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Singular Solitons and Spectral Theory

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References: Novikov's Homepage
www.mi.ras.ru/~snovikov
click Publications, items 175,176,182, 184. New Results were published recently in the Journal of Brazilian Math Society, December 2013, special volume dedicated to the 60th Anniversary of IMPA

Problem: Can one extend the idea of Isospectral Deformation to the Singular Solutions of the KdV Equation?

Is it possible to construct Spectral Theory on the real line for the real singular 1D Schrodinger Operators $L = -\partial_x^2 + u(x,t)$ satisfying to KdV such that time dependence is an isospectral deformation?

Example: Lame-Hermit Singular Potentials and their trigonometric or rational degenerations leading to elliptic (trigonometric, rational) solutions.

Some people investigated ”weakly singular” solutions to KdV. For example, Tao proved about 10 years ago that KdV dynamics is well-posed in the Sobolev spaces H^{-s} for $s \leq 3/10$ at the real x-line (circle). No ideas of Inverse Scattering (Spectral) Transform were used, no claims about isospectral properties has been made. Later Kappeler and Topalov using finite-gap approximation proved this for $s \leq 1$ including isospectral property. This is probably the final limit for the ordinary spectral theory.

However, the well-known fundamental class of exact solutions (outside of the physical theory of solitons and spectral theory in the Hilbert spaces) contains all "singular multisolitons" and "singular finite-gap KdV solutions" (algebro-geometric solutions). We cannot find them in the Sobolev spaces above. They have stronger singularities and the property (below) :

All solutions (for all λ) $L\psi = \lambda\psi$ are x-meromorphic at the real line for all t. This property we take as a definition. Question: Is this true for all locally x-meromorphic KdV solutions? Is this property locally true for the domains in the x, t plane or it is global property for KdV solutions?

Our Results: 1. The class of admissible potentials $u(x)$ consists of C^{∞} functions with special isolated singularities. Such operators are symmetric in the indefinite inner product in the corresponding spaces of ψ-functions.
2. The right analog of the Fourier Transform on the Riemann Surfaces will be defined below. It is an isometry in the indefinite inner product. Even more: We prove a Completeness Theorem for the real algebro-geometric (or singular finite-gap) potentials. For the general real admissible potentials this problem is open.

All algebrogeometric (AG)or singular finite-gap potentials belong to this class including rational and elliptic solutions. Their singularities are)

$u = \frac{n_k(n_k+1)}{(x-x_k)^2} + \Sigma_j b_{jk}(x-x_k)^{2j} + o((x-x_k)^{2n_k})$ for $j \geq 0$

We consider general real potentials with discrete set of such singular finite at every period for periodic case and finite at the whole line for the rapidly decreasing case.

The spectral theory should be developed in the class of ψ-functions which are C^∞ plus isolated singularities at the real line $\psi(x) = \sum_{j\le n_k} q_j (x-x_k)^{-n_k+2j} + o((x-x_k)^{n_k})$ for $j \ge 0$, nearby of every real singularity of potential u for given moment t. We call it $F_{x_1,\dots,x_M;n_1,\dots,n_M} = F_{X;N}$

The inner product in the space $F_{X;N}$ is
$< \psi, \phi >= \int \psi(x)\bar{\phi}(\bar{x})dx$

It is well-defined here using complex contours avoiding singularities because all residues of the product are equal to zero.

This inner product is indefinite.

We consider either functions rapidly decreasing at infinity ($T = \infty$) or quasiperiodic with Bloch-Floquet condition $\psi(x+T) = \varkappa\psi(x), \psi \in F_{X,N}(\varkappa)$ for $|\varkappa| = 1$. The number of negative squares of inner product in the space $F_{X,N}(\varkappa)$ is equal to $m_{X;N} = \Sigma_k[(n_k+1)/2]$; (It is the Integral of KdV dynamics, so the time deformation is isospectral).

KdV and Schrodinger Operator:

$$u_t(x,t) = 6uu_x - u_{xxx}$$

$$L\psi = -\psi_{xx} + u\psi = \lambda\psi$$

Classical Theory: Spectral Theory of Rapidly Decreasing and Periodic Schrodinger Operators L requires NONSINGULARITY of Potential $u(x)$ as well as physical derivation of KdV in the Theory of Solitons.

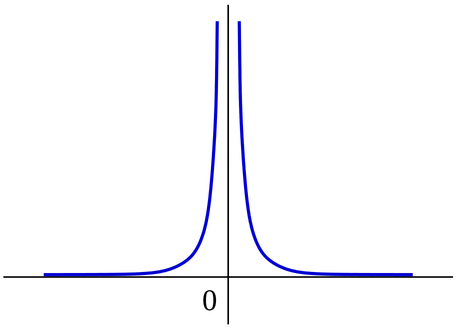
A number of other applications of KdV theory was discovered later which do not require nonsingularity. Huge Literature is dedicated to the singular KdV Solutions. A Theory of Rational and Elliptic Solutions is especially popular.

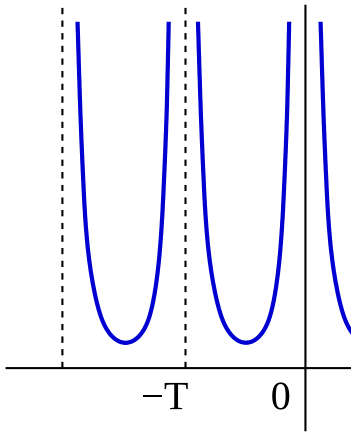
Example: For $j = 1, ..., \frac{n(n+1)}{2}$ there are Real Rational and Elliptic Solutions

$u(x,t) = \sum_j 2/(x - x_j(t))^2$

$u(x,t) = \sum_j 2\wp(x - x_j(t))$

let $u(x,0) = n(n+1)/x^2$





and $u(x,0) = n(n{+}1)\wp(x)$;
(the famous Lame' Potentials.)

Hermit found Spectrum with Dirichlet boundary conditions for $x = 0, T$. Here T is a real period. No spectral theory was constructed on the real line. For $n = 1$ this solution is a SINGULAR TRAVELING WAVE $u = 2\wp(x - at)$ with 2nd order pole in the point $x = at$. Don't Confuse it with NONSINGULAR TRAVELING WAVE $u = 2\wp(x + i\omega' - at)$ where $2i\omega'$ is an imaginary period. This is a first example of periodic finite-gap potentials found in 1950s.

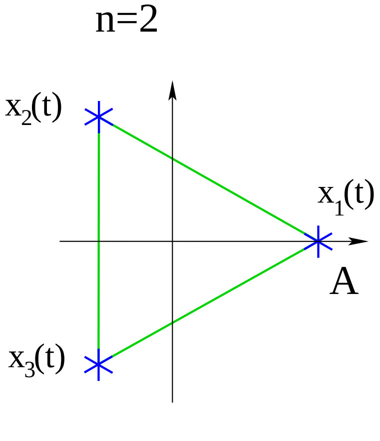
The evolution of Lame' Potentials

$$u(x,0) = n(n+1)\wp(x)$$

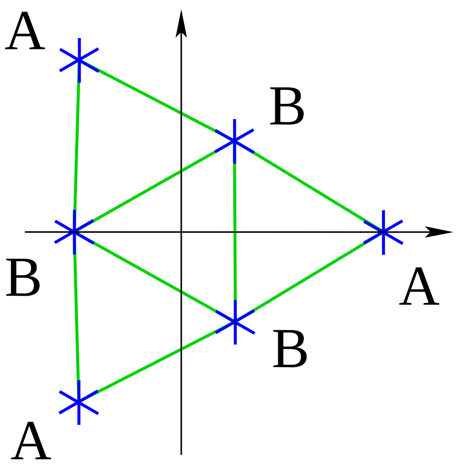
or $u(x,0) = n(n+1)/x^2$ leads to singular solutions

Important Technical Question:

How many real poles these solutions have for $t > 0$?

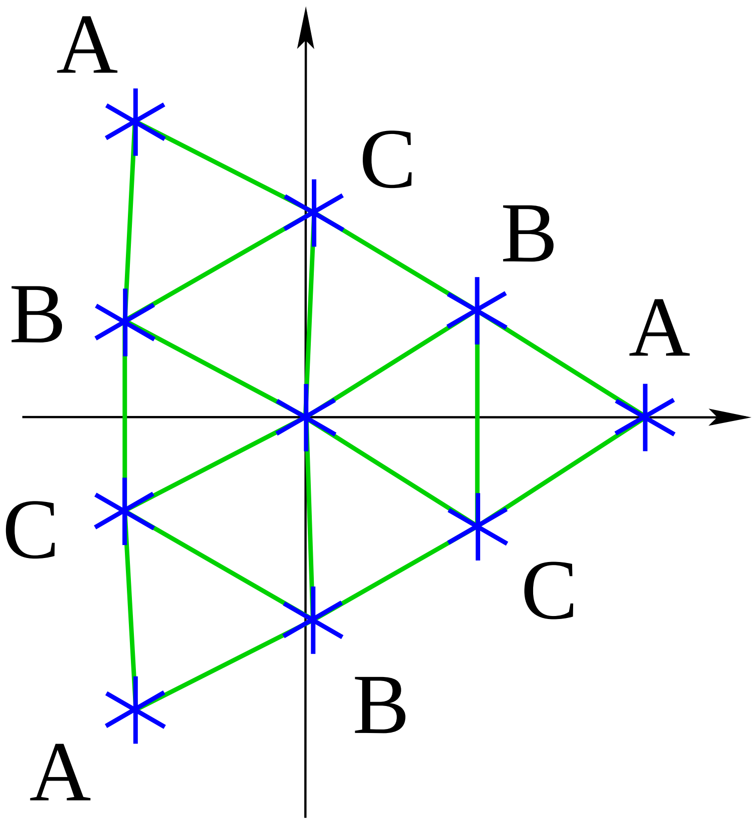


n=3



The orbits of group $Z/3Z$ are marked here. We have $1, 1, 2, 2, 3, ...$ real poles for $n = 1, 2, 3, 4, 5....$

$n=4 \qquad \frac{n(n+1)}{2}=10$



$$x_j \sim r_j t^{1/3}$$

The symmetry group $Z/3Z$ acts here

$r_j \to \zeta r_j, \zeta^3 = 1$

Our Result: The number of real poles is equal to $[(n+1)/2]$. This number is equal to the number of negative squares for the Inner Product in the Spaces of functions on the real line where the operator $L = -\partial_x^2 + u(x,t)$ is symmetric. Physicists Arkad'ev, Polivanov and Pogrebkov constructed earlier some kind of Scattering Theory for the potentials with singularities like $2/(x-x_k)^2$. No spectral theory was discussed.

How Singular Solitons can be used? We used them to define right analog of Fourier Transform on Riemann Surfaces.

What is Fourier Transform on Riemann Surfaces? Which Problems need it? Why singular Solitons are important?

Example: The Fourier/Laurent Series for the contours which are the time-sections of the world-sheet Riemann Surfaces (the "String Diagrams") was constructed by Krichever-Novikov (1986-1990) realizing The Program of Operator Quantization of the Closed Strings). Some Singular solutions to the 2D Toda System were used.

Continuous Fourier Transform on Riemann Surfaces was invented in our works with Grinevich using singular finite-gap solutions to KdV and KP (2003-2010). It is based on the Maximally Singular Baker-Akhiezer (BA) function— i.e. its poles concentrated at the infinite point of Riemann Surface only. We found out that Spectral Problem here is associated with the Indefinite Inner Product for genus $g > 0$.

There are many orthonormal bases in Mathematics and Applications ("Wavelets", for example) but Fourier base has remarkable multiplicative properties.
They are important for Nonlinear Problems. The notion of Resonances is based on multiplication.
It was critical to have bases with good MULTIPLICATIVE properties on Riemann Surfaces for the operator quantization of strings.

Let BA-functions $\Psi(\lambda, x)$ be Maximally Singular.

Example: The classical periodic Lame' operators $-\partial_x^2 + g(g+1)\wp(x)$.

Do Singular Operators have reasonable spectral theory on the whole real line x?

Classical people like Hermit considered spectrum of Lame' potentials only at the interval $[0, T = 2\omega]$ with zero boundary conditions. The inner product needed here is positive. We have to use the whole x-line in order to construct Fourier Transform with good multiplicative properties.

Ψ-functions for smooth real periodic operators do not have good multiplicative properties.

Consider real (may be singular) "finite-gap" periodic operator with spectral curve (Riemann Surface) Γ:

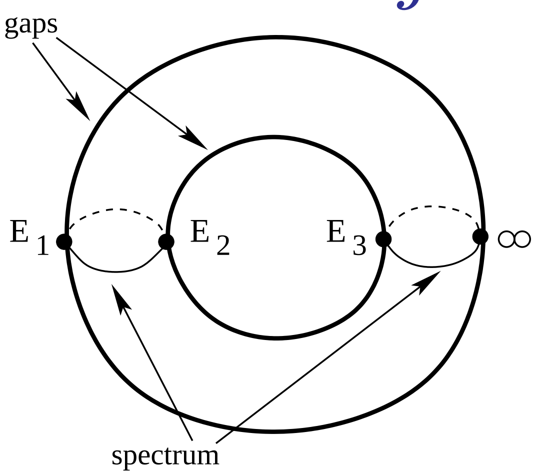
$\mu^2 = (E-E_1)\cdots(E-E_{2g+1})$ with involution $\sigma(E,\mu) = (E,-\mu)$ and poles of Ψ-function $D = \gamma_1, ..., \gamma_g$.

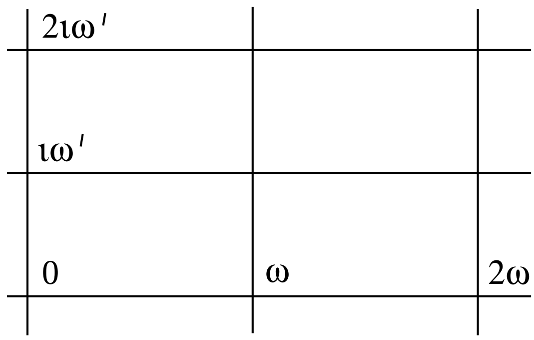
Real case corresponds to the data where Γ and poles are invariant under conjugation $\tau(E,\mu) = (\bar{E},\bar{\mu})$.

The spectrum of operator (see below) is equal to the projection on the complex λ-plane of the τ-invariant Canonical Contour κ_0.

Example 1: Let $g = 1$ (Γ is a torus)

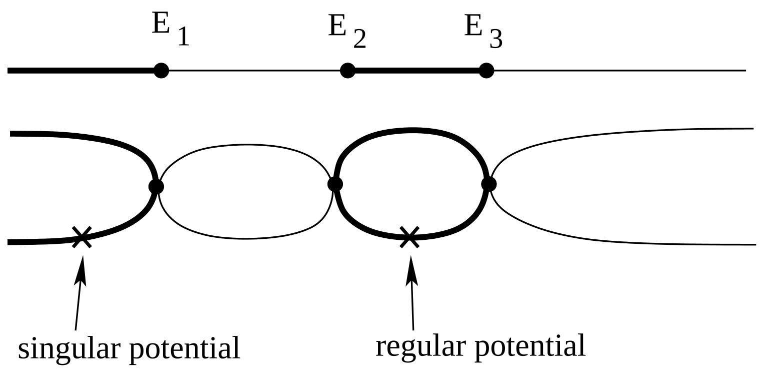
and all E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$ -function in this case is **rectangular with periods** $2\omega, 2i\omega'$.

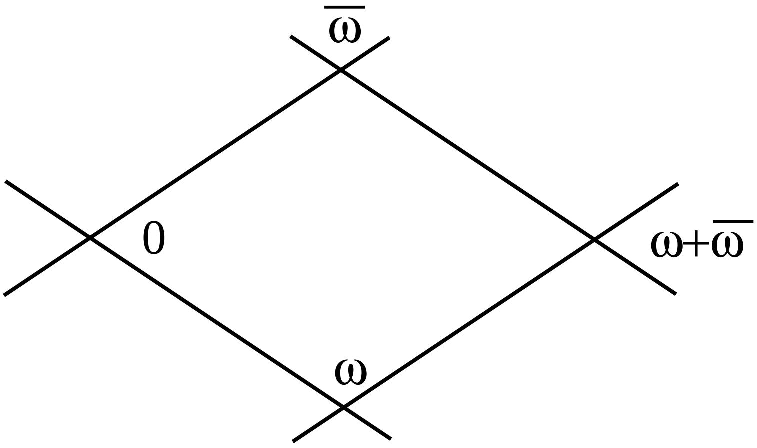
The spectrum is real , and spectral gaps are $[-\infty, E_1]$ and $[E_2, E_3]$, $\tau = id$ at κ_0



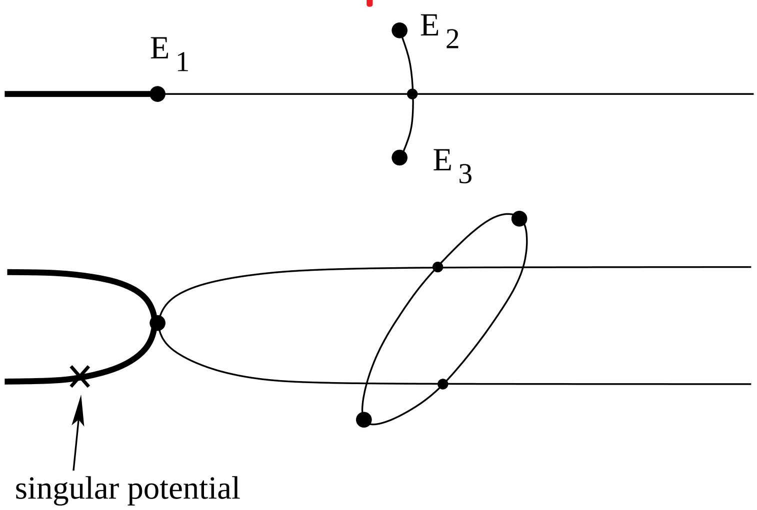
κ_0 is represented by fine lines.

The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For Regular Case it belongs to the finite gap, for the Singular Case it belongs to the infinite gap (They both are the shifted Hermit-Lame Operators but in regular case the shift is imaginary, in singular case the shift is real).

Example 2. Let $g = 1, E_1 \in R, E_3 = \overline{E_2}$:



The lattice of periods is



rombic. κ_0

given by fine lines.

The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E-plane$. It contains complex arc joining $E_2, \bar{E}_2$ and $\tau \neq id$ at κ_0

Define the "spectral measure" $d\mu$. Let λ_j= projection of poles:

$$d\mu = \frac{(E-\lambda_1)...(E-\lambda_g)dE}{2\sqrt{(E-E_1)...(E-E_{2g+1})}}$$

For every smooth function on the contour κ_0 with decay fast at infinity, we define

Direct and Inverse Spectral Transform:

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda)\Psi(\sigma\lambda, x)d\mu(\lambda(E) \tag{1}$$

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_R \tilde{\phi}(x)\Psi(\lambda, x)dx \tag{2}$$

We call it R-Fourier Transform if all $\lambda_j = \infty$; $d\mu^F = dE/2\sqrt{(E-E_1)\dots(E-E_{2g+1})}$, Our base has good multiplicative properties:

$$\Psi(x,\lambda)\Psi(y,\lambda) = l\Psi(x+y,\lambda)$$
$$l = (\partial_z^g + \zeta(z)\partial_z^{g-1} + \ldots)$$
$$\lambda = (E, \pm), z = x + y$$

In the Regular Case $\tau = id$ at κ_0 and measure is positive. This Spectral Transform is an Isometry between the Hilbert spaces with positive inner products

$$< \psi_1, \psi_2 >_{\kappa_0} =$$
$$\int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\tau\lambda)d\mu(\lambda)$$

$$< f_1, f_2 >_R = \int_R f_1(x)\overline{f}_2(x)dx$$

Consider Singular Potentials

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid after a natural regularization.

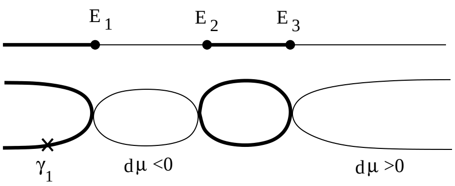
2) Spectral Transform is an isometry between the spaces with indefinite metric described above.

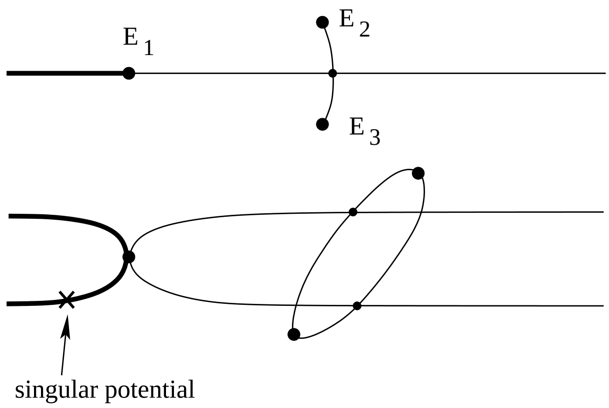
All singularities have a form described above

Example 1. All branching points are real: τ acts identically on κ_0, the form $d\mu$ is negative somewhere. For R-Fourier Transform we have: $d\mu^F/dp > 0$ exactly in every second component starting from the infinite one; So we have $[(g+1)/2]$ "negative" finite components in κ_0.

Example 2. Some pair of branching points is complex adjoint: τ is not identity in the nonreal components of κ_0; So the inner product is nonlocal and therefore indefinite.

We proved Completeness Theorem in the spaces $F_{X,N}(\varkappa)$ which are similar to the Pontryagin-Sobolev spaces





Every function on the line $f(x) \in \mathcal{L}_2(R)$ can be written as a direct integral of the Bloch-Floquet spaces such that $f(\varkappa, x+T) = \varkappa f(\varkappa, x)$. The space $F_{X,N}$ also is a direct integral of Bloch-Floquet spaces $f \in F_{X,N}(\varkappa), |\varkappa| = 1$: Our inner product has r negative squares in the space $F_{X,N}(\varkappa), r = [(g+1)/2]$ for the R-Fourier case.

Remark: Singular Bloch-Floquet eigenfunctions are known for the $k+1$-particle Moser-Calogero operator with Weierstrass pair-wise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. No

one function is known for $k > 1$ serving the discrete spectrum in the space $\mathcal{L}_2$ of the bounded domain inside of poles. Our case corresponds to $k = 1$. We believe that for all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k.