Kolmogorov spectrum in a system of nonlinear oscillators

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The problem of the stochastic behavior of a system of nonlinearly interacting oscillators arises in various branches of physics. If these oscillators are natural oscillations (traveling waves, say) in a slightly nonlinear medium, we speak of weak turbulence (see ref. 1, for example).

Weak turbulence can be described by kinetic equations for the correlation functions. As has been shown in refs. 2-4, these equations have exact, stationary, power-law solutions of Kolmogorov type corresponding to constancy of the energy fluxes or the number of quasiparticles in k space. There is still the question of the conditions for realization of these spectra and the nature of their buildup. This important problem yields to analytic investigation with difficulty so that the use of a digital computer is natural. The present paper contains an exposition of a numerical experiment on the kinetics of weak turbulence for a model system of nonlinear oscillators.

We examine a system of oscillators with the Hamiltonian

$$\mathcal{H} = \sum_n \omega_n a_n a_n^* + \sum_{n,n_1,n_2} V_{n,n_1,n_2}(a_n^* a_{n_1} a_{n_2} + a_n a_{n_1}^* a_{n_2}^*); \qquad (1)$$

$\omega_n = \lambda n$ is the frequency of the oscillators and a_n are complex normal variables.

The following equation of motion for the a_n:

$$\frac{\partial a_n}{\partial t} + i\omega_n a_n = -i \sum_{n_1,n_2} (V_{n n_1 n_2} a_{n_1} a_{n_2} + 2V_{n_1,n,n_2} a_{n_1} a_{n_2}^*). \qquad (2)$$

corresponds to this Hamiltonian.

Taking the average of (2) under the assumption of a small nonlinearity results in a kinetic equation for $N_n = \langle |a_n|^2 \rangle$:

$$\frac{\partial N_n}{\partial t} = 2\pi \sum_{n_1 n_2} |V_{n n_1 n_2}|^2 \{(N_{n_1} N_{n_2} - N_n N_{n_2} - N_n N_{n_1})\,\delta(\omega_n - \omega_{n_1} - \omega_{n_2})$$
$$- 2(N_{n_1} N_{n_2} + N_n N_{n_2} - N_n N_{n_1})\,\delta(\omega_{n_1} - \omega_n - \omega_{n_2})\}. \qquad (3)$$

For a large number of oscillators they can be considered arranged continuously and it is possible to go from summation over n to integration with respect to ω:

$$\frac{dN_\omega}{dt} = \frac{2\pi}{\lambda^2} \int_0^\omega |V_{\omega\omega_1\omega_2}|^2 \{(N_{\omega_1} N_{\omega_2} - N_\omega N_{\omega_1} - N_\omega N_{\omega_2})\,\delta(\omega - \omega_1 - \omega_2)$$
$$+ 2(N_{\omega_1} N_{\omega_2} + N_\omega N_{\omega_1} - N_\omega N_{\omega_2})\,\delta(\omega + \omega_1 - \omega_2)\}\, d\omega_1 d\omega_2. \qquad (4)$$

The kinetic equation (4) has a form completely analogous to the corresponding equation for an isotropic two- or three-dimensional medium after taking the average with respect to the angles. Meanwhile, the original model (1) contains so much smaller a number of oscillators than its equivalent two- or three-dimensional models [the M oscillators in the model (1) correspond to M^2 in the two- and M^3 in the three-dimensional problem]. This circumstance, promising the possibility of going beyond the case of weak turbulence in the future, indeed determines the choice of the model (1).

Equation (4) has the energy integral $\varepsilon = \int \omega N_\omega d\omega$ and the exact stationary solution $N_\omega = T/\omega$, which has the sense of a Rayleigh–Jeans distribution for thermodynamic equilibrium. Moreover, if $V_{\omega\omega_1\omega_2}$ is a homogeneous function

$$|V_{\omega\omega_1\omega-\omega_1}|^2 = \omega^s f(\omega/\omega_1),$$

then (4) has an exact solution of Kolmogorov type (see refs. 3 and 4):

$$N_\omega = cP^{1/2}\omega^{-(s+3)/2},$$

where c = const and P is the energy flux in the high-frequency domain. This solution is realized in the inertial domain in the presence of pumping in the low-frequency region and absorption in the high-frequency region.

The discrete kinetic equation (3) was solved numerically on a BÉSM-6 digital computer of the Computation Center of the Siberian Branch, Academy of Sciences of the USSR for 100 oscillators. It was assumed that $s = 2$, $f(\xi) = \xi^2(1-\xi)^2$. Two cases were examined: the establishing of thermodynamic equilibrium and the development of a Kolmogorov mode. The Cauchy problem with initial conditions of the kind shown in Fig. 1 was solved in the first case.

After a time of the order of unity, the initial distribution went over into a Rayleigh–Jeans spectrum in the whole frequency range except small frequencies $n \leq 10$ (see Fig. 2). As time passed, this domain diminished, tending to zero; the picture remained stationary in the whole remaining domain. The energy integral was conserved with 0.5% accuracy throughout the computation ($t \sim 10$).

This experiment confirms the natural deduction that the system (1) arrives at thermodynamic equilibrium in the absence of damping and pumping.

In the second case dissipative terms were added to Eq. (3):

$$\partial N_n/\partial t \to \partial N_n/\partial t + \gamma_{\mathrm{eff}}^n N_n,$$
$$\gamma_{\mathrm{eff}}^n = -3\exp\left[-\frac{(n-n_1)^2}{16}\right] + 2\sum_{n'=100}^{n+100} V_{n,n',n'-n} N_{n'-n}, \qquad (5)$$
$$n_1 = 10.$$

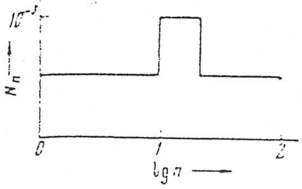

Fig. 1. The initial conditions for the problem of the establishing of thermodynamic equilibrium.

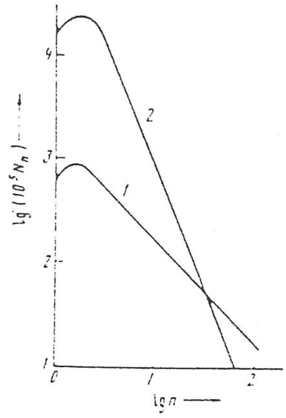

Fig. 2. The frequency distribution for oscillators for t = 5.5. 1) Rayleigh–Jeans spectrum; 2) Kolmogorov-type spectrum.

The first member in (5) corresponds to instability in the low-frequency domain with a maximum increment at n = 10. The second member arises if it is assumed that infinitely strong damping exists for n > 100 in the kinetic equation for an infinite number of oscillators; for n < 100 this member describes energy absorption because of its transfer into the domain n > 100.

The initial condition was selected in the form N_n^0 = const = $5 \cdot 10^{-5}$. A growth in amplitude in the instability domain was observed for t < 4. Energy pumping over to the high-frequency domain (growing of a power-law "tail") started for t ~ 4. The maximum of the distribution function simultaneously started to be shifted towards the low-frequency domain from the instability domain. The distribution N_n is pictured in Fig. 2 in a logarithmic scale for t = 5.5. A Kolmogorov spectrum $N_n \sim 1/n^{5/2}$ is realized with good accuracy in the range 80 > n > 9. An approximation by the hyperbola $N_n = A/n^{\beta}$ yields β = 2.5 ± 0.025; hence N_n varies between 0.7 and 2.10^{-5} (for t = 10) in the whole range 100 > n > 9. In the range n > 80, where damping already appears, N_n decreases rapidly.

The stationary mode was not achieved successfully in the low-frequency domain during times t ≤ 10; the maximum of the distribution function continues to be shifted to smaller and smaller n while growing in value. The numerical experiment we conducted shows that the energy in a system of nonlinear oscillators is distributed according to a Kolmogorov law for pumping in the low-frequency domain and damping at high frequencies.

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