Robust energy transfer mechanism via precession resonance in nonlinear turbulent wave systems

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Outline

According to "folk" tradition . . .

- Strong turbulence $\Longrightarrow$ Large amplitudes
- Wave turbulence theory $\Longrightarrow$ Limit of infinitesimally small amplitudes

However, our research (PRL, 2014, in press) establishes the following: [1] [2]

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  Precession resonance $\Longrightarrow$ strong energy transfers across scales

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- Wave turbulence theory can be developed at intermediate amplitudes
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- We provide abundant evidence of this in a nonlinear PDE:
  Charney-Hasegawa-Mima equation

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Introduction 1/2: Turbulence in Wave Systems

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These systems are characterised by:

- Extreme events, localised in space and time
- Strong nonlinear energy exchanges
- Out-of-equilibrium dynamics: chaos & turbulence

Introduction 2/2: Classical Wave Turbulence Theory

- One of the few consistent theories that deal with nonlinear exchanges
- Widely used in numerical prediction of ocean waves [3]

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In real-life systems, hypotheses of classical wave turbulence do not hold:

- Amplitudes of the carrying fields are **not infinitesimally small**
- Spatial domains have a **finite size**
- Linear wave timescales **are comparable with** nonlinear oscillations' timescales

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Discrete and Mesoscopic Wave Turbulence: A theory in development [4] [5] [6]

- Applications in nonlinear PDEs: Classical fluids – Quantum fluids – Nonlinear optics – Magneto-hydrodynamics – etc.

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$$(\nabla^2 - F)\frac{\partial\psi}{\partial t} + \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0.$$

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- In the plasma case $\psi(\mathbf{x}, t)(\in \mathbb{R})$ is the electrostatic potential
- $F^{-1/2}$ is the ion Larmor radius at the electron temperature
- β is a constant proportional to the mean plasma density gradient
- Periodic boundary conditions: $\mathbf{x} \in [0, 2\pi)^2$

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CHM equation: Fourier representation

- $\psi(\mathbf{x}, t) = \sum_{\mathbf{k}\in\mathbb{Z}^2} A_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ Wavevector: $\mathbf{k} = (k_x, k_y)$

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- Components $A_{\mathbf{k}}(t)\,,\quad \mathbf{k}\in\mathbb{Z}^2$ satisfy the evolution equation

$$\dot{A}_{\mathbf{k}} + i\,\omega_{\mathbf{k}}\,A_{\mathbf{k}} \;=\; \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{Z}^2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,\delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}\,A_{\mathbf{k}_1}\,A_{\mathbf{k}_2} \qquad (1)$$

- $\omega_{\mathbf{k}} = \frac{-\beta k_x}{|\mathbf{k}|^2+F}$ (linear frequencies)
- $Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} = (k_{1x}k_{2y} - k_{1y}k_{2x})\frac{|\mathbf{k}_1|^2-|\mathbf{k}_2|^2}{|\mathbf{k}|^2+F}$ (interaction coefficients)
- δ is the Kronecker symbol

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- The modes $A_{\mathbf{k}}$ interact in **triads**

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- Reality of $\psi \Longrightarrow A_{-\mathbf{k}} = A^*_{\mathbf{k}}$ (complex conjugate)
- The modes $A_{\mathbf{k}}$ interact in **triads**
- A triad is a group of any three spectral modes $A_{\mathbf{k}_1}(t), A_{\mathbf{k}_2}(t), A_{\mathbf{k}_3}(t)$ whose wavevectors satisfy $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$
- Triad's **linear frequency mismatch:** $\omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \equiv \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}$

A Key Observation

$$\boxed{\dot{A}_{\mathbf{k}} + i\,\omega_{\mathbf{k}}\,A_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{Z}^2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,\delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}\,A_{\mathbf{k}_1}\,A_{\mathbf{k}_2}\,.}$$

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We consider inertial-range dynamics, i.e. no forcing and no dissipation: enstrophy cascades to small scales respect enstrophy conservation.

. . . Now, we change gears . . . [7]

[7] V. E. Zakharov, just when the students thought they began to understand the lecture

Truly Dynamical Degrees of Freedom 1/2

CHM equation, Galerkin-truncated to N wavevectors: “Cluster” $\mathcal{C}_N$:

$$\dot{A}_{\mathbf{k}} + i\,\omega_{\mathbf{k}}\,A_{\mathbf{k}} \;=\; \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathcal{C}_N} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,\delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}\,A_{\mathbf{k}_1}\,A_{\mathbf{k}_2}\,, \quad \mathbf{k}\in\mathcal{C}_N$$

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- The **truly dynamical degrees of freedom** are any $N-2$ linearly independent **triad phases** $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \equiv \phi_{\mathbf{k}_1}+\phi_{\mathbf{k}_2}-\phi_{\mathbf{k}_3}$ and the N wave spectrum variables $n_{\mathbf{k}}$

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- The **truly dynamical degrees of freedom** are any $N-2$ linearly independent **triad phases** $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \equiv \phi_{\mathbf{k}_1}+\phi_{\mathbf{k}_2}-\phi_{\mathbf{k}_3}$ and the N wave spectrum variables $n_{\mathbf{k}}$
- These $2N-2$ degrees of freedom form a **closed system**
- Individual phases $\phi_{\mathbf{k}}$ are "slave": obtained by quadrature

Truly Dynamical Degrees of Freedom 2/2

Closed system for the $2N-2$ truly dynamical variables:

$$\dot{n}_{\mathbf{k}} = \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \tag{2}$$

$$\begin{aligned} \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,, \end{aligned} \tag{3}$$

where the second equation applies to any triad $(\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3)$.

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$$
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- $\mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$: "nearest-neighbouring-triad terms"; these are nonlinear terms similar to the first line in Eq. (3)

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$$\begin{aligned} \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin \varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,, \end{aligned} \tag{3}$$

where the second equation applies to any triad $(\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3)$.

- $\mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$: "nearest-neighbouring-triad terms"; these are nonlinear terms similar to the first line in Eq. (3)
- Any dynamical process in the original system results from the dynamics of equations (2)–(3)

Precession Resonance 1/3

$$
\begin{aligned}
\dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\left(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2}\right)^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,,\\
\dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\left(n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2}\right)^{\frac{1}{2}}\left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}}\right]\\
&- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,.
\end{aligned}
$$

Triad phases $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ versus spectrum variables $n_{\mathbf{k}}$:

Precession Resonance 1/3

$$\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin \varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}$$

Triad phases $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ versus spectrum variables $n_{\mathbf{k}}$:

- Wave spectra $n_{\mathbf{k}}$ contribute directly to the energy of the system
- $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ have a contribution that is **more subtle**

Precession Resonance 1/3

$$\boxed{\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}}\left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}}\right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}}$$

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- Wave spectra $n_{\mathbf{k}}$ contribute directly to the energy of the system
- $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ have a contribution that is **more subtle**
- The RHS of $\dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$–equation admits, under plausible hypotheses, a zero-mode (in time): $\boxed{\Omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \equiv \lim_{t\to\infty}\frac{1}{t}\int_0^t \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(t')dt'}$
- This is by definition the **precession frequency** of the triad phase

Precession Resonance 1/3

$$\boxed{\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin \varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}}$$

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- This is by definition the **precession frequency** of the triad phase
- Typically it does not perturb the energy dynamics, except when ...

Precession Resonance 2/3

$$\boxed{\begin{aligned}
\dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\
\dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin \varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\
&- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,.
\end{aligned}}$$

...when the triad precession frequency $\left(\Omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} = \langle \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \rangle\right)$ matches one of the typical nonlinear frequencies of the triad variables, then:

Precession Resonance 2/3

$$\boxed{\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin \varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}} \right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}}$$

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- The RHS of $\dot{n}_{\mathbf{k}}$–equation will develop a zero-mode (in time) $\Longrightarrow$ **sustained growth of energy** in $n_{\mathbf{k}}$, for some wavevector(s) $\mathbf{k}$
- We call this a **triad precession/nonlinear frequency** resonance

Precession Resonance 2/3

$$\boxed{\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}}\left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}}\right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}}$$

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- We call this a **triad precession/nonlinear frequency** resonance

When several triads are involved in precession resonance:
Strong fluxes of enstrophy through the network of interconnected triads, coherent collective oscillations, and cascades towards small scales.

Precession Resonance 3/3

$$
\begin{aligned}
\dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}}\, \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\
\dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}}\left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}}\right] \\
&- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,.
\end{aligned}
$$

- Resonance is accessible via **initial-condition manipulation**
- Simple **overall re-scaling of initial spectrum:** $n_{\mathbf{k}} \to \alpha\, n_{\mathbf{k}}$ *for all* $\mathbf{k}$
- Linear frequency mismatch $\omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ must be nonzero for some triad

Precession Resonance 3/3

$$\boxed{\begin{aligned} \dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,, \\ \dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} &= \sin\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}}\left[\frac{Z^{\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}}{n_{\mathbf{k}_1}} + \frac{Z^{\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}_1}}{n_{\mathbf{k}_2}} - \frac{Z^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}}{n_{\mathbf{k}_3}}\right] \\ &- \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} + \mathrm{NNTT}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}\,. \end{aligned}}$$

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Dimensional analysis explanation:

Precession Resonance 3/3

$$\begin{aligned}
\dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}(n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos\varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,,\\
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Dimensional analysis explanation:

- $\dot{n}_{\mathbf{k}} \propto (n_{\mathbf{k}})^{3/2} \Longrightarrow$ nonlinear frequency: $\Gamma \propto \alpha^{\frac{1}{2}}$

Precession Resonance 3/3

$$
\begin{aligned}
\dot{n}_{\mathbf{k}} &= \sum_{\mathbf{k}_1,\mathbf{k}_2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} (n_{\mathbf{k}}\, n_{\mathbf{k}_1}\, n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \,, \\
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- $\dot{\varphi}^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$ equation $\Longrightarrow$ triad precession: $\Omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \sim C\, \alpha^{\frac{1}{2}} - \omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}$

Precession Resonance 3/3

$$
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$$

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- Therefore, provided $\omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} \neq 0$, $\Omega^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2} = \Gamma$ for some value of α

Results

(I'll show videos if someone asks)

Triggering the mechanism starting from a single triad 1/3

$$\boxed{\dot{A}_{\mathbf{k}} + i\,\omega_{\mathbf{k}}\,A_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{Z}^2} Z^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\,\delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}\,A_{\mathbf{k}_1}\,A_{\mathbf{k}_2}\,.}$$

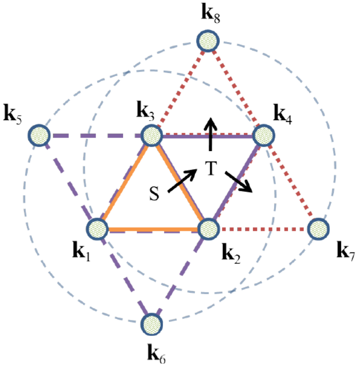
$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$

$\mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4$
$\mathbf{k}_3 + \mathbf{k}_1 = \mathbf{k}_5$
$\mathbf{k}_6 + \mathbf{k}_1 = \mathbf{k}_2$

$\mathbf{k}_5 + \mathbf{k}_6 = \mathbf{k}_4$

$\mathbf{k}_2 + \mathbf{k}_4 = \mathbf{k}_7$
$\mathbf{k}_4 + \mathbf{k}_3 = \mathbf{k}_8$

$\mathbf{k}_7 + \mathbf{k}_1 = \mathbf{k}_8$



Full PDE model is difficult to draw (~ 12 million triads in resolution 128^2)
Pseudospectral, 2/3-**rd dealiased**

Triggering the mechanism starting from a single triad 2/3

- Parameters
  $F = 1, \beta = 10$
- Single triad:
  $\mathbf{k}_1 = (1, -4)$,
  $\mathbf{k}_2 = (1, 2)$,
  $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 = (2, -2)$
- Initial conditions:
  $\varphi^{\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}(0) = \pi/2$,
  $n_{\mathbf{k}_1}(0) = 5.96 \times 10^{-5}\alpha$,
  $n_{\mathbf{k}_2}(0) = 1.49 \times 10^{-3}\alpha$,
  $n_{\mathbf{k}_3} = 1.29 \times 10^{-3}\alpha$,
  where α is a re-scaling parameter
- Initially $n_{\mathbf{k}_a}(0) = 0$ for all other modes

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  $n_{\mathbf{k}_3} = 1.29 \times 10^{-3}\alpha$,
  where α is a re-scaling parameter
- Initially $n_{\mathbf{k}_a}(0) = 0$ for all other modes

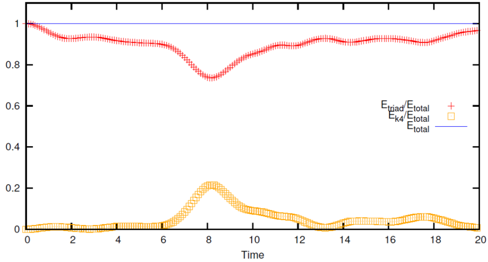
How to quantify a strong transfer?

Use enstrophy conservation

Define transfer efficiency to mode $n_{\mathbf{k}_a}$:

$$Eff_a = \max_{t\in[0,T]} \frac{\mathcal{E}_a(t)}{\mathcal{E}}$$

Example: below, $Eff_4 \sim 20\%$

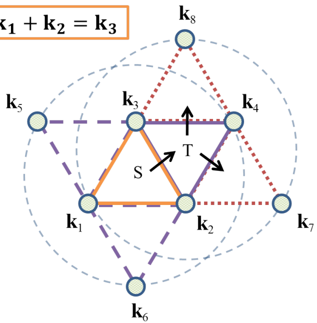


Triggering the mechanism starting from a single triad 3/3

Family of models: Deform the original equations using two positive numbers $\epsilon_1, \epsilon_2 \in [0, 1]$ which multiply the interaction coeffs. $Z^{\mathbf{k}_c}_{\mathbf{k}_a \mathbf{k}_b}$

Z^3_{12} not deformed

$\boxed{\mathbf{k_1} + \mathbf{k_2} = \mathbf{k_3}}$



$Z^4_{23} \to \epsilon_1 Z^4_{23}$

$\boxed{\mathbf{k_2} + \mathbf{k_3} = \mathbf{k_4}}$

$Z^c_{ab} \to \epsilon_2 Z^c_{ab}$, etc.

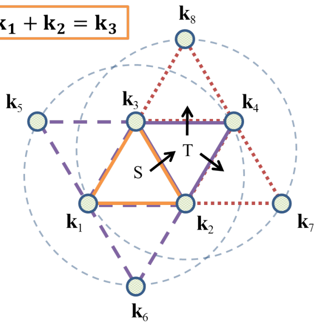
$\boxed{\mathbf{k}_a + \mathbf{k}_b = \mathbf{k}_c}$

Triggering the mechanism starting from a single triad 3/3

Family of models: Deform the original equations using two positive numbers $\epsilon_1, \epsilon_2 \in [0,1]$ which multiply the interaction coeffs. $Z^{\mathbf{k}_c}_{\mathbf{k}_a\mathbf{k}_b}$

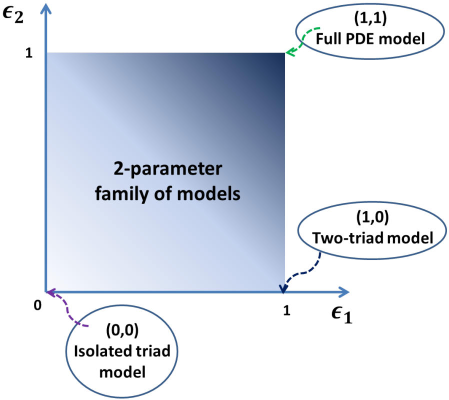
Z^3_{12} not deformed

$\boxed{\mathbf{k_1} + \mathbf{k_2} = \mathbf{k_3}}$



$Z^4_{23} \to \epsilon_1 Z^4_{23}$ $\quad$ $Z^c_{ab} \to \epsilon_2 Z^c_{ab}$, etc.

$\boxed{\mathbf{k_2} + \mathbf{k_3} = \mathbf{k_4}}$ $\quad$ $\boxed{\mathbf{k_a} + \mathbf{k_b} = \mathbf{k_c}}$



Results for two-triad case ($\epsilon_1 \neq 0$, $\epsilon_2 = 0$) **1/3**

- Two connected triads: $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4$, with $\mathbf{k}_4 = (3, 0)$ and $\omega^{\mathbf{k}_4}_{\mathbf{k}_2\mathbf{k}_3} = -\frac{8}{9}$ (freq. mismatch)
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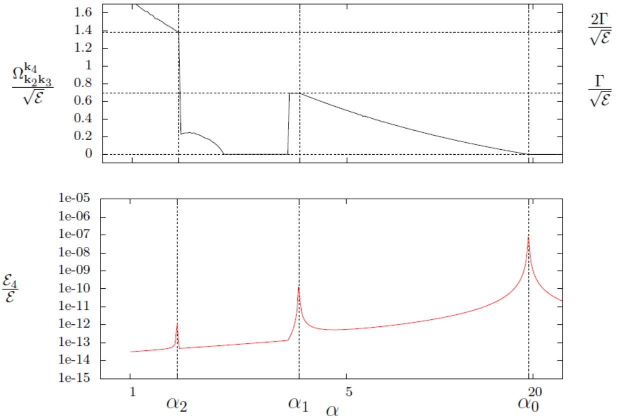
Therefore, initial conditions satisfying

$$\alpha_p = \frac{10.6272}{(0.740382 + p)^2}\,, \quad p = 0, 1, \ldots$$

should show strong transfers towards $n_{\mathbf{k}_4}$.

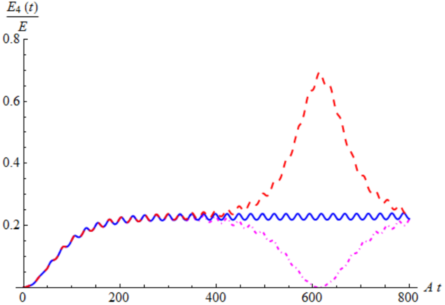
Results for two-triad case ($\epsilon_1 \neq 0$, $\epsilon_2 = 0$) **2/3**

- Integrate numerically evolution equations, from time $t = 0$ to $t = 2000/\sqrt{\mathcal{E}}$.
- Timescale of strong transfer: $t \sim 20/\sqrt{\mathcal{E}}$
- Plots of Triad Precession and Efficiency versus α :



Results for two-triad case ($\epsilon_1 \neq 0$, $\epsilon_2 = 0$) **3/3**

Why the peaks of efficiency? Unstable manifolds! (e.g., periodic orbits)



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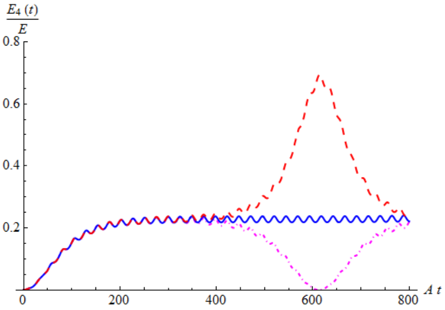
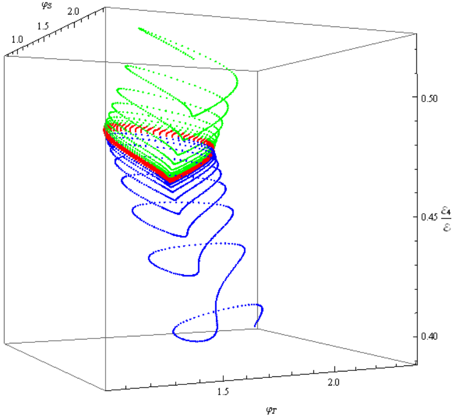
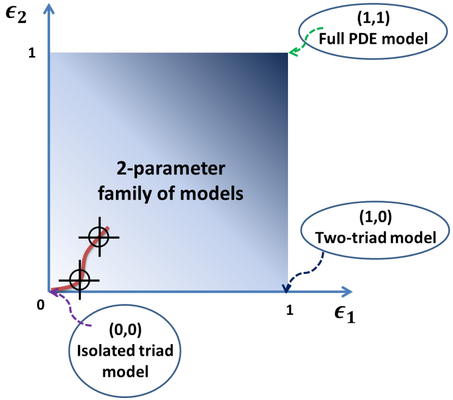




Results for family-model case ($\epsilon_1 \neq 0$, $\epsilon_2 \neq 0$) **1/2**

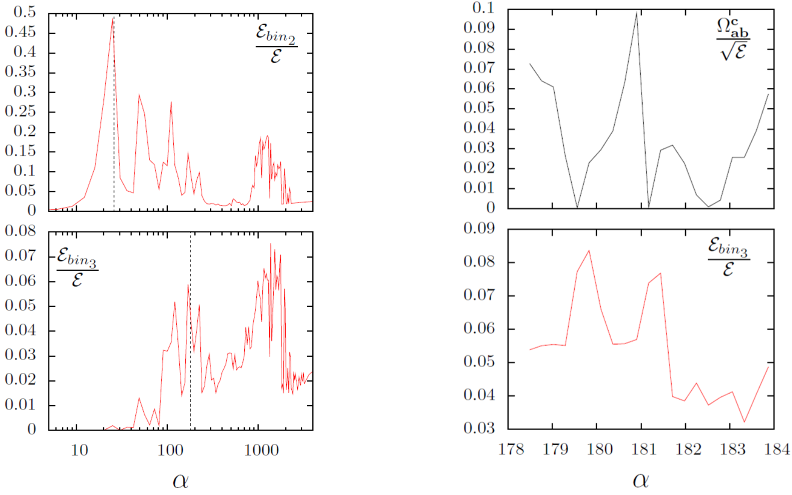
Role of invariant manifolds is **very important:**

- They are **persistent** in parameter space (ϵ_1, ϵ_2)
- We can **"trace"** the invariant manifolds along the parameter space
- **New precession resonances involving new modes**



Results for family-model case ($\epsilon_1 \neq 0$, $\epsilon_2 \neq 0$) **2/2**

Triad initial condition. "Tracing" method until $\epsilon_1 = \epsilon_2 = 0.1$
Pseudospectral method, 128^2 resolution (3500 modes) $\Longrightarrow$ look at "bins"

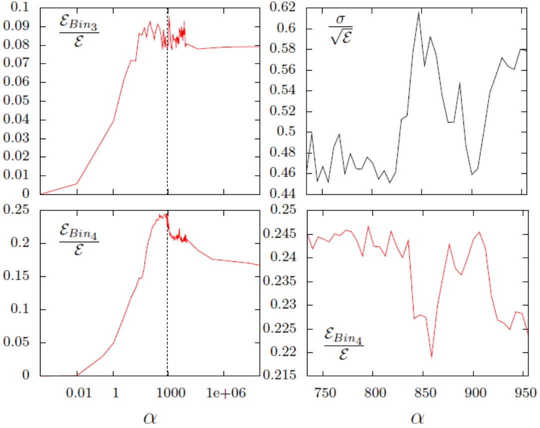


Results for Full-PDE case ($\epsilon_1 = \epsilon_2 = 1$) **1/2**

- We consider a more general large-scale initial condition: $n_{\mathbf{k}} = 0.0321 \times \alpha |\mathbf{k}|^{-2} \exp\left(-|\mathbf{k}|/5\right)$ for $|\mathbf{k}| \leq 8$ and zero otherwise, where α is the re-scaling parameter
- Total enstrophy: $\mathcal{E} = 0.156\alpha$
- Initial phases $\phi_{\mathbf{k}}$ are chosen randomly and uniformly between 0 and 2π
- Direct numerical simulations: pseudospectral method with resolution 128^2 from $t = 0$ to $t = 800/\sqrt{\mathcal{E}}$
- To study cascades, partition the **k**-space in shell bins defined as follows: $Bin_1 : 0 < |\mathbf{k}| \leq 8$, and $Bin_j : 2^{j+1} < |\mathbf{k}| \leq 2^{j+2} \quad j = 2, 3, \ldots$
- Nonlinear interactions lead to successive transfers $Bin_1 \to Bin_2 \to Bin_3 \to Bin_4$

Results for Full-PDE case ($\epsilon_1 = \epsilon_2 = 1$) **2/2**

- Efficiencies of enstrophy transfers from Bin_1 to Bin_3 and Bin_4 have broad peaks
- These correspond to collective synchronisation of precession resonances
- Strong synchronisation is signalled by minima of the dimensionless precession standard deviation $\sigma = \sqrt{\langle \Omega^2 \rangle - \langle \Omega \rangle^2} / \sqrt{\mathcal{E}}$ averaged over the whole set of triad precessions



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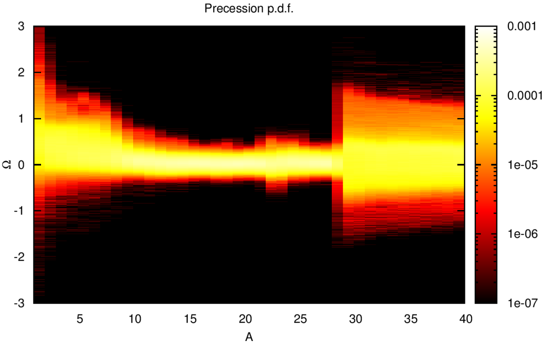
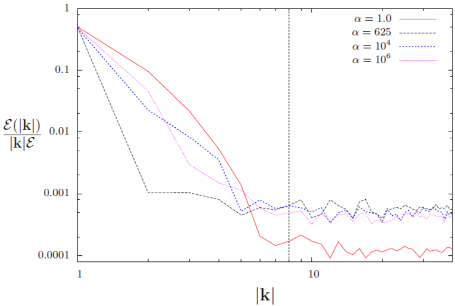


Enstrophy fluxes, equipartition and resolution study (Full-PDE case)

- Time averages ($T = 800/\sqrt{\mathcal{E}}$) of dimensionless enstrophy spectra $\mathcal{E}_k/\mathcal{E}$, compensated for enstrophy equipartition
- In all cases the system reaches small-scale equipartition (Bin_2–Bin_4) quite soon: $T_{\mathrm{eq}} \approx 80/\sqrt{\mathcal{E}}$



- The flux of enstrophy from large scales (Bin_1) to small scales (Bin_4) is 50% **greater in the resonant case ($\alpha = 625$)** than in the limit of very large amplitudes ($\alpha = 10^6$)
- At double the resolution (256^2), the enstrophy cascade goes further to Bin_5 and all above analyses are verified, with Bin_4 replaced by Bin_5

Conclusions and Extensions

- There is vast literature on precession-like resonances in galactic dynamics, notably Pluto precession-orbit resonance and orbital 2 : 5 Saturn-Jupiter resonance [8]
- Critical balance turbulence principle [9] is effectively satisfied at the precession resonance, where we fine-tune a nonlinear frequency with the linear frequency mismatch $\omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$
- **Future work:** precession resonance mechanism in water gravity waves, magneto-hydrodynamics
- Quartet and higher-order systems (Kelvin waves in superfluids, nonlinear optics)
- Including forcing and dissipation

[8] TA Michtchenko and S Ferraz-Mello, *Modeling the 5: 2 mean-motion resonance in the jupiter–saturn planetary system*, Icarus **149** (2001), no. 2, 357–374

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THANK YOU!

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