# Particular solutions to a multidimensional version of $n$-wave type equation.

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Dealing with generalizations of the dressing method, the following problems must be underlined.

1) Possibility to construct the new types of higher dimensional PDEs.

2) Provide the rich manifold of particular solutions.

3) Whether the new PDEs are reducible to the known types of PDEs?

4) If no, then what is the relation (reduction) to the classical algorithm.

5) Applicability.

## The general form of $D^2$-dimensional nonlinear PDE

$$\sum_{m_2=1}^{D}\sum_{m_1=1}^{D} L^{(m_1)}\Big(U_{t_{m_1m_2}} + Upp^T\hat{a}^{(1;m_1 1)}\hat{a}^{(1;1m_2)}s^{-1}U - Us^{-1}\hat{a}^{(1;m_1 1)}\hat{a}^{(1;1m_2)}pp^TU\Big)R^{(m_2)} = 0,$$

$U$ is $N\times N$ field, $L^{(m_1)}$, $R^{(m_2)}$ and $s$ are arbitrary diagonal constant matrices, $\hat{a}^{(1;m_1 1)}$ and $\hat{a}^{(1;1m_2)}$ are diagonal constant matrices depending on $L^{(m_1)}$ and $R^{(m_2)}$, $p$ is $N\times 1$ matrix of units.

**Reduction:**

$$U = U^+,\ \ t_{m_1m_2} = t_{m_2m_1},\ \ m_2 \geq m_1,\ \ L^{(m_1)} = R^{(m_1)},$$
$$\mathrm{Re}(s) = 0,\ \ \mathrm{Im}\,(\hat{a}^{(1;m_1 1)}) = 0,\ \ \hat{a}^{(1;m_1 1)} \equiv \hat{a}^{(1;1m_1)}.$$

**To compare: the completely integrable (2+1)-dim. $n$-wave eq.**

$$\sum_{m_2=1}^{3}\sum_{m_1=1}^{3} L^{(m_1)}\Big(U_{t_{m_1m_2}} + UC^{(m_1m_2)}U\Big)L^{(m_2)} = 0,$$

$C^{(m_1m_2)} = -C^{(m_2m_1)}$, $C^{(m_1m_1)} = 0$, $t_{m_1m_2} = -t_{m_2m_1}$,
$C^{(23)} = L^1$, $C^{(31)} = L^2$, $C^{(12)} = L^3$, (usually $L^{(1)} = I$),
$t_1 = t_{23}$, $t_2 = t_{31}$, $t_3 = t_{12}$, while $t_{m_1m_1}$ disappear.

# THIS PDE IS NOT COMPLETELY INTEGRABLE

**Relation among the parameters $D^2$ (number of independent variables), $N$ (matrix dimensionality) and $K$ (dimensionality of solution space).**

**Arbitrary function of $\tilde{K}$ var., $x_i = x_i(t_{m_1 m_2})$:**

$\int F(\mu_1, \ldots, \mu_{\tilde{K}}) \exp\left(\sum_{i=1}^{\tilde{K}} \mu_k x_i\right)$ .

$U$ depends on $(N - \tilde{K} + 1)(N - \tilde{K} + 2)$ arbitrary scalar functions of $2\tilde{K}$ var., $3N - 1$ arbitrary scalar functions of $\tilde{K}$ var.

The number of independent variables is $D^2 = 4(\tilde{K} + 1)^2$.

**The matrix dimensionality and "the nonlinearity degree":**

Nonlinearity disappears: $N < 2(\tilde{K} + 1)$.

The "partial nonlinearity" : $2(\tilde{K} + 1) \leq N < 2D - 1 = 4\tilde{K} + 3$.

The completely nonlinear PDE: $N \geq 4\tilde{K} + 3$.

## CONTENT

**1. Derivation of nonlinear PDE**

**Lin. int. eq. relating $\Psi(\lambda, \mu; t)$ (the kernel) and $W(\lambda; t)$ $\rightarrow$**
**Lin. PDE for $W(\lambda; t)$ with potential $V(t)$ $\rightarrow$**
**PDE for $V(t)$**

**2. Construction of the kernel $\Psi(\lambda, \mu; t)$; link to the completely integrable (2+1)-dim. $n$-wave equation**

**3. Solutions $U(t)$**

# 1. Derivation of the nonlinear PDEs

## 1.1. Linear integral equation

$$P(\lambda) = W(\mu;t) * \Psi(\mu,\lambda;t) + W(\lambda;t),$$

where $P(\mu)$, $\Psi(\mu,\lambda;t)$, $W(\lambda;t)$ are $N\times N$ matrix functions of arguments, $t_i$ are independent variables of nonlinear PDEs, $t=(t_1,\dots,t_D)$, $\lambda$, $\mu$, $\nu$ are complex vector parameters. Here $*$ means the integral operator:

$$f(\mu) * g(\mu) = \int f(\mu) g(\mu) d\Omega(\mu),$$

$\Omega(\mu)$ is some measure. Introduce the unit $\mathcal{I}(\lambda,\mu)$ operator:

$$f(\lambda,\nu) * \mathcal{I}(\nu,\mu) = \mathcal{I}(\lambda,\nu) * f(\nu,\mu) = f(\lambda,\mu).$$

$$\Rightarrow \quad W = P * (\Psi + \mathcal{I})^{-1}.$$

The $t$-dependence ($B^{(m_1m_2)}$, $A$, $C^{(m_1m_2)}$ are $N\times N$ matrices):

$$\Psi_{t_{m_1m_2}}(\lambda,\mu;t) = \Big(B^{(m_1m_2)}(\lambda,\nu) + A(\lambda)C^{(m_1m_2)}P(\nu)\Big) * \Psi(\nu,\mu;t) -$$
$$\Psi(\lambda,\nu;t) * (B^{(m_1m_2)}(\nu,\mu) - A(\nu)C^{(m_1m_2)}P(\mu)),$$

## 1.2. System of compatible linear equations for $W(\lambda;t)$.

**Theorem 1:** Let matrices $B^{(m_1 m_2)}(\lambda,\mu)$ satisfy the constraints:

$$\sum_{m_1=1}^{D} L^{(m_1)} P(\lambda) * (B^{(m_1 m_2)}(\lambda,\mu) - A(\lambda) C^{(m_1 m_2)} P(\mu)) = 0, \quad (1)$$
$$m_2 = 1,\dots,D,$$

where $L^{(m_1)}$ are some $N\times N$ constant matrices. Then the matrix function $W(\lambda;t)$ is a solution to the following system of linear equations

$$E^{(m_2)}(\lambda;t) := \quad (2)$$
$$\sum_{m_1=1}^{D} L^{(m_1)} \Big( W_{t_{m_1 m_2}}(\lambda;t) + V(t) C^{(m)} W(\lambda;t) +$$
$$W(\mu;t) * (B^{(m_1 m_2)}(\mu,\lambda) + A(\mu) C^{(m_1 m_2)} P(\lambda)) \Big) = 0, \quad m_2 = 1,\dots,D,$$

where

$$V(t) = -2W(\mu;t) * A(\mu).$$

**Proof:** Differentiate equation

$$P(\lambda) = W(\mu;t) * (\Psi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)),$$

with respect to $t_{m_1m_2} \Rightarrow$:

$$\begin{aligned}
&\mathcal{E}^{(m_1m_2)}(\mu;t) := \\
&P(\nu) * (B^{(m_1m_2)}(\nu,\mu) - P(\nu)C^{(m_1m_2)}A(\mu)) = \\
&\quad \tilde{E}^{(m_1m_2)}(\nu;t) * (\Psi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)), \\
&\quad \tilde{E}^{(m_1m_2)}(\lambda;t) = W_{t_{m_1m_2}}(\lambda;t) + V(t)C^{(m_1m_2)}W(\lambda;t) + \\
&\quad W(\mu;t) * (B^{(m_1m_2)}(\mu,\lambda) + A(\mu)C^{(m_1m_2)}P(\lambda)).
\end{aligned}$$

Consider the following combination: $\sum_{m_1=1}^{D_1} L^{(m_1)}\mathcal{E}^{m_1m_2}$. Using the constraint (1), one gets:

$$\sum_{m_1=1}^{D} L^{(m_1)}\mathcal{E}^{m_1m_2} := \sum_{m_1=1}^{D} L^{(m_1)}\tilde{E}^{(m_1m_2)}(\nu;t) * (\Psi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu)) = 0.$$

Finally, invert the operator $*(\Psi(\nu,\mu;t) + \mathcal{I}_1(\nu,\mu))$.

## 1.3. First order nonlinear PDEs for the field $V(t)$

**Theorem 2.** Impose another set of constraints:

$$\sum_{m_2=1}^{D} (B^{(m_1 m_2)}(\lambda,\nu) + A(\lambda) C^{(m_1 m_2)} P(\nu)) * A(\nu) R^{(m_2)} = 0, \quad (3)$$
$$m_1 = 1, \ldots, D,$$

where $R^{(m_2)}$ are some $N \times N$ constant matrices. Then the $N \times N$ matrix functions $V(t)$ is a solution to the following system of nonlinear PDEs:

$$\sum_{m_1,m_2=1}^{D} L^{(m_1)} \left( V_{t_{m_1 m_2}} + V C^{(m_1 m_2)} V \right) R^{(m_2)} = 0. \quad (4)$$

**Proof:** Applying operator $*(-2A)$ to eq.(2)

$$\sum_{m_1=1}^{D} L^{(m_1)} \Big( W_{t_{m_1 m_2}}(\lambda;t) + V(t)C^{(m)}W(\lambda;t) +$$
$$W(\mu;t) * (B^{(m_1 m_2)}(\mu,\lambda) + A(\mu)C^{(m_1 m_2)}P(\lambda)) \Big) = 0, \quad m_2 = 1,\dots,D,$$

from the right one gets the following equation

$$E^{(m_2)}(t) = E^{(m_2)}(\lambda;t) * A(\lambda) := \tag{5}$$
$$\sum_{m_1=1}^{D} L^{(m_1)} \Big( V_{t_{m_1 m_2}} + VC^{(m_1 m_2)}V + U^{(m_1 m_2)} \Big) = 0,$$

which introduces a new set of fields $U^{(m_1 m_2)}$

$$U^{(m_1 m_2)}(t) = -2W(\mu;t) * (B^{(m_1 m_2)}(\mu,\nu) + A(\mu)C^{(m_1 m_2)}P(\nu)) * A(\nu).$$

Due to the external constraints (3), the combinations $\sum_{m_2=1}^{D_2} E^{(m_2)} R^{(m_2)}$ results in eq. (4).

## 2. Construction of the kernel $\Psi$; here $m = \{m_1, m_2\}$.

$$\Psi(\lambda,\mu) = \chi(\lambda,\nu) * \Big(\varepsilon(\nu;t)\mathcal{C}(\nu,\tilde{\nu})\tilde{\varepsilon}(\tilde{\nu},t)\Big) * \tilde{\chi}(\tilde{\nu},\mu). \qquad (6)$$

where $m = (m_1, m_2)$,

$$\varepsilon(\nu;t) = e^{\sum_m T^{(m)}(\nu)t_m}, \quad \tilde{\varepsilon}(\nu;t) = e^{-\sum_m T^{(m)}(\nu)t_m}.$$

Here $\chi$ and $\tilde{\chi}$ are $N \times N$ invertible matrix operators, $T^{(m)}$ is the diagonal $N \times N$ matrix functions of the spectral parameter. The $t$-dependence of $\Psi$ yields:

$$\chi T^{(m)} - (B^{(m)} + AC^{(m)}P) * \chi = 0, \qquad (7)$$
$$T^{(m)}\tilde{\chi} - \tilde{\chi} * (B^{(m)} - AC^{(m)}P) = 0. \qquad (8)$$

Compatibility of these two equations reads:

$$R(\lambda,\mu)T^{(m)}(\mu) - T^{(m)}(\lambda)R(\lambda,\mu) = 2r(\lambda)C^{(m)}\tilde{r}(\mu), \quad \forall m \qquad (9)$$
$$R(\lambda,\mu) = \tilde{\chi} * \chi, \quad r(\lambda) = \tilde{\chi}(\lambda,\nu) * A(\nu), \quad \tilde{r}(\mu) = P(\nu) * \chi(\nu,\mu).$$

## 2.1. Classical (2+1)-dimensional $n$-wave equation

corresponds to the diagonal matrices in eq.(9). Therewith

$$T^{(m_1m_2)}(\lambda) = \lambda C^{(m_1m_2)}, \quad R(\lambda,\mu) = \frac{r(\lambda)\tilde{r}(\mu)}{\mu-\lambda}.$$

$n$-wave equation:

$$[C^{(3)}, U_{t_2}] - [C^{(2)}, U_{t_3}] + C^{(2)}U_{t_1}C^{(3)} - C^{(3)}U_{t_1}C^{(2)} + [[C^{(3)}, U], [C^{(2)}, U]] = 0,$$

$$C^{(23)} \equiv C^{(1)} = I_N, \quad C^{(13)} \equiv C^{(2)}, \quad C^{(12)} \equiv C^{(3)},$$
$$t^{(1)} \equiv t^{(32)}, \quad t^{(2)} \equiv t^{(13)}, \quad t^{(3)} \equiv t^{(12)}.$$

## 2.2. Non-classical multidimensional case.

$$C^{(m)} = \xi\xi^{(m)} - \eta^{(m)}\eta,$$
$$R(\lambda,\mu) = \delta(\lambda-\mu) + r(\lambda)\xi\eta\tilde{r}(\mu),$$

where $\xi$ and $\eta^{(m)}$ are $N\times 1$ constant matrices, and $\xi^{(m)}$ and $\eta$ are $1\times N$ constant matrices. Substitute into the compatibility condition $R(\lambda,\mu)T^{(m)}(\mu) - T^{(m)}(\lambda)R(\lambda,\mu) = 2r(\lambda)C^{(m)}\tilde{r}(\mu), \;\; \forall m \Rightarrow$

$$r(\lambda)\eta^{(m)} = \frac{1}{2}T^{(m)}(\lambda)r(\lambda)\xi,$$
$$\xi^{(m)}\tilde{r}(\mu) = \frac{1}{2}\,\eta\tilde{r}(\mu)T^{(m)}(\mu).$$

Its solution reads:

$$T^{(m)}_{\alpha}(\lambda) = 2\frac{\sum_{\gamma=1}^{N} r_{\alpha\gamma}(\lambda)\eta^{(m)}_{\gamma 1}}{\sum_{\gamma=1}^{N} r_{\alpha\gamma}(\lambda)\xi_{\gamma 1}}.$$

$$\tilde{r}(\lambda) = r^{T}(\lambda), \;\; \eta = \xi^{T}, \;\; \xi^{(m)} = (\eta^{(m)})^{T}.$$

**Nonlinear equation ($m = \{m_1, m_2\}$):**

$$\sum_{m_1,m_2=1}^{D} L^{(m_1)} \left( V_{t_{m_1 m_2}} + V\xi(\eta^{(m_1 m_2)})^T V - V\eta^{(m_1 m_2)}\xi^T V \right) R^{(m_2)} = 0.$$

**Constraints:**

$$\sum_{m_1} L^{(m_1)} r^T * R^{-1} T^{(m)} = 0, \quad m_2 = 1, \ldots, D,$$

$$\sum_{m_2} T^{(m)} R^{-1} * r R^{(m_2)} = 0, \quad m_1 = 1, \ldots, D.$$

$$T_\alpha^{(m)}(\lambda) = 2\frac{\sum_{\gamma=1}^{N} r_{\alpha\gamma}(\lambda)\eta_{\gamma 1}^{(m)}}{\sum_{\gamma=1}^{N} r_{\alpha\gamma}(\lambda)\xi_{\gamma 1}}, \quad R(\lambda, \mu) = \delta(\lambda - \mu) + r(\lambda)\xi\eta\tilde{r}(\mu).$$

To resolve constraints: $r(\lambda) = \sum_{j=1}^{K} g^{(j)}(\lambda) a^{(j)}$, $g^{(j)}$ – arbitrary independent scalar functions; $\tilde{K} = K - 1$.

**NOTICE:**

In general, solution of the above constraints leads to the following relation: $N = D^2 = (K^2 + 1)^2$, i.e. $\boldsymbol{N = D^2 \sim K^4}$

Thus, if we have arbitrary function of $\tilde{K}$ variables, then both the dimensionality of equation and the matrix dimensionality $\sim \tilde{K}^4$, $\tilde{K} = K - 1$.

**This means that the system is far from the complete integrability.**

But there is a particular solution to the constraints decreasing $N$ as follows:

Nonlinearity disappears from PDE: $N < 2K$.

The ”partial nonlinearity” in PDE: $2K \leq N < 2D - 1 = 4K - 1$.

The completely nonlinear PDE: $N \geq 4K - 1$.

Therewith, as before, $D^2 = (K^2 + 1)^2$

This particular solution of constraints effects on the nonlinear part of PDE.

## RESULT

The nonlinear equation may be written as:

$$\sum_{m_1,m_2=1}^{D} \hat{L}^{(m_1)} \Big( U_{t_{m_1 m_2}} + U p\, p^T \hat{a}^{(1;m_1 1)} \hat{a}^{(1;1 m_2)} s^{-1} U - U s^{-1} \hat{a}^{(1;m_1 1)} \hat{a}^{(1;1 m_2)} p\, p^T U \Big) \hat{R}^{(m_2)} = 0.$$

Here $U$

$$U = \frac{s^{-1} \hat{a}^{(1)} V (\hat{a}^{(1)})^T s^{-1}}{1+Q}, \qquad V(t) = -2W(\mu; t) * A(\mu),$$

where $Q$ is a constant (not arbitrary), $s$ is arbitrary diagonal matrix, and $\hat{a}^{(1)}$ is a constant matrix (not arbitrary)

$$\hat{a}^{(1;m_1 1)}_{\beta} = -\frac{\Delta^{(m_1)}_{\beta}}{\Delta_{\beta}}, \;\; \hat{a}^{(1;1 m_2)}_{\alpha} = -\frac{\tilde{\Delta}^{(m_1)}_{\alpha}}{\tilde{\Delta}_{\alpha}}, \;\; m_1, m_2 = 2, \dots, D, \;\; \hat{a}^{(1;11)}_{\beta} = 1,$$

The admittable reduction:

$$U = U^+, \;\; t_{m_1 m_2} = t_{m_2 m_1}, \;\; m_2 \geq m_1, \;\; L^{(m_1)} = R^{(m_1)},$$
$$\text{Re}(s) = 0, \;\; \text{Im}\,(\hat{a}^{(1;m_1 1)}) = 0, \;\; \hat{a}^{(1;m_1 1)} \equiv \hat{a}^{(1;1 m_1)}.$$

$$\boldsymbol{\Delta}_\beta = \begin{vmatrix} \hat{L}^{(2)}_{\beta-K+1} & \hat{L}^{(3)}_{\beta-K+1} & \cdots & \hat{L}^{(D)}_{\beta-K+1} \\ \hat{L}^{(2)}_{\beta-K+2} & \hat{L}^{(3)}_{\beta-K+2} & \cdots & \hat{L}^{(D)}_{\beta-K+2} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{L}^{(2)}_{\beta} & \hat{L}^{(3)}_{\beta} & \cdots & \hat{L}^{(D)}_{\beta} \\ \hat{L}^{(2)}_{\beta+1} & \hat{L}^{(3}_{\beta+1} & \cdots & \hat{L}^{(D)}_{\beta+1} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{L}^{(2)}_{\beta+K-1} & \hat{L}^{(3)}_{\beta+K-1} & \cdots & \hat{L}^{(D)}_{\beta+K-1} \end{vmatrix},$$

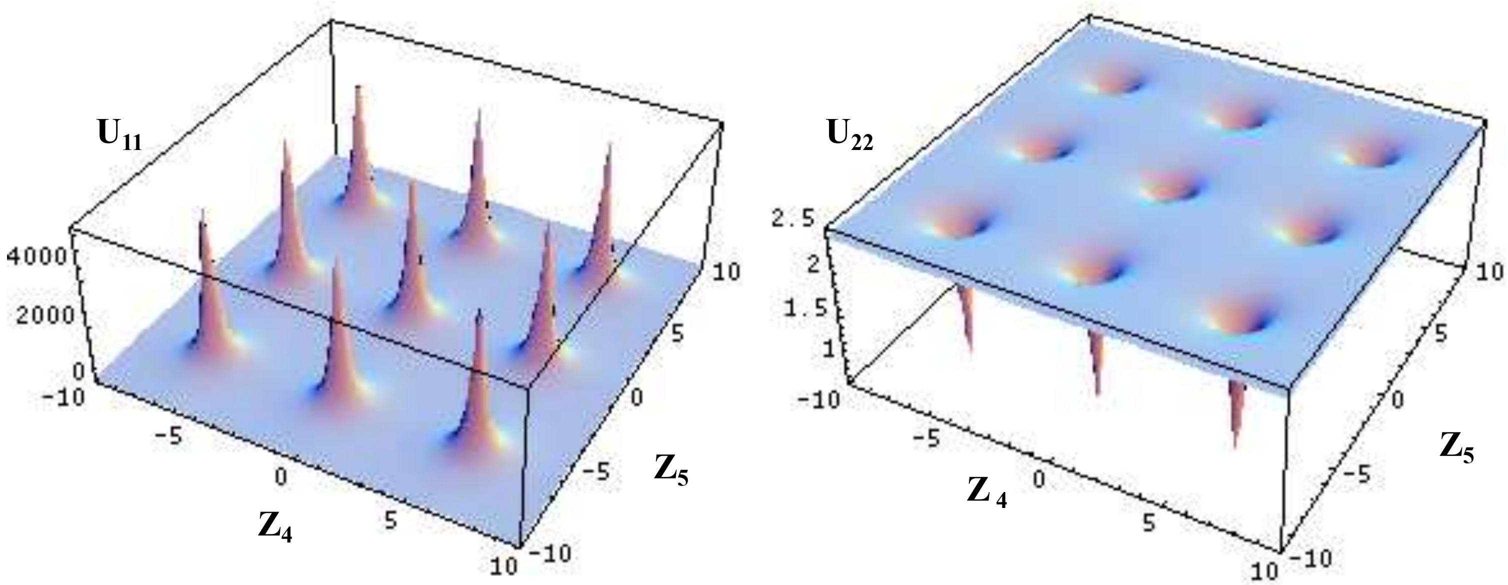
$$\tilde{\boldsymbol{\Delta}}_\alpha = \begin{vmatrix} \hat{R}^{(2)}_{\alpha-K+1} & \hat{R}^{(3)}_{\alpha-K+1} & \cdots & \hat{R}^{(D)}_{\alpha-K+1} \\ \hat{R}^{(2)}_{\alpha-K+2} & \hat{R}^{(3)}_{\alpha-K+2} & \cdots & \hat{R}^{(D)}_{\alpha-K+2} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{R}^{(2)}_{\alpha} & \hat{R}^{(3)}_{\alpha} & \cdots & \hat{R}^{(D)}_{\alpha} \\ \hat{R}^{(2)}_{\alpha+1} & \hat{R}^{(3)}_{\alpha+1} & \cdots & \hat{R}^{(D)}_{\alpha+1} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{R}^{(2)}_{\alpha+K-1} & \hat{R}^{(3)}_{\alpha+K-1} & \cdots & \hat{R}^{(D)}_{\alpha+K-1} \end{vmatrix}.$$

## 3. Example of solution:

## $K = 2$, $D = 4$, $N = 6$, elements $U_{11}$ and $U_{22}$

**Reduction:**

$$U = U^{+}, \quad t_{m_1 m_2} = t_{m_2 m_1}, \quad m_2 \geq m_1, \quad L^{(m_1)} = R^{(m_1)},$$
$$\text{Re}(s) = 0, \quad \text{Im}\,(\hat{a}^{(1;m_1 1)}) = 0, \quad \hat{a}^{(1;m_1 1)} \equiv \hat{a}^{(1;1 m_1)}.$$




$$L^{(1)} = I_N, \quad L_{\alpha}^{(m_1)} = \begin{cases} \alpha^{m_1 - 1}, & m_1 = 2,3,4, \quad \alpha = 1,2,3 \\ (2-\alpha)^{m_1 - 1}, & m_1 = 2,3,4, \quad \alpha = 4,5,6 \end{cases} .$$

$\varepsilon(t) \equiv \varepsilon(p_1, t) = \text{diag}(e^{iX_1}, \ldots, e^{iX_6})$, $X_i = X_i(t_{m_1 m_2})$, $Z_i = X_{i+1} - X_1$, $i = 1, \ldots, 5$, $Z_1 = Z_2 = Z_3 = 0$.