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Energy Portrait of Rogue Waves

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Processes of the concentration of energy at the formation of rogue waves have been studied in computer experiments based on the exact hydrodynamic equations for an ideal fluid. The distribution of anomalies of waves both in height and in energy has been found in the computer experiment. Correlation between the energy concentration and height of anomalously large surface waves has been revealed. The results can be used to estimate the danger of anomalously large surface waves.

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1. INTRODUCTION

Rogue waves are sudden single surface waves with large heights (up to 30 m). Their suddenness and high amplitude pose serious danger to marine facilities. Studies of these waves began in the last decades. Any general theory of extreme waves is still absent. Rogue waves are both of applied and theoretical interest [1, 2].

Computer experimental studies of rogue waves have become most popular in recent years [3–8]. The statistics of rogue waves appearing through the nonlinear dynamics of surface waves of an ideal fluid were studied in numerical experiments reported in our preceding works [9, 10]. The processes of concentration of energy and momentum are quantitatively estimated in the computer experiments in this work. The capabilities of the numerical simulation of rogue waves make it possible "to look into" the processes responsible for their formation.

2. COMPUTER EXPERIMENTS

In our computer experiments, we considered waves traveling in one direction, which corresponds to ripples in the ocean. The experiments were based on the numerical solution of the Euler equation for an ideal fluid with a free surface and infinite depth in two-dimensional geometry:

$$-\infty < y < \eta(x, t),$$
$$0 < x < 2\pi.$$

The boundary conditions in the variable x were 2π periodic. It was assumed that a flow is potential and the fluid is incompressible. Therefore,

$$v(x, y, t) = \nabla\Phi(x, y, t), \quad \operatorname{div} v = 0.$$

Thus, the potential of the velocity field of the fluid Φ satisfies the Laplace equation

$$\Delta\Phi(x, y, t) = 0.$$

For numerical calculations, we used the equations in conformal variables obtained in [11]. Mathematical problems of the correctness of these equations, as well as problems of numerical calculations, were considered in [12–14]. The equations in conformal variables make it possible to perform the necessary computer experiments with a high accuracy in long time intervals and to observe the appearance of rogue waves.

We performed the conformal mapping of the region occupied by the fluid onto the lower half-plane, where the coordinates are specified as $w = u + iv$. This mapping is specified by the function $z = z(w)$, $z = x + iy$.

The dynamic equations written in the Dyachenko variables

$$R = \frac{1}{z'_w}, \quad V = i\frac{\partial\Phi}{\partial z}$$

have the form

$$\dot{R}(u, t) = i(UR' - U'R) - \alpha R'''',$$
$$\dot{V}(u, t) = i(UV' - B'R) + g(R - 1) - \alpha V'''' + F,$$

$$U = P(V\bar{R} + \bar{V}R),$$

$$B = P(V\bar{V}).$$

Here, $P = \frac{1}{2}(1 + iH)$ is the projection operator on the lower half-plane, where H is an analog of the Hilbert operator for the periodic case, which is given by the expression

$$H[f](u) = \frac{1}{2\pi}\text{v.p.}\int_0^{2\pi}\frac{f(u')}{\tan[(u'-u)/2]}du'.$$

In these equations, dissipation was used to take into account the possibility of the collapse of waves in the course of evolution, as well as energy pumping. Energy pumping served for the maintenance of the same energy during the entire experiment and was switched on when the energy of the system decreased below the necessary level. This dissipation was also used in our preceding work [10]. The form of pumping was modified in order to make it physically meaningful. Pumping has the meaning of a surface force that acts on the free surface and is proportional to the slope of the surface:

$$F = F_w\left|\frac{dy}{dx}\right|.$$

The coefficients α and F_w were chosen empirically so as to maintain a given energy level in the system. The use of dissipation and pumping allows calculations in an almost unlimited time interval.

Rogue waves are usually identified by the amplitude criterion

$$\nu(t) = \frac{H_{\max}(t)}{H_s(t)} \geq \nu^* = 2.1,$$

where $H_{\max}$ is the maximum height and H_s is the significant height of waves (average of one-third of the highest waves). The critical value $\nu^* = 2.1$ was chosen empirically and was used in many studies of rogue waves [1].

3. ENERGY CHARACTERISTICS OF INDIVIDUAL WAVES

In our computer experiments, we considered a train of waves periodic in the spatial variable. In this case, the profile of the free surface is specified by the function $y = y(x, t)$, which is 2π-periodic in the variable x. The region between two local minima of the free surface is called an individual wave.

The energy E (kinetic energy T and potential energy U), as well as the absolute value of the momentum I (absolute values of the horizontal, I_x, and vertical, I_y, projections of the momentum), can be calculated for each wave by hydrodynamic formulas. In addition, we also calculated the geometric characteristics of individual waves (amplitude A, steepness M, and curvature C).

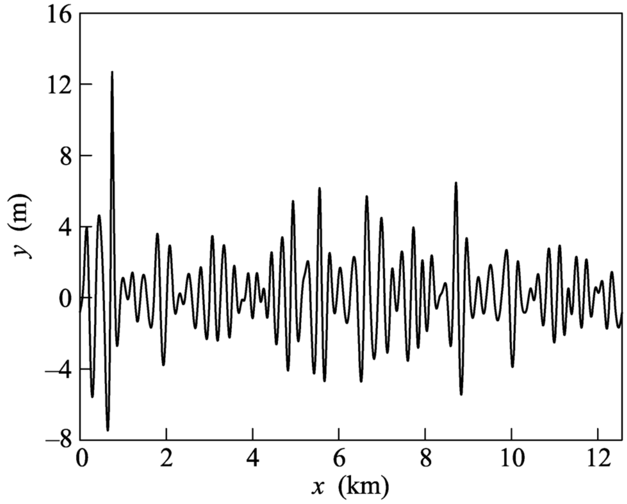

Fig. 1. Profile of a typical rogue wave.

For each characteristic, we calculated the ratio of its maximum value to the average value for all waves in the train at a fixed time; this ratio is called the concentration of this characteristic:

$$C_X = \frac{X_{\max}}{\frac{1}{N}\sum_{i=1}^{N}X_i},$$

where X_i ($i = 1, \ldots, N$) are the characteristics of individual waves and $X_{\max} = \max_i X_i$.

4. EXAMPLE OF THE FORMATION OF A ROGUE WAVE

Numerous computer experiments on the simulation of the nonlinear dynamics of surface waves were performed in our previous works [9, 10] in order to examine the statistics of the occurrence of rogue waves. When extreme waves are formed, the processes of concentration of energy and momentum are observed. Concentration often occurs according to the same scenario.

We consider below the characteristic example of the formation of a rogue wave. In this experiment, the initial field of waves consisted of a train of 53 waves traveling in one direction. The square of average steepness was $\mu^2 = 3.72 \times 10^{-3}$. The duration of the experiment was approximately 3525 periods. A rogue wave appeared after 1412 periods. The profile of the free surface at this time is shown in Fig. 1. The functional ν reached the value

$$\nu(t^*) = 2.26.$$

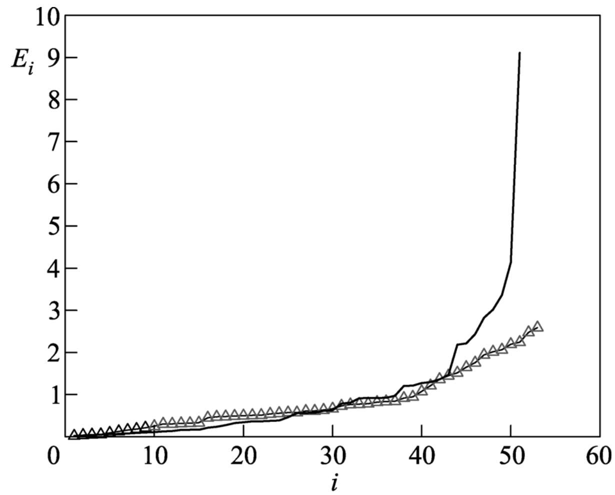

Fig. 2. Energy concentration at the (thin line with triangles) initial time and (thick line) time of formation of the rogue wave.

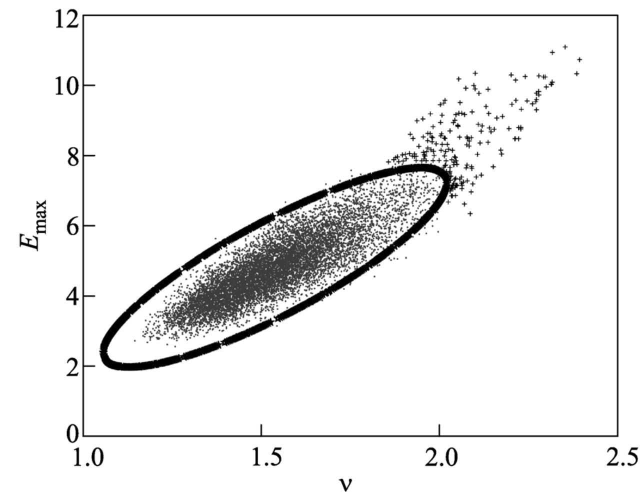

Fig. 3. Joint distribution of the parameters ν_{max} and $C_{E_{max}}$.

The concentration of the total energy at the time of formation of the rogue wave is shown in Fig. 2 together with the energy concentration at the initial time. It can be seen that the distribution of the concentration at the initial time is almost linear, whereas the largest fraction of the energy at the time of formation of the rogue wave is concentrated in the largest wave. The concentration of the energy is $C_E = 9.1$. A similar picture is also characteristic of the momentum of the rogue wave.

We now consider the joint two-dimensional distribution of the parameter ν and the concentration of the total energy C_E for the largest wave at each time of the computer experiment. In the overwhelming majority (about 98%) of realizations, ν_{max} and $C_{E_{max}}$ satisfy normal probability distributions: $\nu_{max} \sim N(1.54, 0.026)$ and $C_{E_{max}} \sim N(4.8, 0.895)$. The coefficient of correlation between these quantities is $r = 0.85$.

We plot the scattering ellipse for this example, which has the eccentricity $e = 0.985$ and ellipticity $k = 0.029$ (Fig. 3). The points inside the ellipse correspond to $\{\nu_{max}, C_{E_{max}}\}$ pairs satisfying a normal two-dimensional distribution. The points outside the ellipse correspond to $\{\nu_{max}, C_{E_{max}}\}$ pairs that do not satisfy a normal distribution and correspond to anomalous waves having either an anomalously high height or an anomalously high energy. It is noteworthy that their deviations only toward higher values are observed.

Figure 4 shows the histogram of the average values of the maximum concentrations of the energy and momentum at the time of appearance of the rogue wave according to the results of a large series of computer experiments with various parameters of the initial waves.

It is also possible to construct a regression correlation between the concentrations of the height and energy of a rogue wave. In the results of numerous computer experiments, we determined the concentrations of the height and energy for rogue waves at the time when the parameter ν for them was maximal. The resulting regression correlation is shown in Fig. 5. The linear regression equation has the form $y = 4.1x - 4.4$. The linear determination coefficient is $R^2 = 0.834$.

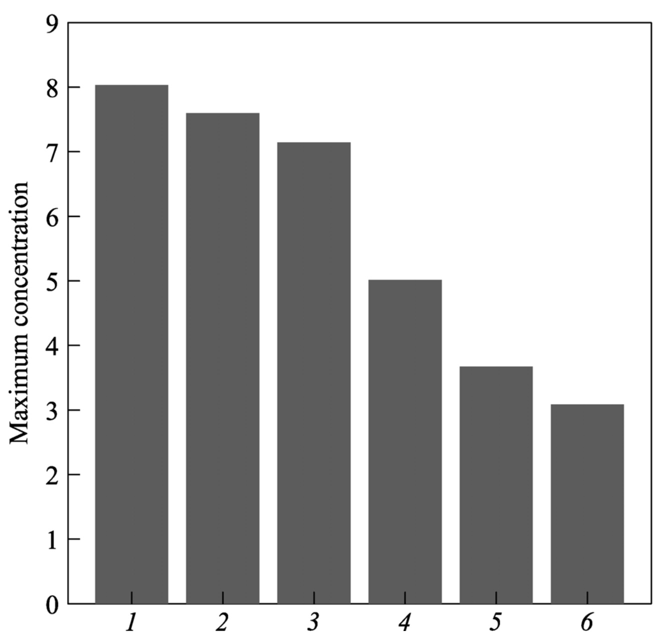

Fig. 4. Average maximum concentrations of the (*1*) kinetic energy, (*2*) total energy, (*3*) potential energy, (*4*) vertical momentum component, (*5*) horizontal momentum component, and (*6*) absolute value of the momentum.

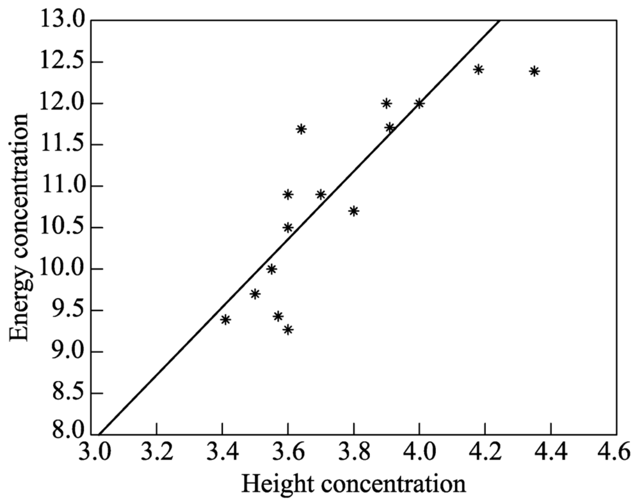

Fig. 5. Regression correlation between the concentrations of the height and energy of rogue waves.

5. CONCLUSIONS

In summary, nonlinear processes of local concentration of the energy at the time of formation of extreme surface waves, i.e., rogue waves, have been considered. Quantitative estimates of these processes have been obtained. It has been shown that a single anomalously large wave can concentrate the energy an order of magnitude higher than the average energy of neighboring waves.

Significant correlation between the amplitude criterion of the rogue wave and energy concentration has been demonstrated. This allows a new insight into the definition of a rogue wave.

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