Happy 75 !!!

and

Great Achivements ...

Whitham type equations revisited: critical points and Lauricella functions

B. Konopelchenko
Lecce, Italy

joint work with
Y. Kodama (USA) and W. Schief (Austr.)

"Solitons, Collapses and Turbulence.."
August 05, 2014.

Hydrodynamic type systems

$$\frac{\partial \beta_i}{\partial t_2} = \lambda_i(\beta)\frac{\partial \beta_i}{\partial t_1} \qquad i = 1, \dots, N$$

Long history...
Whitham (1974)
Dubrovin & Novikov (1983, 1989)
Tsarev (1985, 1991)
H. Flaschka, M.G. Forest & D.W. McLaughlin (198
M.G. Forest & J.-E. Lee, (1986)
I.M. Krichever, (1988, 1994) ...

1. Orthogonal and conjugate nets
2. Riemann surfaces

Hamiltonian and semihamiltonian systems.

Properties. (semi-hamiltonian) ?.

1. Generalized hodograph equation:

Tsarev (1991)

$$M_i \equiv t_1 + \lambda_i(\beta) t_2 - \omega_i(\beta) = 0, \quad i = 1..,N$$

where

$$\frac{\frac{\partial \omega_i}{\partial \beta_k}}{\omega_k - \omega_i} = \frac{\frac{\partial \lambda_i}{\partial \beta_k}}{\lambda_k - \lambda_i} =: B_{ik} \quad (i \neq k)$$

2. Infinite family of symmetries

$$\frac{\partial \beta_i}{\partial t_\alpha} = \lambda^\alpha_i(\beta) \frac{\partial \beta_i}{\partial t_1}, \quad i = 1,..,N; \ \alpha = 2,3,\ldots$$

where

$$\frac{\frac{\partial \lambda^\alpha_i}{\partial \beta_k}}{\lambda^\alpha_k - \lambda^\alpha_i} = B_{ik}.$$

3. Infinite set of conservation laws

$$\frac{\partial P}{\partial t_2} = \frac{\partial Q_1}{\partial t_1}$$

$\Rightarrow$

$$\frac{\partial Q_\alpha}{\partial \beta_i} = \lambda^\alpha_i(\beta)\frac{\partial P}{\partial \beta_i} \qquad (i=1\dots N)$$

$\Rightarrow$

$$\frac{\partial^2 P}{\partial \beta_i \partial \beta_k} = B_{ik}\frac{\partial P}{\partial \beta_i} + B_{ki}\frac{\partial P}{\partial \beta_k} \qquad (i \neq k)$$

$\Rightarrow$ $B_{ik} = \frac{\partial}{\partial \beta_k} \ln B_i$ and B_i obey the Darboux system. (not Egoroff!)

. . . . Tsarev (1992).

$\Rightarrow$

$$M_i \equiv t_1 + \frac{\frac{\partial Q_1}{\partial \beta_i}}{\frac{\partial P}{\partial \beta_i}} - \frac{\frac{\partial \hat{Q}}{\partial \beta_i}}{\frac{\partial P}{\partial \beta_i}} = 0$$

equivalent to

$$\sum_\alpha t_\alpha \frac{\partial Q_\alpha}{\partial \beta_i} = 0 \qquad i=1\dots n$$

$\Rightarrow$ gen. hodograph eqs. – critical points of

$$\Theta(\vec{x}, t) = \sum_\alpha t_\alpha Q_\alpha(\vec{x}).$$

Whitham (1974) Dubrovin (1997) Lorenzoni (2006) Konopel. & Martínez Alonso, Medina (2010)...
Flaschka (1980) Krichever (1994) ....

General scheme.

1. Family of functions $\theta(\vec{x}, \vec{t})$.
2. At critical point $d_x \theta = 0$ $\left(\frac{\partial\theta}{\partial p_i} = 0\right)$ second diff. $d_x^2 \theta$ is diagonal.
3. Simple characterization of θ - solutions of linear equations.

Natural candidates:

$$\frac{\partial^2 \theta}{\partial x_i \partial x_k} = A_{ik}(x) \frac{\partial \theta}{\partial x_i} + A_{ki}(x) \frac{\partial \theta}{\partial x_k}$$

$i \neq k$
$i, k = 1, 2, \dots, n$

=> compatibility condition

$A_{ik} = \frac{\partial \ln H_i}{\partial x_k}$ and Darboux equation for H_i.

Conjugate nets. (Darboux Eisenhart, Tsárev (1993))

So.

$$\theta(\vec{x}, t) = \sum_{\alpha=1}^{M} t_\alpha \theta_\alpha(x)$$

or

$$\theta(\vec{x}, t) = \sum_{\alpha=1}^{n} t_\alpha \theta_\alpha(\vec{x}) + \tilde{\theta}(\vec{x}).$$

Critical points β^i:

$$\sum_{\alpha=1}^{n} t_\alpha \frac{\partial \theta_\alpha}{\partial \beta_i} + \frac{\partial \tilde{\theta}}{\partial \beta_i} = 0, \qquad i=1,\dots,N.$$

- generalized hodograph type equation.

regular sector – unique solvability

$$\operatorname{Jac}\left|\frac{\partial^2\theta}{\partial\beta_i\partial\beta_k}\right| = \prod_{i=1}^{N} \frac{\partial^2\theta}{\partial\beta_i^2} \neq 0$$

So all $\frac{\partial^2\theta}{\partial\beta_i^2} \neq 0$, $\quad i=1,\dots N.$

$\Rightarrow$ ODEs

$$\frac{\partial\beta_i}{\partial t_\alpha} = -\frac{\frac{\partial\theta_\alpha}{\partial\beta_i}}{\frac{\partial^2\theta}{\partial\beta_i^2}} \qquad i=1,\dots,N, \quad \alpha=1,\dots,n.$$

compatible – common solution $\Rightarrow$

PDEs

$$\frac{\partial\beta_i}{\partial t_\alpha} = \lambda_i^\alpha(\vec{\beta})\frac{\partial\beta_i}{\partial t_1} \qquad i=1,\dots,N \quad \alpha=1,\dots n$$

with

$$\lambda_i^\alpha(\beta) = \frac{\frac{\partial\theta_\alpha}{\partial\beta_i}}{\frac{\partial\theta_1}{\partial\beta_i}}.$$

Properties:

1. Infinite set of symmetries
2. Infinite set of conservation laws

$$\frac{\partial Q_\alpha}{\partial t_\gamma} = \frac{\partial Q_\gamma}{\partial t_\alpha} \qquad \alpha = 1 \qquad I_\gamma = \int Q_\gamma(\beta)\,dt,$$

3. Semi-hamiltonianity

$$\frac{\frac{\partial \lambda_i^\alpha}{\partial \beta_k}}{\lambda_k^\alpha - \lambda_i^\alpha} = A_{ki}\,\frac{\frac{\partial \theta_1}{\partial \beta_k}}{\frac{\partial \theta_1}{\partial \beta_i}} = \frac{\partial}{\partial \beta_k}\ln\left(\frac{\frac{\partial \theta_1}{\partial \beta_i}}{H_i}\right) \qquad i \neq k$$

$$\Rightarrow \frac{\partial}{\partial \beta_l}\left(\frac{\frac{\partial \lambda_i^\alpha}{\partial \beta_k}}{\lambda_k^\alpha - \lambda_i^\alpha}\right) = \frac{\partial}{\partial \beta_k}\left(\frac{\frac{\partial \lambda_i^\alpha}{\partial \beta_l}}{\lambda_l^\alpha - \lambda_i^\alpha}\right)$$

So. – semi-hamiltonian system of hydrodynamic type.

Conjugate nets – critical points – families of PDEs.

Solutions of the linear Darboux system:

$\bar\partial$-dressing method – Zakharov & Manakov (1985), Konopelchenko (1993)

Algebro-geometric approach – Krichever (1997)

...

$\Rightarrow$ integrable PDEs.

Euler-Poisson-Darboux equations. Lauricella functions.

Simple (simplest ?) example of conjugate nets.

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = \frac{1}{x_i - x_k}\left(\varepsilon_k \frac{\partial f}{\partial x_i} - \varepsilon_i \frac{\partial f}{\partial x_k}\right)$$

$i \neq k$.

and

$$H_i = \text{const} \prod_{k \neq i} (x_i - x_k)^{-\varepsilon_k}.$$

EPD equations $E(\varepsilon_1, \ldots, \varepsilon_N)$.

1. Classical diff. geometry ... (Darboux book)
2. Multidimensional generalizations of classical hypergeometric functions.

Lauricella (1893).

EPD $E(\frac{1}{2})$ equations in theory of Whitham equations.

Kudashev & Sharapov (1991) Gurevich, Krylov, El (1991).
Tian (1994) ... ε-system (Pavlov, 1992, 2003)

d multicomp. KdV - Konop., Martinez Alonso, Medina (2010 ...),
vortex filaments - Konop., Ortenzi (2012).

Solutions of EPD.

$$f(x) = \sum_{k=1}^{N} \int_{\Gamma_k} f_k(z) \prod_{i=1}^{N} (z - x_i)^{-\varepsilon_i} dz$$

arbitrary functions $f_k(z)$
arbitrary contours Γ_k in $\mathbb{C}$.

Lauricella type functions.

properties, monodromy ....
e.g. E. Looijenga, 2007.

$\Theta_\alpha(x)$ as solutions of EPD' -
- class of integrable systems

1. only one nonzero function

$$f_k = \sum_\alpha t_\alpha \lambda^{\alpha + \varepsilon_1 + \dots + \varepsilon_N}$$

and $\Gamma_k = \Gamma_\infty$. $\Rightarrow$

$\Theta_\alpha(x)$ - polynomials in β_i.

At $\varepsilon_i = \frac{1}{2}$ - disp. limit of N-comp. KdV equation (Konopel, M.A. Medina 2010).

-

in particular ε- systems (Pavlov, 2003)

2. Choice

$$\theta(\vec{x},\vec{t}) = t_1 \sum_{i=1}^{N} \varepsilon_i x_i + t_2 \int_{x_1}^{x_2} \prod_{i=1}^{N} (z - x_i)^{-\varepsilon_i} dz + \ldots$$

One key PDE

$$\frac{\partial \beta_i}{\partial t_2} = \lambda_i(\beta) \frac{\partial \beta_i}{\partial t_1}$$

with $\lambda_i = \frac{1}{\varepsilon_i} \frac{\partial}{\partial \beta_i} \left(\int_{\beta_1}^{\beta_2} \prod_{i=1}^{N} (z - \beta_i)^{-\varepsilon_i} dz \right.$.

At $N = 3, 4$, $\varepsilon_i = \frac{1}{2}$ - Pavlov (1992).

3. Choice

$$\theta = t_1 \sum_{i=1}^{N} \varepsilon_i x_i + t_2 F_D(a, \varepsilon_1, \ldots, \varepsilon_N, b; x_1, \ldots x_N) + \tilde{\theta}$$

where

$$F_D \equiv \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_1^\infty z^{\varepsilon_1 + \ldots + \varepsilon_N - b - 2} (z-1)^{b-a-1} \prod_{i=1}^{N} (z - x_i)^{-\varepsilon_i} dz$$

- Lauricella's N-dim. hypergeom. function (1893)

$$\frac{\partial \beta_i}{\partial t_2} = \frac{1}{\varepsilon_i} \frac{\partial F_D(\beta)}{\partial \beta_i} \frac{\partial \beta_i}{\partial t_1} \qquad i = 1, \ldots N$$

Lauricella type functions as seed functions

EPD equations $\longrightarrow$ linear Darboux eqs.

Transformations: gauge, Darboux, Levy...

Simplest:
Take two solutions f_1 and f_2 of the same EPD equations. Then

$$\theta = \frac{f_1}{f_2}$$

is a solution of linear Darboux

$$\frac{\partial^2 \theta}{\partial x_i \partial x_k} = A_{ik} \frac{\partial \theta}{\partial x_i} + A_{ki} \frac{\partial \theta}{\partial x_k}$$

with $A_{ik} = \frac{\partial}{\partial x_k} \ln \left(\frac{\prod_{k \neq i} (x_i - x_k)^{-\varepsilon_k}}{f_2(x)} \right)$.

Functions $\theta = \sum_\alpha t_\alpha \theta_\alpha(x)$,
Critical points $\longrightarrow$ integrable PDEs of hydrodynamic type

Particular case (arbitrary N)

$$f_1 = t_1 + t_2 U + \tilde{f}$$

$$f_2 = \int_{x_1}^{x_2} dz \prod_{i=1}^{N} (z - x_i)^{-\varepsilon_i} \equiv I(\beta)$$

where $U = \sum_{i=1}^{N} \varepsilon_i x_i$, $\hat{f}$ - solution of EPD $E(\varepsilon_1, \dots, \varepsilon_N)$. $\Rightarrow$

$$\theta = \frac{t_1 + t_2 U + \tilde{f}}{I} = t_1 \frac{1}{I} + t_2 \left(\frac{U}{I}\right) + \frac{\tilde{f}}{I}$$

Critical points $\to$ system with characteristic speeds

$$\lambda_i(\beta) = \frac{\frac{\partial}{\partial \beta_i}\left(\frac{U}{I}\right)}{\frac{\partial}{\partial \beta_i}\left(\frac{1}{I}\right)} = U - \frac{I}{\frac{\partial I}{\partial \beta_i}} \frac{\partial U}{\partial \beta_i}$$

At $N = 3, 4$ and $\varepsilon_i = \frac{1}{2}$ - classical one-phase Whitham eqs for KdV and NLS eqs.

Gurevich, Krylov, El (1991), Kudashev (1991) JETP Lett..

Levy' transform

$N \geq 3$ - Konop., Schief 1993

At $N \geq 5$ and $\varepsilon_i = \frac{1}{2}$ -
- semi-hamiltonian hydrodynamic type systems with characteristic speeds parametrized by hyperelliptic integrals

!!! - $\boxed{\text{not}}$ multi-phase Whitham equations for KdV or NLS.

Appropriate $\Theta(x,t)$ — solutions of certain linear Darboux

How to construct? -
- Transformation of Lauricella type functions

i.e. EPD $\rightarrow$ Darboux.

Extended N+1-dim. EPD system

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = \frac{1}{x_i - x_k}\left(\varepsilon_k \frac{\partial f}{\partial x_i} - \varepsilon_i \frac{\partial f}{\partial x_k}\right) \quad i \neq k$$

$$\frac{\partial^2 f}{\partial x_i \partial y} = \frac{1}{x_i - y}\left(\varepsilon_k \frac{\partial f}{\partial x_i} - (1-g)\frac{\partial f}{\partial y}\right).$$

g integer ≥ 1 $\qquad i = 1, \dots N$

Why auxilliary variable y?

Levy' transformation w.r.t. y.

Seed solution

$$f_k = \int_{\Gamma_k} \frac{(z-y)^{g-1}}{\prod_{i=1}^{N}(z-x_i)^{s_i}}\, dz$$

Γ_k - arbitrary contours

$g-1$ - fold Levy' transformation of any solution φ of extended EPD

$$\varphi_{g-1} = \frac{\begin{vmatrix} \varphi & \varphi_y & \cdots & \varphi_{(g-1)y} \\ \hat{f} & \hat{f}_y & \cdots & \hat{f}_{(g-1)y} \end{vmatrix}}{\begin{vmatrix} \hat{f}_y & \cdots & \hat{f}_{(g-1)y} \end{vmatrix}},$$

where $\hat{f} := (f_2, \dots, f_g)^T$ and

$$\varphi_{(m)y} \equiv \frac{\partial^m \varphi}{\partial y^m}.$$

Levy transforms of

$$\varphi_0 = \sum_{i=1}^{N} \varepsilon_i x_i + (1-g)y \,, \quad \text{and } f_1.$$

are

$$\varphi^0_{g-1} = \frac{\sum_{i=1}^{N} \varepsilon_i x_i \left| \hat{f}_y \cdots \hat{f}_{(g-1)y} \right| + (g-1) \left| \hat{f} - y\hat{f}_y, \hat{f}_{yy} \cdots \hat{f}_{(g-1)y} \right|}{\left| \hat{f}_y \cdots \hat{f}_{(g-1)y} \right|}$$

and

$$\varphi^1_{g-1} = \frac{\left| f \cdots f_{(g-1)y} \right|}{\left| \hat{f} \cdots \hat{f}_{(g-1)y} \right|}$$

where $f := (f_1, f_2, \dots, f_g)^T$.

Since f_k are polynomials in y of degree $g-1$ $\Rightarrow$

$$\varphi^0_{g-1} = \frac{\sum_i \varepsilon_i x_i \left|\hat{I}_{g-2} \cdots \hat{I}_0\right| - \left|\hat{I}_{g-1}, \hat{I}_{g-3}, \cdots \hat{I}_0\right|}{\left|\hat{I}_{g-2} \cdots \hat{I}_0\right|}$$

$$\varphi^1_{g-1} = \frac{\left|\hat{I}_{g-1} \cdots \hat{I}_0\right|}{\left|\hat{I}_{g-2} \cdots \hat{I}_0\right|}$$

where

$$\hat{I}^k_e = \int_{\Gamma_k} \frac{z^e}{\prod_{i=1}^{n} (z - x_i)^{\varepsilon_i}} dz$$

1. φ^0_{g-1} and φ^1_{g-1} do not depend on y!
2. φ^0_{g-1} and φ^1_{g-1} are solutions of the same linear Darboux system.

Introduction of y - trick - to construct the from simpler seed solutions.

Now the function

$$\theta(x; t_1, t_2) = \frac{t_1 + t_2 \varphi^0_{g-1} + \tilde{\varphi}}{\varphi^1_{g-1}}$$

where $\tilde{\varphi}$ obeys the same linear Darboux system as φ^0_{g-1} and φ^1_{g-1}

Critical points of θ β_i:

dependence on t_1 and t_2 $\Rightarrow$ integrable hydrodynamic type system

$$\frac{\partial \beta_i}{\partial t_2} = \lambda_i(\beta) \frac{\partial \beta_i}{\partial t_1} \qquad i = 1, \dots, N$$

with characteristic speeds

$$\lambda_i(\beta) = \frac{\frac{\partial}{\partial \beta_i}\left(\frac{\varphi^0_{g-1}}{\varphi^1_{g-1}}\right)}{\frac{\partial}{\partial \beta_i}\left(\frac{1}{\varphi^1_{g-1}}\right)} = \varphi^0_{g-1} - \frac{\varphi^1_{g-1}}{\frac{\partial \varphi^1_{g-1}}{\partial \beta_i}} \cdot \frac{\partial \varphi^0_{g-1}}{\partial \beta_i}$$

$$i = 1, \dots, N$$

Levy transform

Arbitrary N, ε_i and Γ_K.

Particular cases: $N = 2g+1$, all $\boxed{\varepsilon_i = \frac{1}{2}}$

Γ_k - standard b_k cycles for the Riemann surfaces of genus g.

Denote

$$H_1 = \frac{|\hat{I}_{g-2} \cdots \hat{I}_0|}{|\hat{I}_{g-1} \cdots \hat{I}_0|}, \quad H_2 = -\frac{|\hat{I}_{g-1}, \hat{I}_{g-3} \cdots \hat{I}_0|}{|\hat{I}_{g-1} \cdots \hat{I}_0|}.$$

and

$$\Gamma_0 = 1, \quad \Gamma_1 = V = \frac{1}{2}\sum_{i=1}^{2g+1} \beta_i$$

$\Rightarrow$ characteristic speeds.

$$\lambda_i(\beta) = \frac{\frac{\partial}{\partial \beta_i}(H_1 \Gamma_1 + H_2 \Gamma_0)}{\frac{\partial H_1}{\partial \beta_i}}, \qquad i = 1, \dots, 2g+1$$

- g-phase Whitham equations for KdV (Tian, 1994), (Flaschka et al 1980).

Similar formulas for $N = 2g+2$, $\boxed{\varepsilon_i = \frac{1}{2}}$

- g-phase Whitham eqs for NLS (Forest, Lee (1986))

deformations of hyperelliptic curves

• $\varepsilon_i = \frac{1}{K}$ $\quad i = 1, \dots, N$

Deformation of (K, N) curves

$$p^K = \prod_{i=1}^{N} (z - \beta_i) \qquad \text{"cyclic"}$$

- coisotropic deformations

•• $\varepsilon_i = \frac{K_i}{K}$ $\quad i = 1, \dots N$

Deformation of singular (K, N) curves

$$p^K = \prod_{i=1}^{N} (z - \beta_i)^{K_i}$$

••• Irrational ε_i

Deformations of non-algebraic curve

Example: $\varepsilon_i = \frac{\sqrt{2}}{\pi}$ $\quad i = 1, \dots, N$

Curve

$$p^{\pi} = \prod_{i=1}^{N} (z - \beta_i)^{\sqrt{2}}.$$

Discretisation.

Discretise – what?

Standard method of discretisation of soliton equations –
– Backlund, Darboux ... discrete transformations

seems not. Indications?

generalised hodograph equation.

classical : $N = 2$

$$t_1(\beta_1, \beta_2), \quad t_2(\beta_1, \beta_2)$$

$$\frac{\partial t_1}{\partial \beta_1} + \lambda_2(\beta) \frac{\partial t_2}{\partial \beta_1} = 0,$$

$$\frac{\partial t_1}{\partial \beta_2} + \lambda_1(\beta) \frac{\partial t_2}{\partial \beta_2} = 0$$

$\Rightarrow$ discretisation of "dependent" variables β_1, β_2. : $\beta_i \to \delta_i n_i$

at arbitrary N – N times $t_1, t_2, \dots, t_N$
$N-1$ -commut. flows.

So:

1. Functions $\theta(\bar{n})$ of discrete variables $n_1, \dots, n_N$.

2. Critical points $\Delta_i \theta = 0$

$$\Delta_i \theta(n) \equiv \theta(n_i + 1, \dots) - \theta(n_i \dots)$$

3. Linear difference equations to characterise θ!

Thus:

1. Solutions θ of linear discrete Darboux system

$$\Delta_i \Delta_k \theta = A_{ik} \Delta_i \theta + A_{ki} \Delta_k \theta$$

$i \neq k$
$i, k = 1, \dots, N$

2. Functions

$$\theta(\bar{n}, \bar{t}) = \sum_{\alpha=1}^{m} t_\alpha \theta_\alpha(\bar{n}) + \tilde{\theta}(\bar{n}).$$

3. Critical points

$$\Delta_i \theta = 0 \quad \Rightarrow$$

generalised discrete hodograph eqs.

$$\sum_{\alpha=1}^{m} t_\alpha \lambda_i^\alpha + \mu_i = 0$$

where $\lambda_i^\alpha = \dfrac{\Delta_i \theta_\alpha}{\Delta_i \theta_1}$, $\mu_i = \dfrac{\Delta_i \tilde{\theta}}{\Delta_i \theta_1}$

So, the starting point:
the system

$$\Delta_i \Delta_k \Theta = A_{ik} \Delta_i \Theta + A_{ki} \Delta_k \Theta. \quad i \neq k$$

Bogdanov, Konopel. (1995)
- discrete Darboux system

Doliwa & Santini (1997),
Konopel. & Schief (1998) -
- geometrical interpretation.

Solutions -
- $\bar{\partial}$-dressing method - Bogdanov & Konopel. (1995)
- algebro-geometric approach -
  - Akhmetshin, Krichever & Volvovski (1999)

....

Our approach:

1. Discretisation of the Euler-Poisson-Darboux equations and their solutions.
2. The use of discrete Levy transformation.

Discretization of EPD equations:

$$\Delta_i \Delta_k \varphi = \frac{1}{\delta^i(n^i+\nu^i) - \delta^k(n^k+\nu^k)} \left(\delta^k \varepsilon_k \Delta_i \varphi - \delta^i \varepsilon_i \Delta_k \varphi \right)$$

$i \neq k.$

δ^i - lattice parameters

ν^i - parameters allowing to place points at edges, faces ....

continuous limit $x^i = \delta^i(n^i+\nu^i)$, $\delta^i \to 0$

Extended EPD system

$$\Delta_i \frac{\partial \varphi}{\partial y} = \frac{1}{y - \delta^i(n^i+\nu^i)} \left(\delta^i \varepsilon_i \frac{\partial \varphi}{\partial y} - (1-g) \Delta_i \varphi \right).$$

$\varphi = \varphi(n, y).$

Separable solutions

$$\varphi = \mathcal{g}(y,\lambda) \prod_{i=1}^{N} \rho_i(n,\lambda)$$

where λ - constant of separation

$$\Delta_i \rho_i = \frac{\delta^i \varepsilon_i}{\lambda - \delta^i(n^i + \nu^i)} \rho_i, \quad \frac{\partial \rho}{\partial y} = \frac{1-g}{\lambda - y} \rho$$

$$\Rightarrow \quad \rho = (\lambda - y)^{g-1}$$

$$\varphi \to (\lambda - y)^{g-1} \prod_{i=1}^{N} (\lambda - x^i)^{-\varepsilon_i} \qquad \delta_i \to 0.$$

As in continuous case

(g-1)-fold Levy transform of

$$\varphi^0 = \sum_{i=1}^{N} \varepsilon_i \delta^i n_i + (1-g) y, \quad \text{and } \varphi^1$$

where

$$\varphi_k = \oint_{b_k} (\lambda - y)^{g-1} \prod_{i=1}^{N} \rho_i(n, \lambda) \, d\lambda$$

$$k = 1, \dots, g$$

$\Rightarrow$

$$\varphi^0_{g-1} = \frac{\sum_{i=1}^{N} \varepsilon_i \delta^i n_i \left| \hat{I}_{g-2} \cdots \hat{I}_0 \right| - \left| \hat{I}_{g-1}, \hat{I}_{g-3} \cdots \hat{I}_0 \right|}{\left| \hat{I}_{g-2} \cdots \hat{I}_0 \right|}$$

$$\varphi^1_{g-1} = \frac{\left| \hat{I}_{g-1} \cdots I_0 \right|}{\left| \hat{I}_{g-2} \cdots \hat{I}_0 \right|}, \qquad \hat{I}^k_e = \oint_{b_k} \lambda^e \prod_{i=1}^{N} \rho_i(\lambda) \, d\lambda.$$

φ^0_{g-1}, φ^1_{g-1} do not depend on y and solve the same discrete linear Darboux system

So, as in continuous case we take

$$\theta = \frac{t_1 + t_2\,\varphi^0_{g-1} + \tilde{\varphi}}{\varphi^1_{g-1}}$$

$\Rightarrow$ discrete characteristic speeds

$$\lambda_i = \frac{\Delta_i\left(\frac{\varphi^0_{g-1}}{\varphi^1_{g-1}}\right)}{\Delta_i\left(\frac{1}{\varphi^1_{g-1}}\right)} = \underbrace{\varphi^0_{g-1} - \frac{\varphi^1_{g-1}}{\Delta_i \varphi^1_{g-1}} \cdot \Delta_i \varphi^0_{g-1}}_{\text{discrete Levy transform}}$$

Particular case $N = 2g+1$, $\varepsilon_i = \frac{1}{2}$

$$r_0 = 1$$

$$\lambda_i = \frac{\Delta_i (H_1 r_1 + H_2 r_0)}{\Delta_i H_1} \qquad r_1 = \sum_{i=1}^{2g+1} \varepsilon_i \delta^i R^i$$

at the continuous limit $\delta^i \to 0$ characteristic speeds of g-phase Whitham equation for KdV.

Explicit form of $\rho_i(n,\lambda)$:

equation

$$\rho_i(n_i+1,\dots) = \frac{\lambda - \delta^i(n^i+\nu^i-\varepsilon_i)}{\lambda - \delta^i(n^i+\nu^i)}\,\rho_i(n)$$

=>

$$\rho_i = (\delta^i)^{-\varepsilon_i}\,\frac{\Gamma(\xi^i - n^i - \nu^i + 1)}{\Gamma(\xi^i - n^i - \nu^i + \varepsilon_i + 1)} \qquad \xi^i = \frac{\lambda}{\delta^i}$$

where $\Gamma(z)$ is classical Gamma function

- discretisation of $(\lambda - x^i)^{-\varepsilon_i}$. $x^i = \delta(n^i+\nu^i)$

Indeed

$$\lim_{|z|\to\infty}\left(z^{b-a}\,\frac{\Gamma(z+a)}{\Gamma(z+b)}\right) = 1, \quad |\arg z| < \pi$$

- essentially "discretization" of "power" proposed by Gelfand et al (1992)

Particular case $N=2g+1$, $\varepsilon_i=\frac{1}{2}$, $\delta^i=\delta$

$$\varphi \equiv \prod_{i=1}^{2g+1} \rho_i(\lambda)\,d\lambda = \prod_{i=1}^{2g+1} P(\xi, n^i, \nu^i), \quad \xi=\frac{\lambda}{\delta}$$

where

$$P(\xi, n, \nu) = \frac{1}{\delta^{1/2}} \frac{\Gamma(\xi-n-\nu+1)}{\Gamma(\xi-n-\nu+\frac{3}{2})}.$$

Ordering

$$n^1 < n^2 < \dots < n^{2g} < n^{2g+1}$$

and choice

$$\nu^{2k}=\frac{1}{2}, \quad \nu^{2k+1}=0.$$

So

$$\varphi = \frac{1}{\delta^{g+\frac{1}{2}}} \prod_{k=0}^{g} \frac{\Gamma(\xi-n^{2k+1}+1)}{\Gamma(\xi-n^{2k+1}+\frac{3}{2})} \cdot \prod_{k=1}^{g} \frac{\Gamma(\xi-n^{2k}+\frac{1}{2})}{\Gamma(\xi-n^{2k}+1)}.$$

$\Rightarrow$ has no poles and zeros in the region

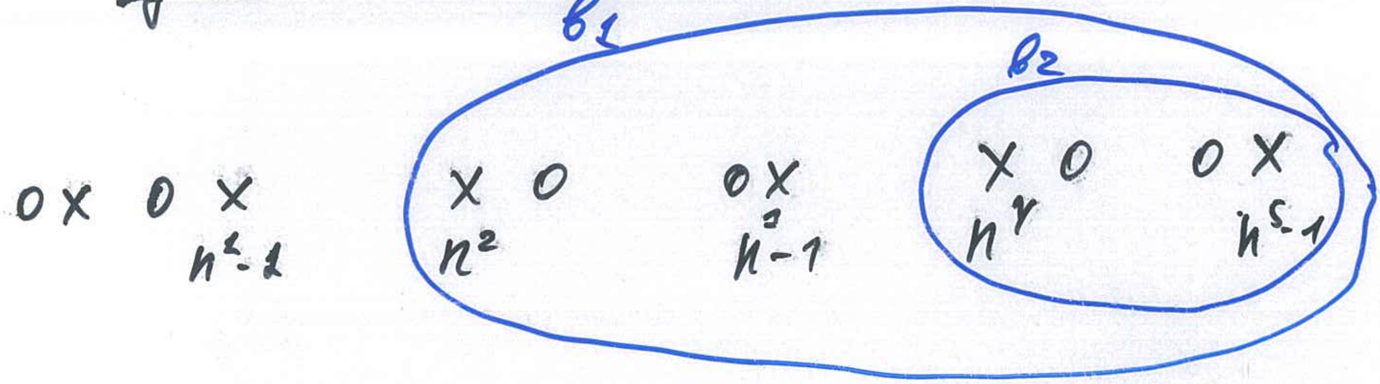
$$\bigcup_{k=1}^{g+1} (n^{2k-1}-1, n^{2k}), \quad n^{2g+2}=\infty$$

Intervals $(n^{2k-1}-1, n^{2k})$, $k=1,\dots,g$

$(n^{2g+1}-1, \infty)$

$\rightarrow$ cuts (x^{2k-1}, x^{2k}), (x^{2g+1}, ∞) in continuous case.

$\underline{g=2}$



o – zeros, x – poles b_1, b_2 – cycles

$$\hat{I}^k_e := \oint_{b_k} \lambda^e \varphi(\lambda)\, d\lambda \qquad k=1,\dots,g, \quad e=0,\dots,g-1$$

– "discrete" analogues of hyperelliptic integrals.

$$\oint_{b_k} \frac{\lambda^e}{\prod_{i=1}^{2g+1}(\lambda - x^i)^{1/2}}\, d\lambda$$

Explicit form of $\hat{I}^k_e$ by Cauchy theorem and $\mathrm{res}(\Gamma(z), z=-e) = \frac{(-1)^e}{e!}$

At $g=1$

$$\hat{I}^1_0 = \frac{\delta}{2\pi i}\oint_{b_1} \varphi(\xi)\,d\xi = \delta \sum_{k=n^2}^{n^3-1} \mathrm{res}\big(\varphi(\xi), \xi = k\big) =$$

$$= \frac{1}{\delta^{1/2}} \sum_{k=n^2}^{n^3-1} \frac{\Gamma(k-n^1+1)}{\Gamma(k-n^2+\frac{3}{2})} \cdot \frac{\Gamma(k-n^2+\frac{1}{2})}{\Gamma(k-n^2+1)} \cdot \frac{\mathrm{res}\big(\Gamma(\eta), \eta = k-n^3+1\big)}{\Gamma(k-n^3+\frac{3}{2})}.$$

$\Rightarrow$ discrete characteristic speeds of one-phase Whitham eq. for KdV.

Comparison with continuous limit

$$\Phi^0 = \frac{2}{\pi\sqrt{x^3-x^1}}\, K\left(\sqrt{\frac{x^3-x^2}{x^3-x^1}}\right).$$

where K - complete elliptic integral of the first kind.

at the points $x^1 = \delta n^1$; $x^2 = \delta(n^2+\frac{1}{2})$, $x^3 = \delta n^3$ there is virtually no difference between the values of $\hat{I}_0$ and Φ^0!

Very good discretisation of elliptic integrals! also at $g>1$.