Freak-waves and their simulation

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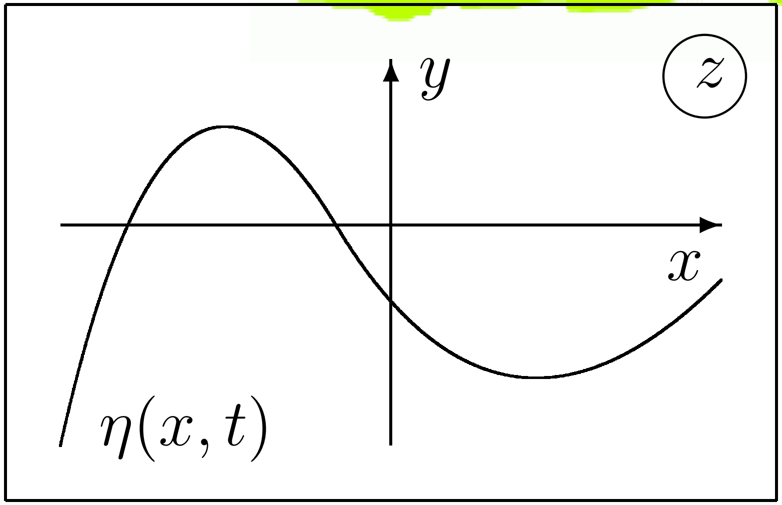
Potential Flow of 2D Ideal Fluid



irrotational

$$\triangle\phi(x,y,t)=0$$

Boundary conditions: $\left[\begin{array}{l}\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho},\\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y\end{array}\right|$ at $y=\eta(x,t)$.

$$\frac{\partial\phi}{\partial y} = 0, y\to-\infty,$$

$$\frac{\partial\phi}{\partial x} = 0, |x|\to\infty, \text{ or periodic}$$

Conformal mapping

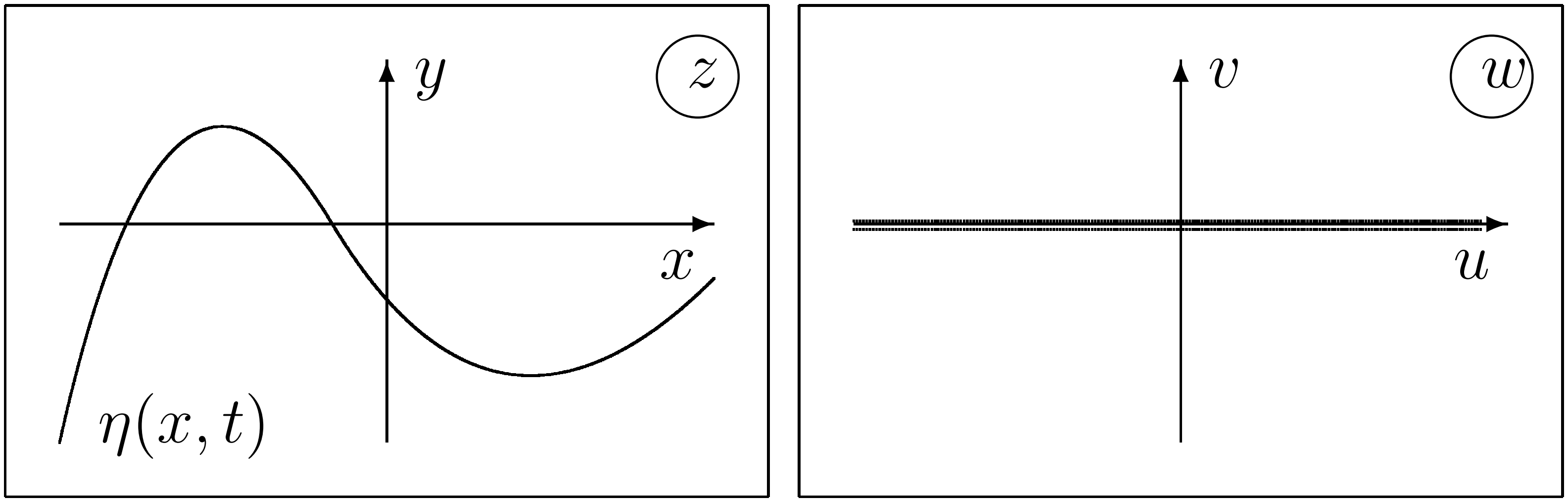
Domain on Z-plane $Z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \le \eta(x,t)\,,$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \le 0\,,$$

W-plane $W = u + iv$.



Equations for Z and Φ

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$Z = x + iy,$ and complex velocity potential $\Phi = \Psi + i\hat{H}\Psi.$

$$
\begin{aligned}
Z_t &= iUZ_u, \\
\Phi_t &= iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|Z_u|^2}\right) + ig(Z - u).
\end{aligned}
$$

U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|Z_u|^2}\right). \qquad u \to w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$.

Cubic equations for R and V

Surface dynamics (and the fluid bulk!) is described by two analytic functions, $R(w,t)$ and $V(w,t)$. They are related to conformal mapping Z and complex velocity potential:

$$R = \frac{1}{Z_w}, \qquad \Phi_w = -iVZ_w.$$

For R and V dynamic equations have the simplest form:

$$\begin{aligned} R_t &= i\left[UR' - U'R\right], \\ V_t &= i\left[UV' - B'R\right] + g(R-1). \end{aligned}$$

Complex transport velocity U is defined as

$$U = \hat{P}(V\bar{R} + \bar{V}R), \qquad \text{and} \qquad B = \hat{P}(V\bar{V}).$$

4th order Hamiltonian

$$H = \frac{1}{2}\int (g\eta^2 + \psi\hat{k}\psi)dx \qquad -\frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \\ +\frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \ldots$$

here $\eta(x,t)$ - is the shape of a surface,

$\psi(x,t) = \phi(x, z = \eta(x,t), t)$ - is a potential at the surface.

Classical variables Ψ, η

Normal complex variable a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

Three-waves and four-wave resonances are absent

$$\begin{aligned} k &= k_1 + k_2, & k + k_1 &= k_2 + k_3 \quad \text{NO!} \\ \omega_k &= \omega_{k_1} + \omega_{k_2}, & \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3} \end{aligned}$$

Cubic and fourth order nonresonat terms can be *excluded* by canonical transformation:

$$a_k \to b_k.$$

Compact Hamiltonian

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int |b'|^2 \left[\frac{i}{2}(bb'^* - b^* b') - \hat{k}|b|^2 \right] dx.$$

Corresponding dynamical equation is

$$\begin{aligned} i\frac{\partial b}{\partial t} = \hat{\omega}_k b \quad &+ \quad \frac{i}{4}\left[b^* \frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime} \frac{\partial}{\partial x} b^2) \right] \\ &- \quad \frac{1}{2}\left[b \cdot \hat{k}(|b'|^2) - \frac{\partial}{\partial x}(b' \hat{k}(|b|^2)) \right]. \end{aligned}$$

Transformation from b_k to η_k and ψ_k

$$\begin{aligned}
\eta(x) &= \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{\frac{1}{4}}\mathbf{b}(x) + \hat{k}^{\frac{1}{4}}\mathbf{b}(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{\frac{1}{4}}\mathbf{b}(x) - \hat{k}^{\frac{1}{4}}\mathbf{b}^*(x)]^2 + \\
\psi(x) &= -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{1}{4}}\mathbf{b}(x) - \hat{k}^{-\frac{1}{4}}\mathbf{b}(x)^*) + \\
&+ \frac{i}{2}[\hat{k}^{\frac{1}{4}}\mathbf{b}^*(x)\hat{k}^{\frac{3}{4}}\mathbf{b}^*(x) - \hat{k}^{\frac{1}{4}}\mathbf{b}(x)\hat{k}^{\frac{3}{4}}\mathbf{b}(x)] + \\
&+ \frac{1}{2}\hat{H}[\hat{k}^{\frac{1}{4}}\mathbf{b}(x)\hat{k}^{\frac{3}{4}}\mathbf{b}^*(x) + \hat{k}^{\frac{1}{4}}\mathbf{b}^*(x)\hat{k}^{\frac{3}{4}}\mathbf{b}(x)] + O(\mathbf{b}^3)
\end{aligned}$$

Fully Nonlinear and Compact Eqs

$$
\begin{aligned}
R_t &= i\left[UR' - U'R\right], \\
V_t &= i\left[UV' - B'R\right] + g(R-1).
\end{aligned}
$$

$U = \hat{P}(V\bar{R} + \bar{V}R),$ and $B = \hat{P}(V\bar{V}).$

$$
\begin{aligned}
i\frac{\partial \mathbf{b}}{\partial t} = \hat{\omega}_k \mathbf{b} \quad &+ \quad \frac{i}{4}\left[\mathbf{b}^* \frac{\partial}{\partial x}(\mathbf{b}'^2) - \frac{\partial}{\partial x}(\mathbf{b}^{*\prime} \frac{\partial}{\partial x}\mathbf{b}^2)\right] \\
&- \quad \frac{1}{2}\left[\mathbf{b} \cdot \hat{k}(|\mathbf{b}'|^2) - \frac{\partial}{\partial x}(\mathbf{b}'\hat{k}(|\mathbf{b}|^2))\right].
\end{aligned}
$$

Modulational instability of wave train

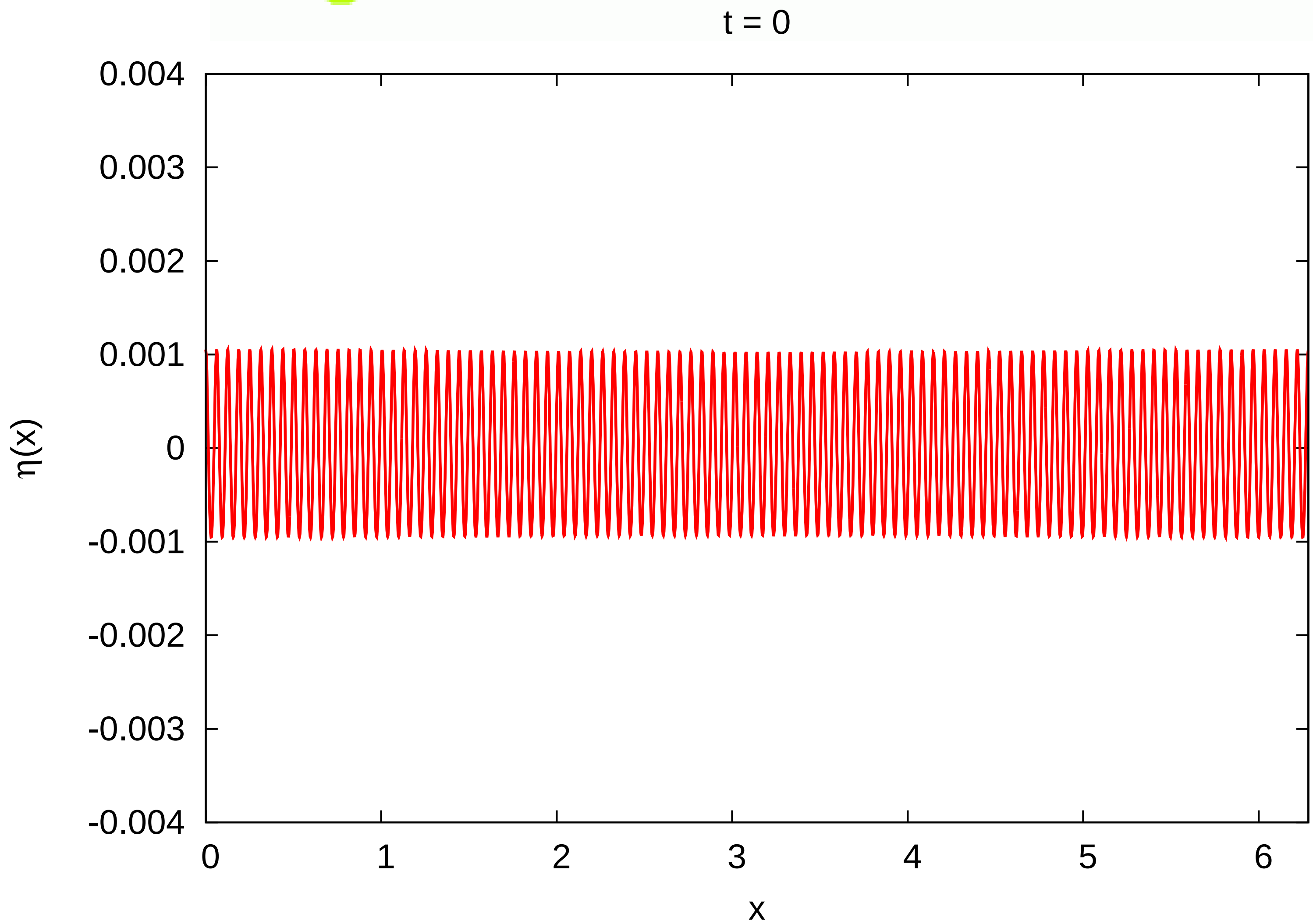


Figure 1. Initial wavetrain.

Breather

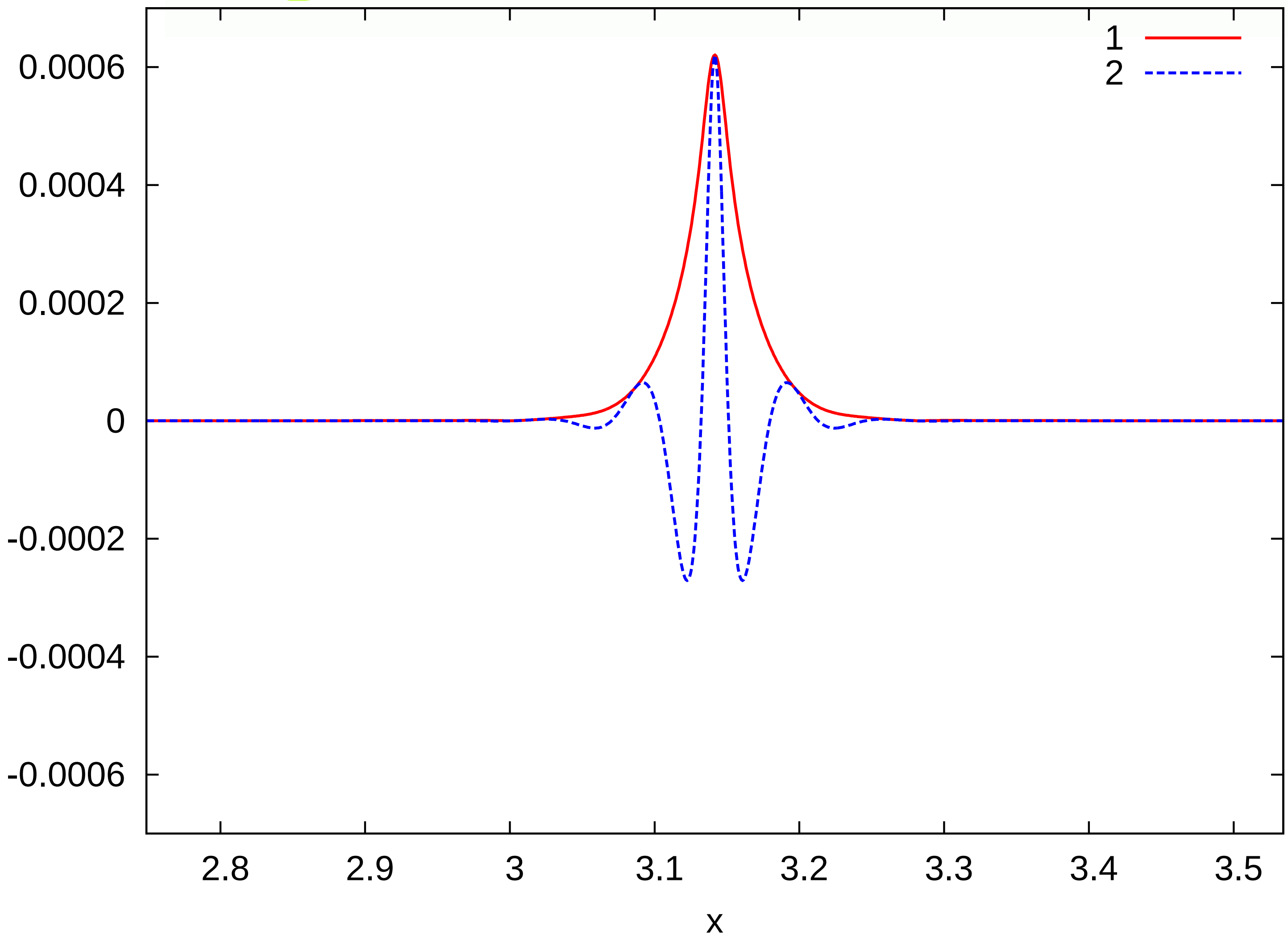


Figure 2. Breather and Envelope

Giant Breather, k-ω spectrum

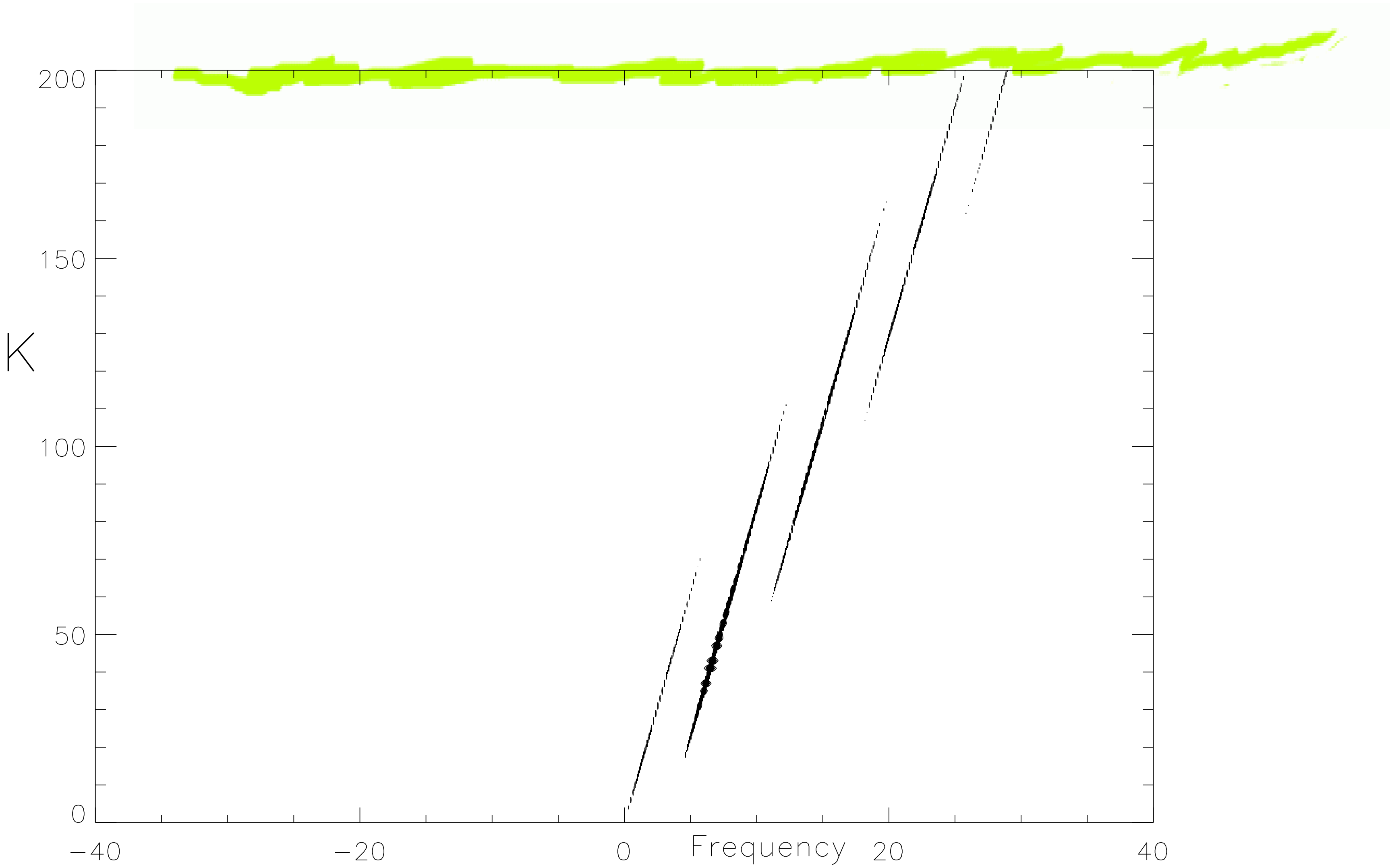


Figure 3. Negative frequency is absent!

2D surface = 3D water

Generalization of the compact equation for the "almost" 2-D water waves, or "almost" 3-D fluid

$$\mathcal{H} = \int \omega_{\vec{k}} |\mathbf{b}_{\vec{k}}|^2 dk_x dk_y + \frac{1}{4} \int |\mathbf{b}'_x|^2 \left[\frac{i}{2} (\mathbf{b}\mathbf{b}'^*_x - \mathbf{b}^* \mathbf{b}'_x) - \hat{k}_x |\mathbf{b}|^2 \right] dx dy.$$

Surface $\eta(x, y)$ ***and steepness*** $\frac{\partial \eta}{\partial x}(x, y)$ ***at*** $t = 313$.