Variational principle and stationary mirror structures in a plasma with pressure anysotropy

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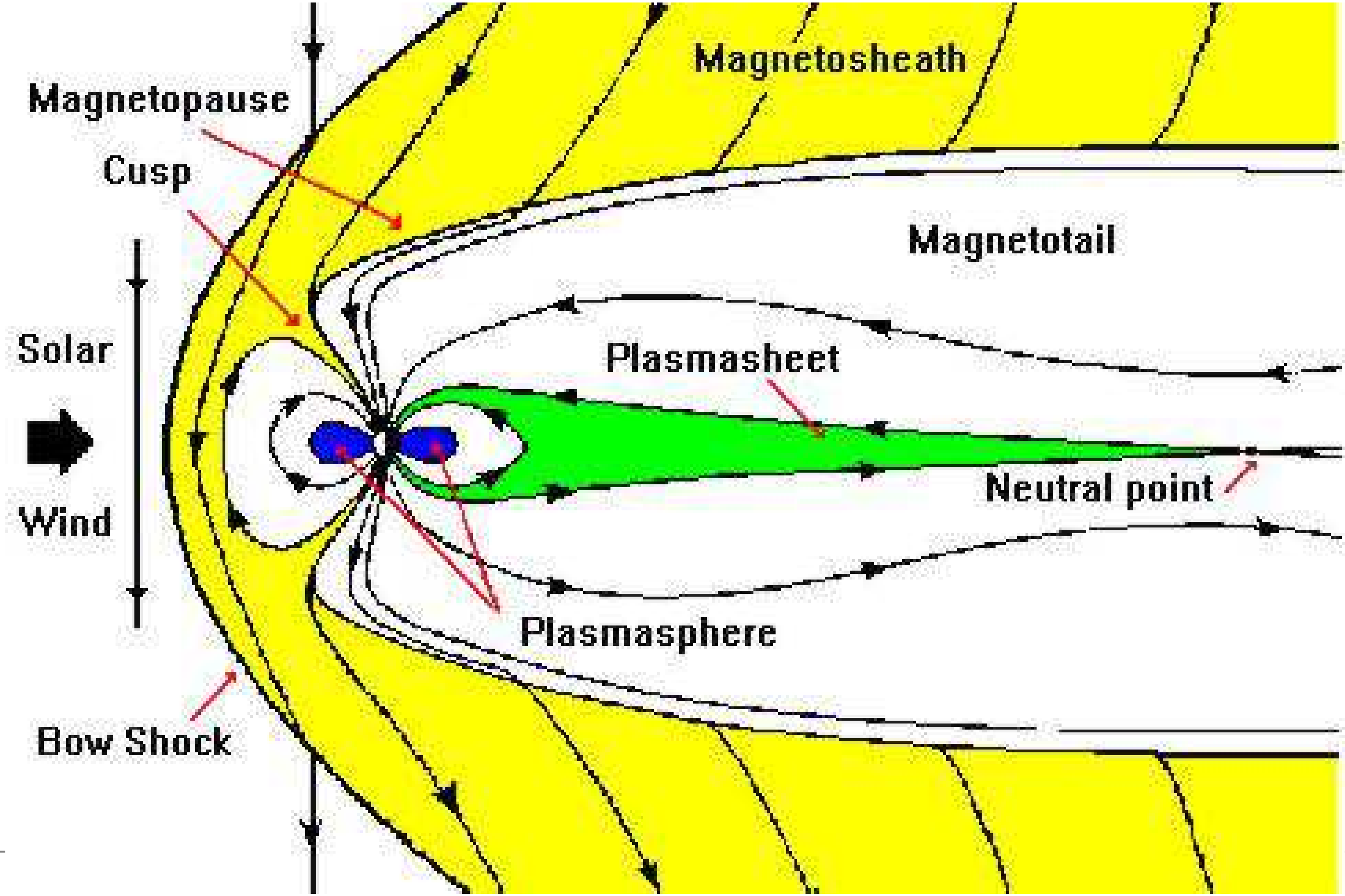
OUTLINE

- Introduction: history and experimental data
- Main goals
- Anisotropic Grad-Shafranov equations and parallel identity
- Variational principle and its matching with the weakly nonlinear regime
- Adiabatic approximation and lump solution for KPII
- Numerical solutions

Introduction

- Cigar-like magnetic structures, elongated along the direction of the ambient magnetic field, are commonly observed in planetary magnetosheaths close to the magnetopause, in the solar wind and even near comets (see, for instance, data obtained first time near the Halley comet).

Introduction



Introduction

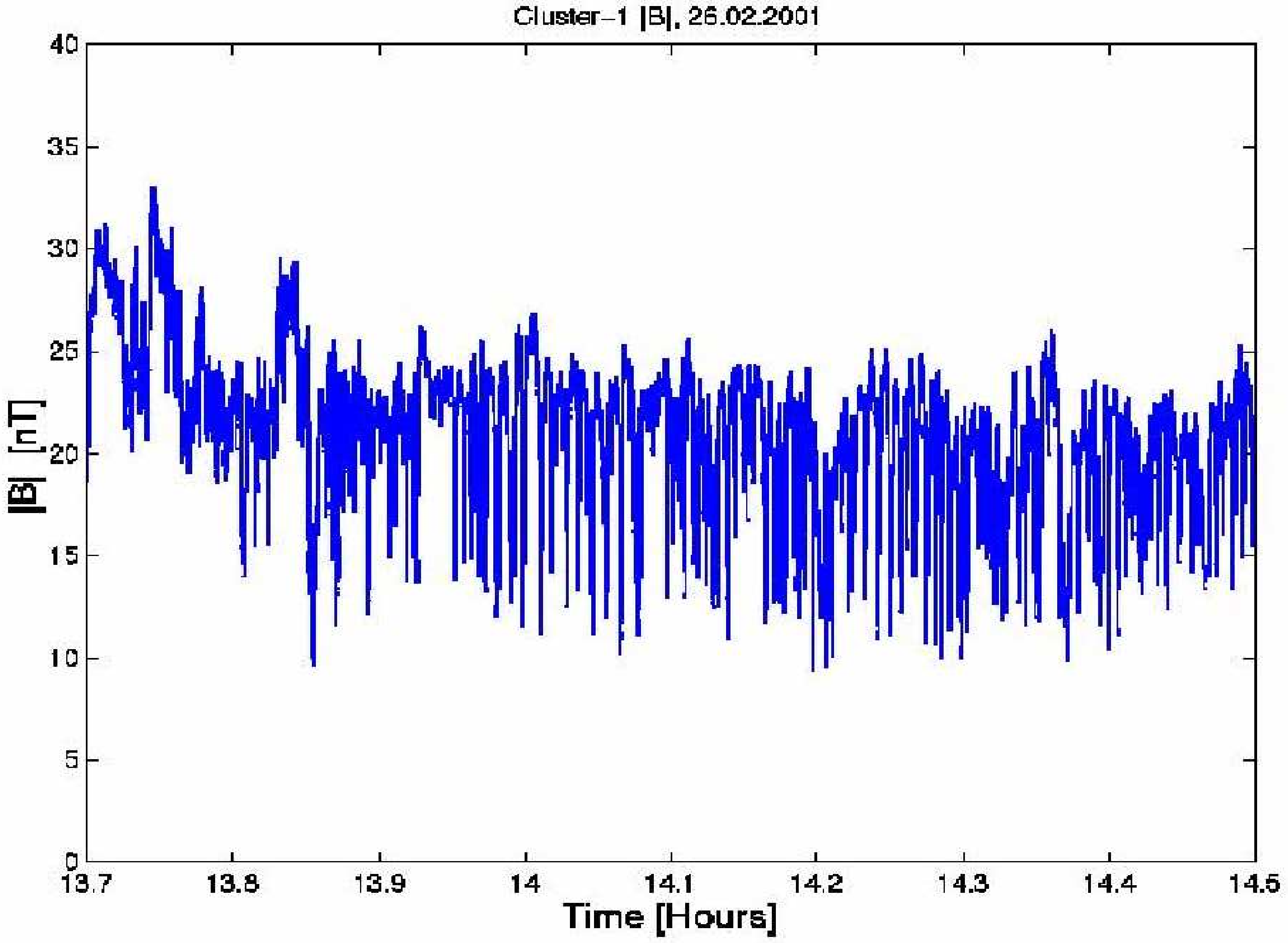
- In these regions the plasma is characterized by a relatively large β and a transverse (usually ionic) temperature $T_\perp$ larger than the parallel one $T_{||}$, such that the condition for mirror instability

$$T_\perp/T_{||} - 1 > \beta_\perp^{-1}$$

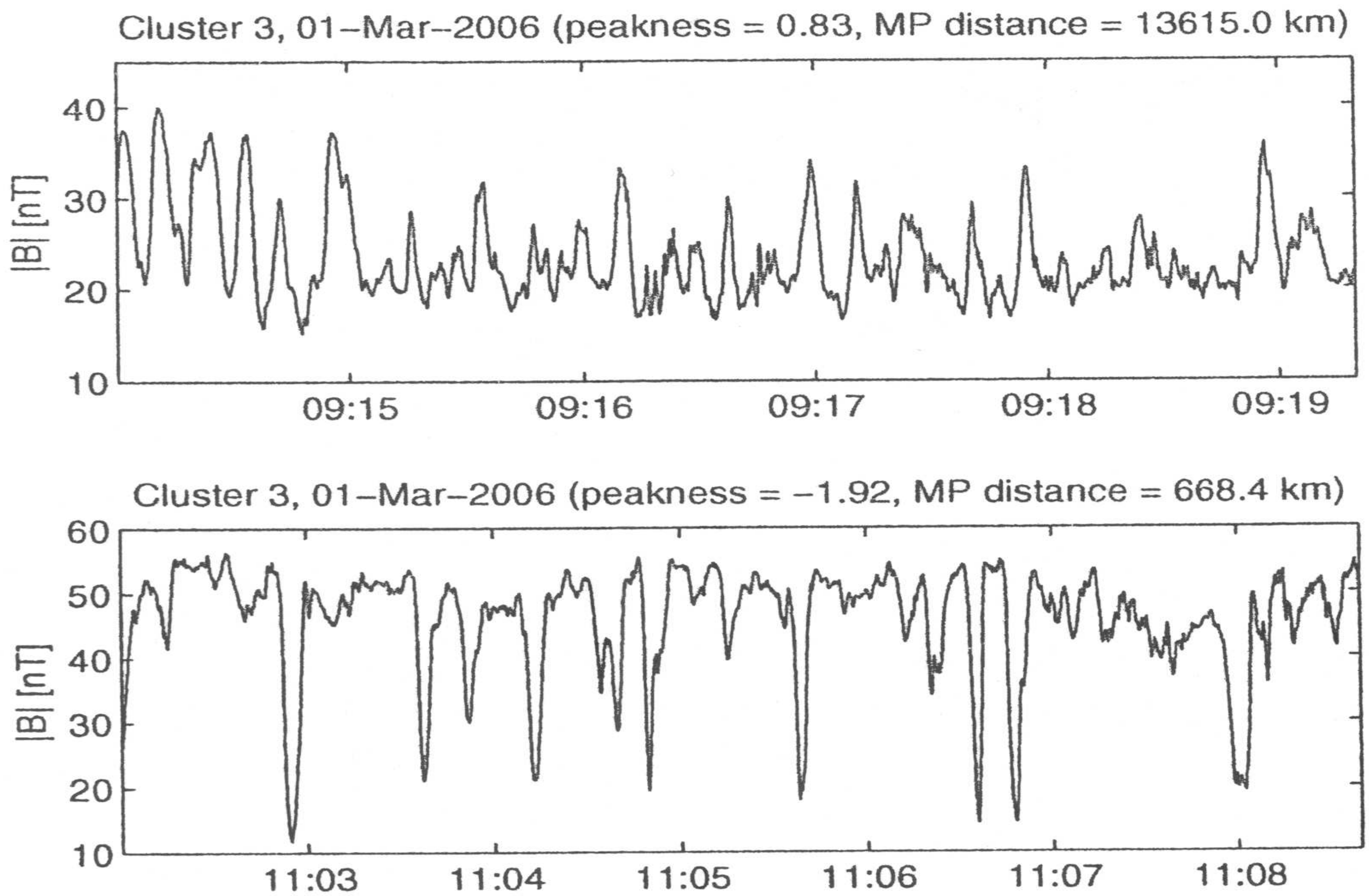
is fulfilled.

- Here $\beta_\perp = 8\pi p_\perp/B^2$ (similarly, $\beta_{||} = 8\pi p_{||}/B^2$), where $p_\perp$ and $p_{||}$ are the perpendicular and parallel plasma pressures respectively. For this reason, these magnetic structures are often associated with the nonlinear development of the mirror instability, and called mirror structures.

Introduction



Introduction



Introduction

- The mirror instability (MI) is a kinetic instability first predicted by Vedenov and Sagdeev in 1957 using the expansion $\omega/\omega_{ci} \ll 1$. The growth rate

$$\gamma = |k_z| v_{T_\| i} \frac{2\beta_\|}{\sqrt{\pi}\beta_\perp} \left[\frac{\beta_\perp}{\beta_\|} - 1 - \frac{1}{\beta_\perp} - \frac{k_z^2}{k_\perp^2} \left(1 + \frac{\beta_\perp - \beta_\|}{2} \right) \frac{1}{\beta_\perp} \right].$$

  Here ion distribution function $f(v_\|, v_\perp)$ is assumed bi-Maxwellian and electrons cold.
- The applicability condition of MI $\gamma/k_z \ll v_{T_\| i}$ means that

$$\varepsilon = \frac{2\beta_\perp}{2 + \beta_\perp - \beta_\|} \left(\frac{\beta_\perp}{\beta_\|} - 1 - \frac{1}{\beta_\perp} \right) \ll 1.$$

Introduction

- The MI has later on been extensively studied both analytically and numerically, usually by means of particle-in-cell (PIC) simulations. γ at fixed angle turns out to increase linearly with the wavenumber k. As first shown by Hasegawa (1969), the linear instability is arrested at large k by finite ion Larmor radius (FLR) effects. Later this effect was studied in details by many authors (Hall, Pokhotelov, Sagdeev, Balikhin, etc.).

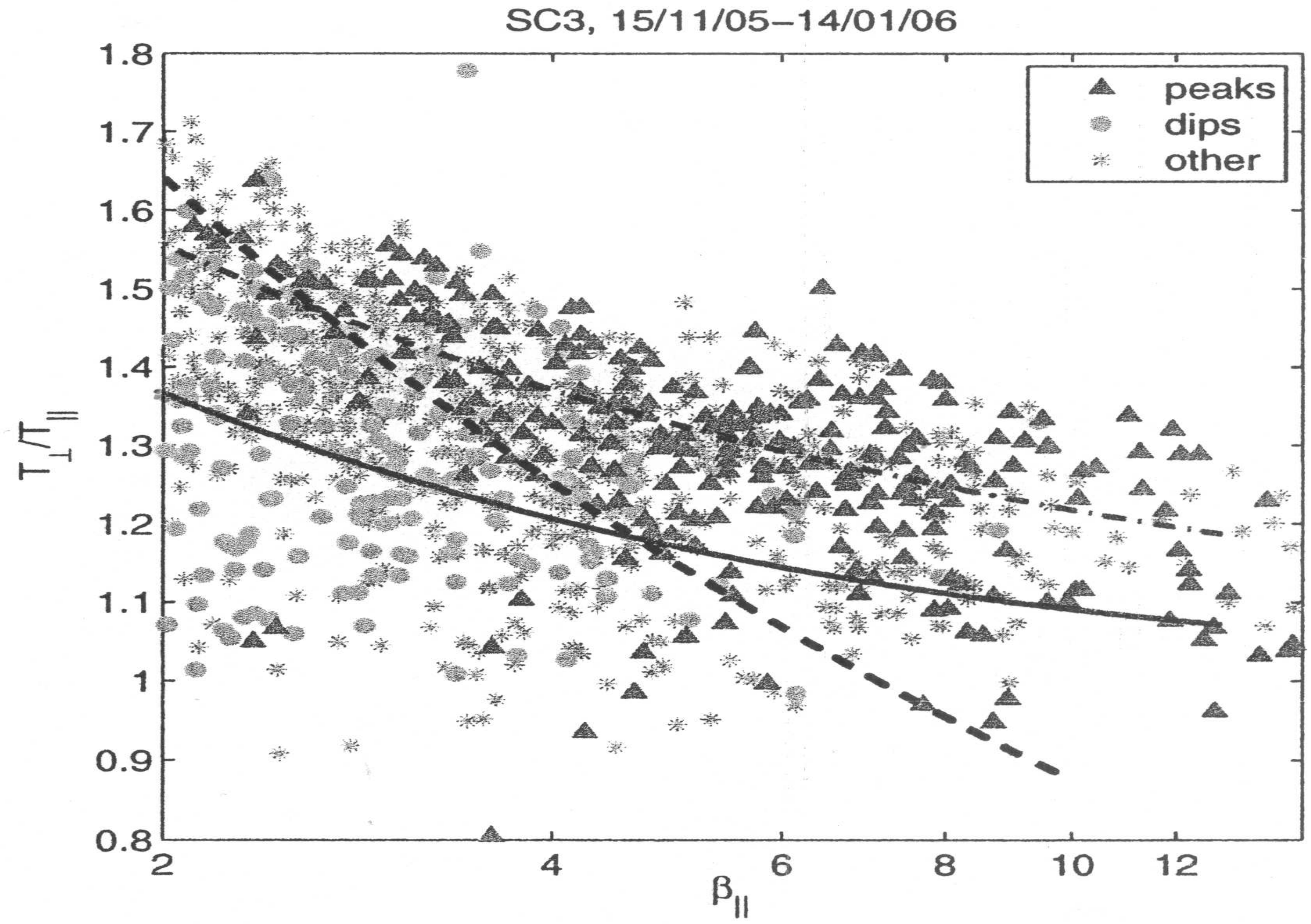
Introduction

- According to observations (Soucek, et. al., 2007), more than half of mirror structures are magnetic holes associated with maxima of the density and pressure fluctuations. A typical depth of magnetic holes is about 20% from the mean magnetic field value and can sometimes reach 50 %. Their characteristic width is of the order of a few ion Larmor radii, and they display an aspect ratio of about 7 to 10.
- Other mirror structures are humps with relatively large enhancement of the magnetic field amplitude, and are associated with minima in the plasma density. The presence of magnetic holes or humps was correlated with the relatively small or large value of β.

Introduction

- A quantitative characterization of the statistically dominant type of magnetic structures was presented by Genot, et. al. (2007) and Soucek et. al. (2007), by measuring the skewness of the magnetic fluctuations whose negative or positive sign reflects the preference towards magnetic holes or humps respectively. It turns out that there exists a clear statistical correlation between the skewness and the distance to the mirror instability threshold. Slightly above threshold, quasi-sinusoidal fluctuations dominate, while at further distance (which often corresponds to larger values of β), magnetic humps are preferably observed.

Introduction



Introduction

- Magnetic holes are mainly observed both below or slightly above threshold. Mirror structures are also observed when the plasma is linearly stable, which may be viewed as the signature of a bistability regime resulting from a subcritical bifurcation. As well known, for such a bifurcation, non trivial stationary states below threshold are linearly unstable, while above threshold, initially small-amplitude solutions undergo a sharp (asymptotically blowing-up) transition to a large-amplitude state.

Main goals

- The main goal of this paper is to study stationary localized structures resulting from the balance of magnetic and (both parallel and perpendicular) thermal pressures, whose simplest description is provided by anisotropic MHD.
- Isotropic MHD equilibria are classically governed by the Grad-Shafranov (GS) equation. We here revisit this approach in the case of anisotropic electron and ion fluids where the perpendicular and parallel pressures are given by equations of state appropriate for the static character of the solutions.

Main goals

- In this case, as well known (Grad, Shafranov, etc.), the parallel component of the equation is satisfied identically that allows to formulate a variational principle with a free energy given by the space integral of the parallel tension:

$$F = \int \left(\frac{B^2}{8\pi} - p_{\|} \right) d\mathbf{r}.$$

- However, the MHD stationary equations, at least in the two-dimensional geometry, turn out to be ill-posed due to breaking. Therefore these equations require some regularization. For nonlinear mirror modes, such regularization originates from finite Larmor radius (FLR) corrections.

Main goals

- This free energy can be matched with the weakly nonlinear expression found perturbatively in our papers (PRL, JETP Letters, 2007). It gives a key how one to seek for stationary mirror structures.
- In our previous papers we demonstrated that at the weakly nonlinear stage of the MI the system above the MI threshold has a blow-up behavior so that the final saturated states should have amplitude of order 1. I.e. we have subcritical bifurcation. In order to find possible stationary states we should use exact nonlinear equations.

Anisotropic Grad-Shafranov equations

The pressure balance equation for a static anisotropic MHD equilibrium is of the form

$$\frac{\partial}{\partial x_j}\Pi_{ij} = 0$$

where the tension tensor

$$\Pi_{ij} = \Pi_\perp\left(\delta_{ij} - b_i b_j\right) + \Pi_{||} b_i b_j, \quad \mathbf{b} = \mathbf{B}/B,$$

$$\Pi_\perp = p_\perp + B^2/(8\pi)$$

and

$$\Pi_{||} = p_{||} - B^2/(8\pi).$$

Anisotropic Grad-Shafranov equations

In usual notations these equations read as

$$0 = -\nabla \cdot \mathbf{P} + \frac{1}{c}\left[\mathbf{j} \times \mathbf{B}\right],$$

where the current $\mathbf{j}= \frac{c}{4\pi}\nabla\times\mathbf{B}$, and the pressure tensor $\mathbf{P}$ is

$$P_{ij} = p_\perp\left(\delta_{ij} - b_i b_j\right) + p_\| b_i b_j.$$

The solvability conditions read $\mathbf{B}\cdot(\nabla\cdot\mathbf{P})=0,$ and $\mathbf{j}\cdot(\nabla\cdot\mathbf{P})=0$.
The pressures $p_\perp = \Sigma_\alpha p_{\perp\alpha}$ and $p_\| = \Sigma_\alpha p_{\|\alpha}$ where the partial pressures are expressed as

$$p_{\perp\alpha} = m_\alpha B^2 \int \mu f_\alpha dv_\| d\mu,$$

$$p_{\|\alpha} = m_\alpha B \int v_\|^2 f_\alpha dv_\| d\mu.$$

Anisotropic Grad-Shafranov equations

The distribution functions f_α satisfy the stationary drift kinetic equations

$$v_\parallel \nabla_\parallel f_\alpha - \left(\mu \nabla_\parallel B + \frac{e_\alpha}{m_\alpha} \nabla_\parallel \phi\right) \frac{\partial f_\alpha}{\partial v_\parallel} = 0,$$

$\nabla_\parallel = \mathbf{b} \cdot \nabla$ denotes the gradient along $\mathbf{B}$, $v_\parallel$ the parallel component of the particle velocity, ϕ the electric potential, and $\mu = v_\perp^2 / (2B)$ the adiabatic invariant (a parameter). These equations are supplemented by

$$\sum_\alpha e_\alpha B \int f_\alpha dv_\parallel d\mu = 0,$$

that allows one to eliminate ϕ.

Identity in the parallel direction

We consider partial solutions of the kinetic equations depending on the particle energy $W_\alpha = v_\parallel^2/2 + \mu B + (e_\alpha/m_\alpha)\phi$ and μ. In general, the solution can also depend on integrals which label the magnetic field lines. The choice $f_\alpha = f_\alpha(W_\alpha, \mu)$ can be matched with the solution for weakly nonlinear mirror modes. In this case $p_{\perp,\alpha}$ and $p_{\parallel,\alpha}$ are functions of B only.

The anisotropic pressure balance equation reads

$$-\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \left[1 + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\right]\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}$$

$$+\mathbf{B}(\mathbf{B}\cdot\nabla)\left(\frac{p_\perp - p_\parallel}{B^2}\right) = 0.$$

Identity in the parallel direction

Projection along the magnetic field gives

$$-\nabla_{\|} p_{\|} - \frac{4\pi\left(p_{\perp} - p_{\|}\right)}{B^2} \nabla_{\|} \frac{B^2}{8\pi} = 0,$$

which coincides with Eq. (9.2) (Shafranov,1966). It is possible to prove that the solvability condition reduces to an identity by means of both stationary kinetic equations and the quasi-neutrality condition. Since the pressures depend on B only, the identity also can be written as

$$\frac{p_{\perp} - p_{\|}}{B} = -\frac{dp_{\|}}{dB}.$$

Identity in the parallel direction

The existence of this identity means that for stationary states, the pressure balance provides only two scalar equations which, together with the condition $\nabla \cdot \mathbf{B} = 0$, leads to a closed system of three equations for the three magnetic field components.

Defining $\nabla_\perp = \nabla - B^{-2}\mathbf{B}(\mathbf{B}\cdot\nabla)$, the perpendicular component reads

$$-\nabla_\perp p_\perp + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\nabla_\perp \frac{B^2}{8\pi} + \left[1 + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\right] \frac{\Big[[\nabla\times\mathbf{B}]\times\mathbf{B}\Big]}{4\pi} = 0,$$

which coincides with Eq. (9.3) of Shafranov's review (1966).

Anisotropic Grad-Shafranov equations

In two dimensions, defining the stream function ψ, such that $B_x = \partial\psi/\partial y$, $B_y = -\partial\psi/\partial x$, due to the identity, the equation for B_z admits integration (like in Grad-Shafranov):

$$\frac{B_z}{4\pi}\left(1 + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\right) = f(\psi).$$

When $p_\perp - p_\parallel = 0)$, we have $B_z = B_z(\psi),$ in full agreement with the Grad-Shafranov reduction.

The equation for ψ has the form, which can be viewed analogous to the Grad-Shafranov equation:

$$\left(\nabla\psi\cdot\nabla(p_\perp + \frac{B_z^2}{8\pi})\right) - \frac{(p_\perp - p_\parallel)}{2B^2}\left(\nabla\psi\cdot\nabla\left(B^2 - B_z^2\right)\right)$$
$$= -\frac{(B^2 - B_z^2)}{4\pi}\left[1 + \frac{4\pi}{B^2}(p_\perp - p_\parallel)\right]\Delta\psi,$$

Variational principle

In the purely two-dimensional geometry when $B_z = 0$ the equation for ψ reduces to

$$\nabla \cdot \left\{ \left[1 + \frac{4\pi}{B^2}(p_\perp - p_\|) \right] \nabla\psi \right\} = 0.$$

Due to the identity, this equation follows from the variational principle $\delta\mathcal{F} = 0$ with

$$\mathcal{F} = \int \left(\frac{B^2}{8\pi} - p_\| \right) dx\, dy \equiv - \int \Pi_\|\, dx\, dy.$$

Thus, all the two-dimensional stationary states in anisotropic MHD are stationary points of the functional $\mathcal{F}$. Its density is a function of B only. In the special case of cold electrons, this free energy turns out to identify with the Hamiltonian of the static problem (Passot, Ruban, & Sulem, 2006).

Variational principle

Similar equations arise in the context of pattern structures in thermal convection. As shown by Ercolani, Indik, Newell, Passot (2006) , such equations represent integrable hydrodynamic systems. As in the usual one-dimensional gas dynamics, these systems display breaking phenomena where the solution looses its smoothness at finite distance, due to the formation of folds. As a consequence, these models require some regularization. In plasma case, this procedure, corresponding to the replacement of $\mathcal{F} \to \mathcal{F} + (\nu/2) \int (\Delta\psi)^2 \, dxdy$ with $\nu > 0$, originates from finite Larmor radius (FLR) corrections, which are, however, beyond the drift approximation.

Adiabatic approximation and lump for KPII

Mirror instability represents one of the slowest instabilities known in plasma physics. Its characteristic frequencies are much smaller than the ion gyro-frequency. Therefore it is natural to apply the adiabatic arguments to construct the equation of state. In the weakly nonlinear regime, as it was demonstrated in our papers, the transition from the initial homogeneous state is slow in time, so that, to leading order, the distribution function f_α as a function of μ and W_α retains its form during the evolution. Therefore, the function $f_\alpha(\mu, W_\alpha)$ can be determined by matching with the initial distribution function $f_\alpha^{(0)}$.

Adiabatic approximation: matching

In the bi-Maxwellian case

$$f_\alpha^{(0)} = A_\alpha \exp\left[-\frac{v_\parallel^2}{v_{\parallel\alpha}^2} - \frac{\mu B_0 m_\alpha}{T_{\perp\alpha}}\right],$$

which corresponds to $\phi = 0$ and$W_\alpha = \frac{v_\parallel^2}{2} + \mu B_0$. Here $T_{\parallel\alpha}$ and $T_{\perp\alpha}$ are the initial perpendicular and transverse temperatures, B_0 the initial homogeneous magnetic field, and $v_{\parallel\alpha} = \left(2T_{\parallel\alpha}/m_\alpha\right)^{1/2}$ the parallel thermal velocity. As a result of the matching, we get

$$f_\alpha(\mu, W_\alpha) = A_\alpha \exp\Big[-\frac{2W_\alpha}{v_{\parallel\alpha}^2} + \mu B_0 m_\alpha\Big(\frac{1}{T_{\parallel\alpha}} - \frac{1}{T_{\perp\alpha}}\Big)\Big].$$

Adiabatic approximation and lump for KPII

This function has the Boltzmann form with respect to W_α but display, at fixed W_α, an exponential growth relatively to μ when $T_{\perp\alpha} > T_{\|\alpha}$, a necessary condition for MI. Such a growth, however, leads to a singular behavior of

$$p_\| = n_0(T_{\|i} + T_{\|e})\frac{1+u}{(1+a_e u)^{c_e}\,(1+a_i u)^{c_i}},$$

where $u = B/B_0 - 1$, $a_\alpha = T_{\perp\alpha}/T_{\|\alpha}$ is the anisotropy parameter f_α, and $c_\alpha = T_{\|\alpha}(T_{\|e} + T_{\|i})^{-1}$. The singularities at $u = -a_\alpha^{-1}$ correspond to

$$B_s = B_0\frac{a_\alpha - 1}{a_\alpha} < B_0.$$

For cold electrons, $p_\| = n_0 T_\|(1+u)(1+au)^{-1}$ displays a pole singularity.

Adiabatic approximation: regularization

The above singularities are presumably related to an overestimated contribution from large μ, corresponding either to small B or to large a transverse kinetic energy. In both cases, the applicability of the drift approximation breaks down and we are thus led to introduce some cut-off type correction near μ_α^*. In a simple variant, we take

$$f_\alpha = \tilde{C}_\alpha \exp(-m_\alpha W_\alpha / T_{\|\alpha})$$

at $\mu > \mu_\alpha^*$, with some positive constant $\tilde{C}_\alpha$, and f_α retains its original form for $\mu \leq \mu_\alpha^*$.

KP lump

Next, it is possible to see $\mathcal{F}$ has the meaning of a free energy because in the weakly nonlinear regime the behavior of the mirror modes can be described by a generalized gradient model, that in the 2D geometry reads

$$u_t = -\widehat{|k_y|}\frac{\delta F}{\delta u}, \quad F = \int \left[\frac{1}{2}\left(-\varepsilon u^2 + u\left(\frac{\partial_y}{\partial_x}\right)^2 u + u_x^2\right) + \frac{\lambda}{3}u^3\right] dxdy.$$

Here $u = (B - B_0)/B_0$, ε is the distance from the threshold, the third term originates from the FLR, and nonlinear coupling coefficient $\lambda > 0$ for bi-Maxwellian distributions. Irreversibility is connected with the positiveness of operator $\widehat{|k_y|}$: F can decrease in time function (the latter provides by Landau ion damping).

KP lump

If one expands the functional $\mathcal{F}$ in series with respect to u (assuming $\psi = -B_0(x+\varphi)$ with $\varphi \to 0$ as $|\mathbf{r}| \to \infty$ putting $u \approx \varphi_x$), so that the mean magnetic field B_0 is directed along the y-axis) it is easy to establish that its expansion coincides with F.

In particular, the quadratic term in this expansion defines the MI threshold (for any distribution function!). For instance, from here one can easily get the Vedenov-Sagdeev' answer:

$$\varepsilon = \frac{2\beta_\perp}{2+\beta_\perp-\beta_{||}}\left(\frac{\beta_\perp}{\beta_{||}} - 1 - \frac{1}{\beta_\perp}\right).$$

KP lump

As well known, as $\epsilon \to 0$, MI develops in quasi-transverse directions:$\varphi_x \gg \varphi_y$. In the stationary case φ is determined from

$$L\varphi - \lambda\partial_x \left(\varphi_x^2\right) = 0$$

where $L = -\varepsilon\partial_{xx} + \partial_{yy} - \partial_{xxxx}$ is elliptic or hyperbolic depending on the sign of ε. For $\varepsilon < 0$, L is elliptic and thus invertible in the class of functions vanishing at infinity. In this case the solution identifies with the soliton for KP equation called lump (Petviashvili, 1976). In our notation, it reads (Zakharov, Manakov, et al, 1979)

$$\varphi_x = -\frac{12|\varepsilon|}{\lambda}\frac{(3 + \varepsilon^2 y^2 - |\varepsilon|x^2)}{[3 + \varepsilon^2 y^2 + |\varepsilon|x^2]^2}.$$

This function vanishes algebraically at infinity like r^{-2}.

Numerical results

The Petviashvili method allows to construct localized stationary solution: $\delta F = 0$.

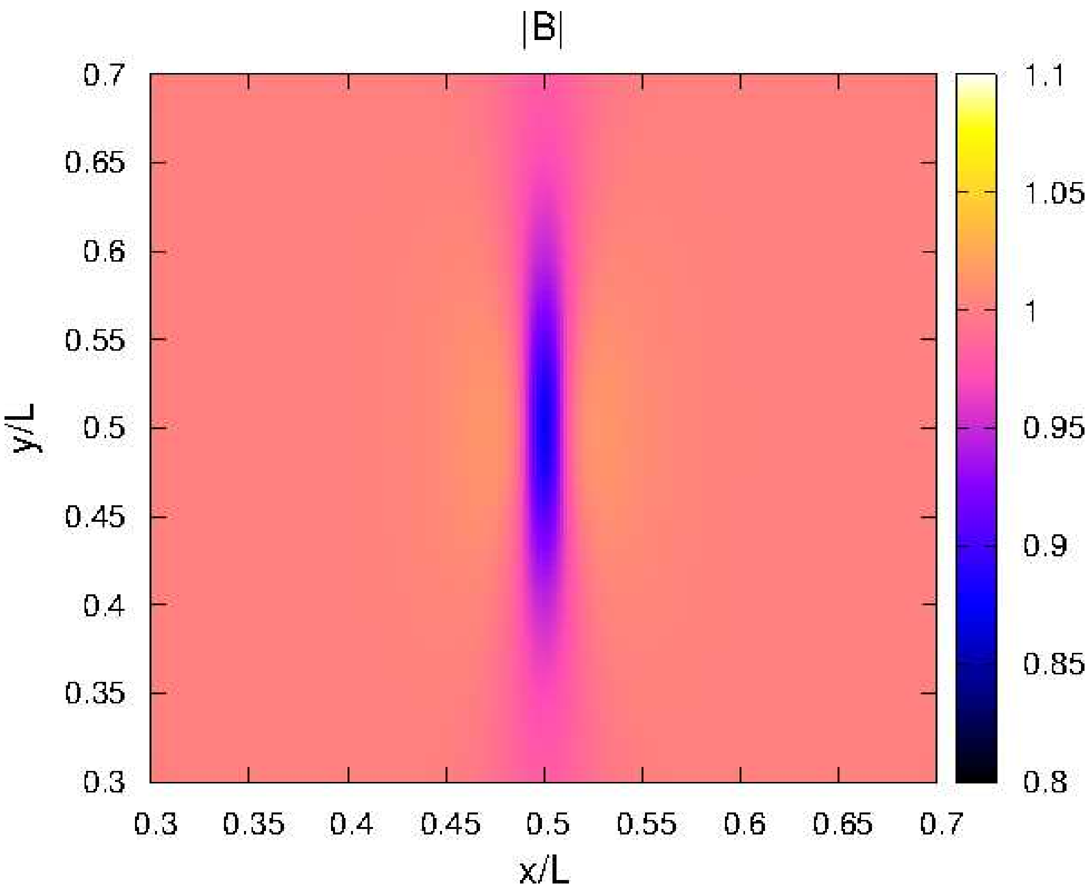


Figure 5: Fig. 1. Localized solution for $\varepsilon = -0.002$.

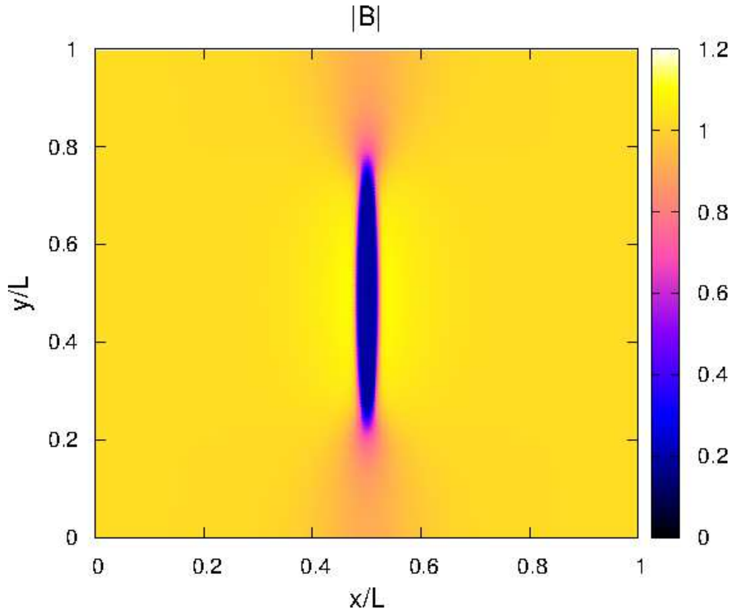
Numerical results

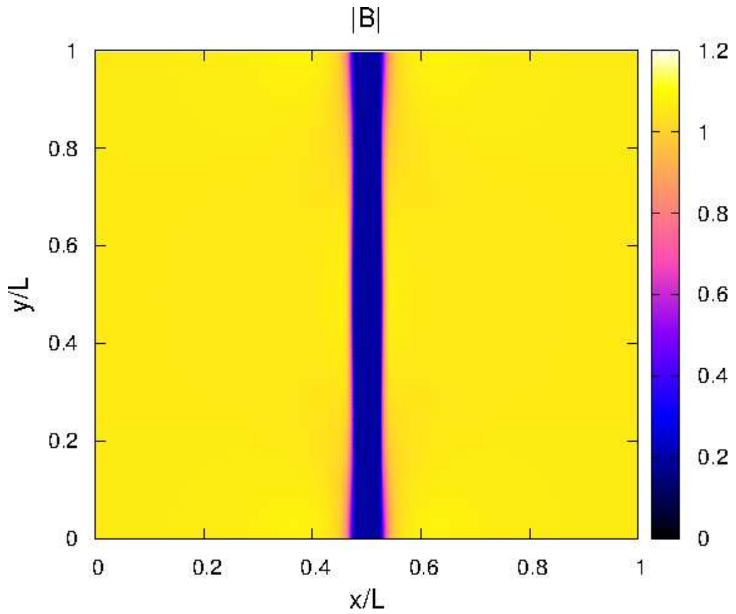
The second approach is a generalization of the well known gradient method which corresponds to a dissipative dynamics along an auxiliary time-like variable τ of the form

$$\varphi_\tau = -\widehat{\Gamma}(\delta\mathcal{F}/\delta\varphi),$$

with a positive definite linear operator $\widehat{\Gamma}$. The operator $\widehat{\Gamma}$ was taken in a form giving stable computation, namely $\Gamma(k_x, k_y) = 1/[k_x^2 + k_y^2 + \nu(k_x^2 + k_y^2)^2]$.
It is clear that attractors in the phase space of the above dynamical system are stable solutions of the equation. Unstable solutions however cannot be found by this method.

Numerical results





Numerical results

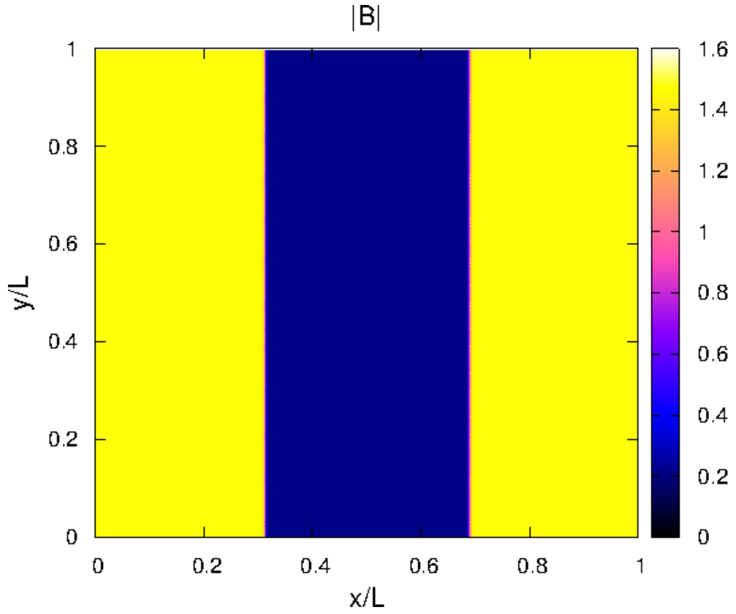


Figure 6: Formation of a stable 1D solution in a gradient computation

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