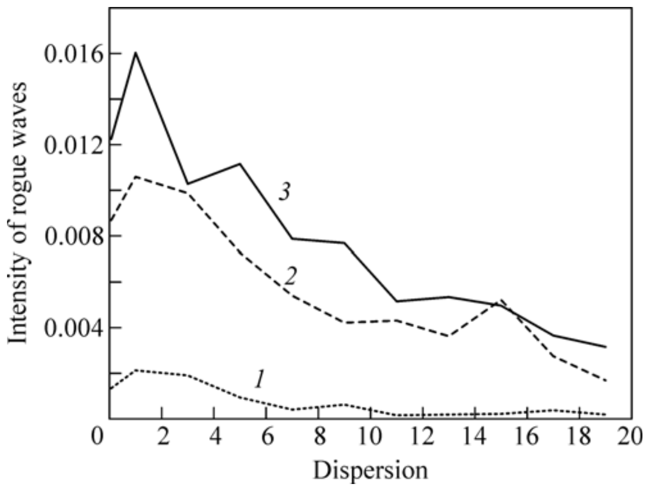


**Fig. 2.** Intensity of the occurrence of rogue waves for $\mu^2 =$ (*1*) $2.06 \times 10^{-3}$, (*2*) $3.08 \times 10^{-3}$, and (*3*) $4.10 \times 10^{-3}$.

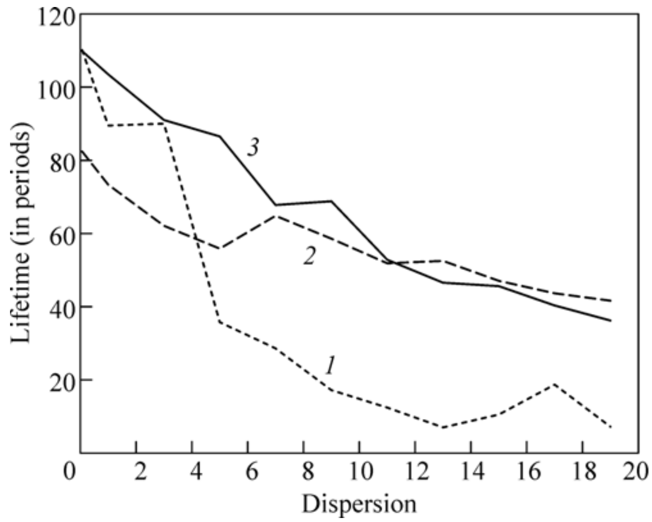


**Fig. 3.** Average existence time of rogue waves for $\mu^2 =$ (*1*) $2.06 \times 10^{-3}$, (*2*) $3.08 \times 10^{-3}$, and (*3*) $4.10 \times 10^{-3}$.

A rogue wave in an individual computer experiment can appear several times. For this reason, we use the stochastic model of the occurrence of rogue waves in the form of the Poisson distribution

$$P(n, T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}.$$

This distribution is the probability of the occurrence of $n$ rogue waves in $T$ periods. Figure 2 shows the parameter $\lambda$, which has the meaning of the intensity of the occurrence of rogue waves for various parameters of initial waves. According to Little's formula, the average number of rogue waves appearing in $T$ periods is

$$N^* = \lambda T.$$

Natural experiments aimed at detecting rogue waves near the southern shore of Sakhalin Island were reported in [15]. The Poisson distribution was also used in those experiments. The estimates obtained in them qualitatively coincide with our results.

The existence time of rogue waves was estimated in computer experiments. Figure 3 shows the average lifetimes (in periods) of rogue waves.

The occurrence of rogue waves is usually accompanied by the collapse of a wave, which is manifested in sharp energy dissipation. Figure 4 shows the dependences of the maximum energy dissipation (in percent) on the initial parameters at the collapse of a rogue wave.

The proposed modification of the Dyachenko equations including pumping and dissipation makes it possible to perform computer experiments, on one hand, stable to the collapse of waves and, on the other hand, with maintenance of a given energy level in the system. Using these experiments based on the complete nonlinear equations, we constructed the Poisson distribution for the occurrence of rogue waves. The obtained statistic certainly depends on the parameters of the spectra of initial waves, in particular, on the number of individual waves in the wave train under consideration. However, our results confirm the stochastic character of the occurrence of rogue waves, which was mentioned in many works, and make it possible to obtain other characteristics of rogue waves.

The intensity (in the sense of a Poisson process) of the occurrence of rogue waves decreases with an increase in dispersion (Fig. 2) in agreement with the conclusions made in other works devoted to the study of rogue waves. The average existence time of a rogue

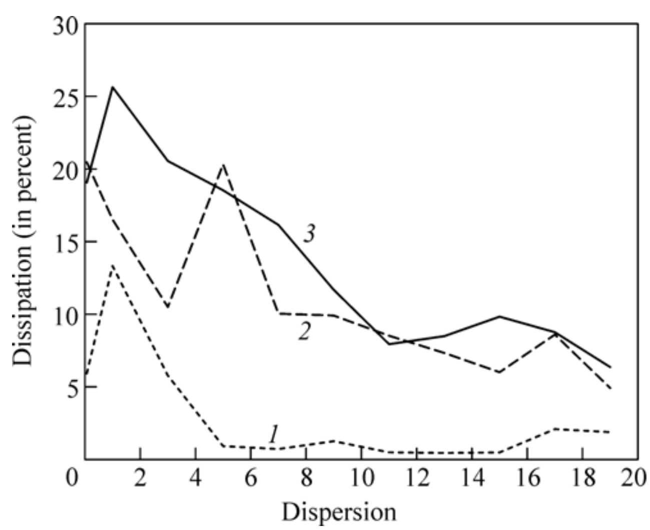


**Fig. 4.** Maximum energy dissipation at the collapse of a rogue wave for $\mu^2 =$ (*1*) $2.06 \times 10^{-3}$, (*2*) $3.08 \times 10^{-3}$, and (*3*) $4.10 \times 10^{-3}$.

wave (Fig. 3) also decreases with an increase in dispersion. It is worth noting that the existence time of a rogue wave at the minimum slope in the case of a narrow spectrum is close to values at a larger slope and this time decreases sharply with an increase in the width of the spectrum.

The maximum energy dissipations at the collapse of a rogue wave shown in Fig. 4 indicate that this wave can contain a significant part of the energy of the entire system. Performing a numerical experiment, A.I. Smirnova calculated the total energy of each individual wave and concluded that a typical rogue wave contains about 20% of the energy of waves.

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