An instability of wave turbulence causing the formation of pulses

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- The Majda-McLaughlin-Tabak model
- Instability of wave turbulence
- Energy transfer by radiating pulses

Majda-McLaughlin-Tabak (MMT) model

for weakly nonlinear waves

$$\left(i\frac{\partial}{\partial t} - \mathcal{L}\right)\psi(x,t) = \lambda\psi(x,t)|\psi(x,t)|^2$$

- complex wave amplitude $\psi(x,t)$
- linear operator $\mathcal{L}\exp(ikx) = \omega_k \exp(ikx)$,
- dispersion $\omega_k = \sqrt{|k|}$.

(Fourier modes $a_k = \int_{-L/2}^{L/2} \psi(x,t)\exp(-ikx)dx/\sqrt{2\pi}$
Large system size L with periodic boundary conditions)

A.J. Majda, D.W. McLaughlin, E.G. Tabak, J. Nonlinear Sci. 6, 9 (1997),

Conserved quantities of the MMT equation

- Hamiltonian or 'energy'

$$\begin{aligned} E &= \sum_k \omega_k |a_k|^2 + (\lambda/2) \int_{-L/2}^{L/2} |\psi|^4 dx \\ &= E_2 + E_4 \end{aligned}$$

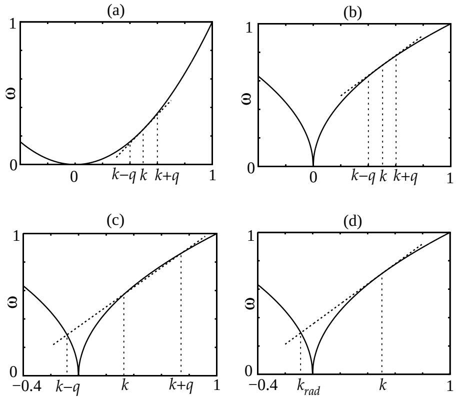
- waveaction

$$N = \sum_k |a_k|^2$$

- momentum

$$P = \sum_k k |a_k|^2$$

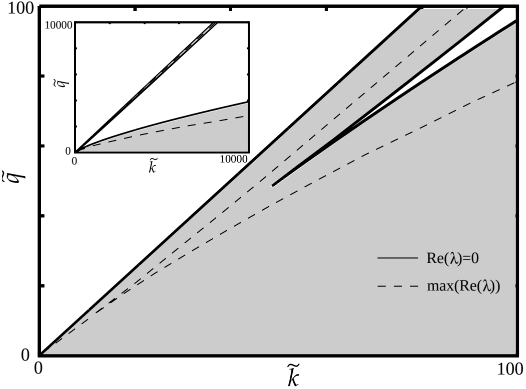
Possible resonance of the modes at k and $k \pm q$



(a) $\omega = k^2$ (nonlinear Schrödinger equation)
(b) $\omega = \sqrt{|k|}$ for $k \gg q$ (MMT)
(c) $\omega = \sqrt{|k|}$ for $k \sim q$ (MMT)
(d) radiating quasisolitons for MMT

Instability by short modulations for $\lambda = 1$

- Monochromatic wave $\psi = (A + \delta a)\exp(ikx)$
- Modulation $\delta a = \delta a_+ \exp(iqx) + \delta a_- \exp(-iqx)$



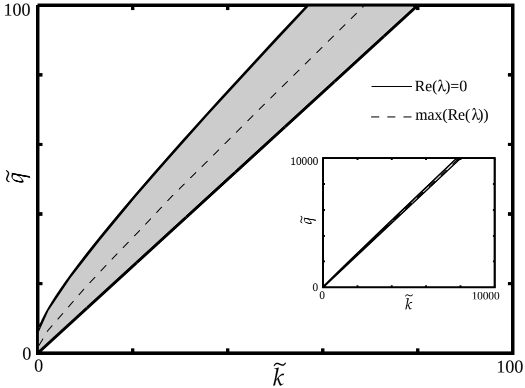
- carrier wave $\tilde{k} = k/A^4$,
- modulation $\tilde{q} = q/A^4$

- Shaded areas: Instabilities at $q \sim \sqrt{k}$ and at $q \approx 5k/4$
- No collapses, $E_2 > 0$, $E_4 > 0$

Instability by short modulations for $\lambda = -1$

- Monochromatic wave $\psi = (A + \delta a)\exp(ikx)$
- Modulation $\delta a = \delta a_+ \exp(iqx) + \delta a_- \exp(-iqx)$

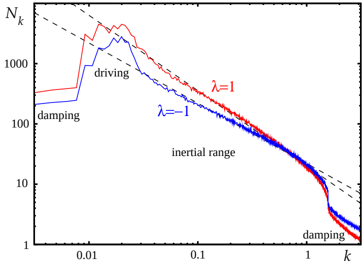
- carrier wave $\tilde{k} = k/A^4$,
- modulation $\tilde{q} = q/A^4$



- Shaded area: Instability at $q \approx 5k/4$
- Collapses are observed, $E_2 > 0$, $E_4 < 0$

Damped and driven MMT equation

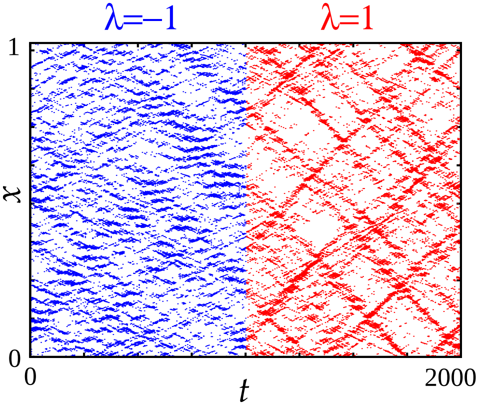
- Driving force
  at moderate wavenumbers
- Damping at high
  and at low wavenumbers
- $N_k = \langle |a_k|^2 \rangle$



- $\lambda = -1$: Kolmogorov-Zakharov spectrum $N_k \sim k^{-1}$
  $\rightarrow$ wave turbulence
- $\lambda = 1$: Steeper spectrum $N_k \sim k^{-1.25}$
  $\rightarrow$ unknown mechanism of turbulence

Contour plot of regions with high amplitudes:

Switching the sign $\lambda = -1$ from to $\lambda = 1$



Wave turbulence - Coherent structures

Envelope equation for wave turbulence

- Ensemble average $\langle u(\mathbf{x}, t) u^*(\mathbf{x} + \mathbf{r}, t) \rangle$ depends on $\mathbf{x}$
- Slow spatial variations of the waveaction

$$N(\mathbf{k}, \mathbf{x}, t) = \int \langle u(\mathbf{x}, t) u^*(\mathbf{x} + \mathbf{r}, t) \rangle \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{r}$$

- Kinetic equation is extended by a Vlasov term

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \mathbf{k}} \frac{\partial N}{\partial \mathbf{x}} - \frac{\partial \tilde{\omega}}{\partial \mathbf{x}} \frac{\partial N}{\partial \mathbf{k}} = T_4[N]$$

- Nonlinear by the renormalized frequency

$$\tilde{\omega}(k, \mathbf{x}, t) = \omega_k + 2\lambda \int N(\mathbf{p}, \mathbf{x}, t) d\mathbf{p}$$

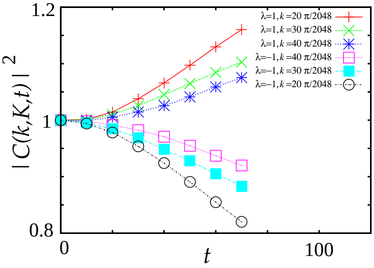
Breaking of the spatial homogeneity symmetry of wave turbulence

- Linearization $N(\mathbf{k}, \mathbf{x}, t) = N_0(k) + \Delta N(\mathbf{k}, \mathbf{x}, t)$
- Kolmogorov-Zakharov spectrum $N_0(k)$
- Modulation $\Delta N(\mathbf{k}, \mathbf{x}, t) = a(\mathbf{k}) \exp(i\mathbf{K} \cdot \mathbf{x} - i\Omega t)$

Stability:

- $\lambda = 1$:
  - unstable in one dimension $\sim$ negative Landau damping
  - no instability in two dimensions
- $\lambda = -1$:
  - no instability

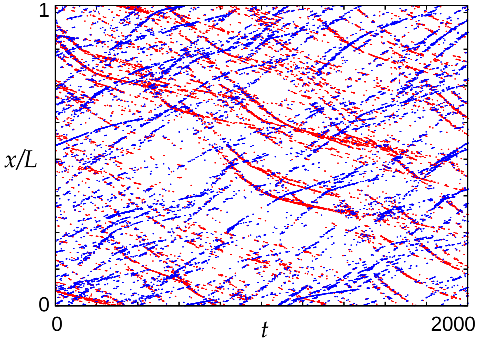
Growth of correlations for $\lambda = 1$; decay for $\lambda = -1$



- time evolution of the correlation $|C(k,K,t)|^2 = |\langle A_k(t) A^*_{k+K}(t)\rangle|^2 / |\langle A_k(0) A^*_{k+K}(0)\rangle|^2$ with $K = 6\pi/2048$ for an ensemble of 400,000 trajectories
- initial conditions contain a small correlation on top of a KZ spectrum

Formation of coherent structures for $\lambda = 1$:

A gas of solitary waves



Pattern of solitary waves ('pulses') with high positive or negative momenta

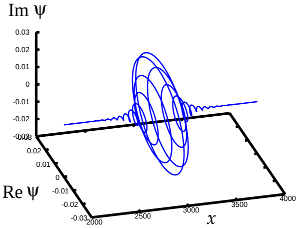
Quasisolitons for $q \ll k_m$

- slow modulation by the envelope $\phi(x,t)$

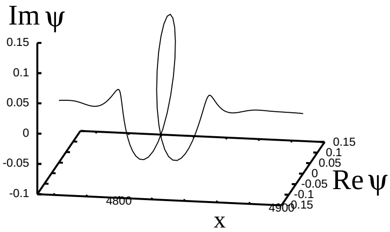
$$\psi(x,t) = \phi(x,t)\exp(ik_m x - i\omega_m t)$$

- soliton solution

$$\phi^{sol}(x,t) = q\sqrt{-\omega''_m}\exp(i\omega''_m q^2 t/2)\operatorname{sech}(q(x - \omega'_m t))$$



Narrow pulse with $q \sim k_m$

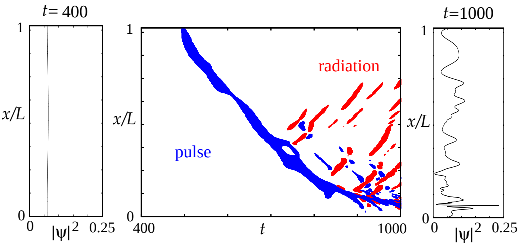


- Shape of a pulse

$$\psi^{(f)} = q\sqrt{\omega_m}{k_m}^{-1} f(\theta)\exp(i\alpha)\exp(i\Omega t)$$

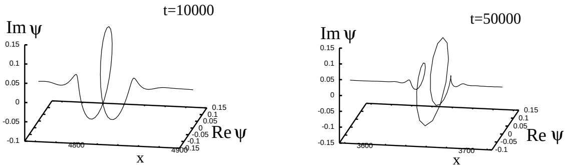
$\theta_x = q$, $\theta_t = -qv$,
$\alpha_x = k_m$, $\alpha_t = k_m v$

Pulses emerge from the instability at $q \sim k_m$



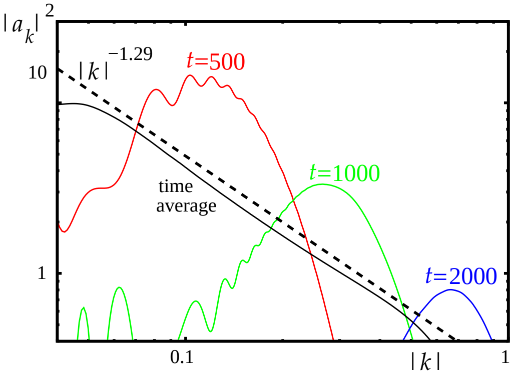
- The pulse-speed decays
- The pulse emits radiation

A radiating pulse evolves in time



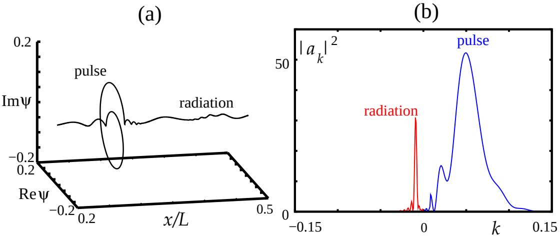
- The pulse narrows in real space - q increases
- The number of loops increases - k_m increases

Time-average of an evolving pulse is MMT-like



- Pulse in k-space at three different times
- Time-average spectrum $\langle |a_k^{(f)}|^2 \rangle \sim k^{-1.29}$

Radiation: Resonant driving of linear waves



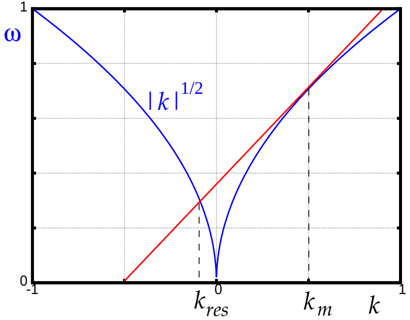
linear wave: driving force by the pulse:

$$i\dot{a}_k - \omega_k a_k = |T_k| \exp(-i\Lambda_k t)$$

with

$$T_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^{(pulse)} |\psi^{(pulse)}|^2 \exp(-ikx) dx$$

Doppler-shifted phase frequency $\Lambda_k = \Omega + kv$



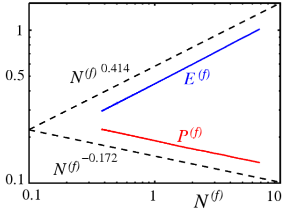
Resonance $\Lambda_{k_{res}} = \omega_{k_{res}}$
at $k_{res} \approx -(\sqrt{2} - 1)^2 k_m$

Evolution of energy and momentum

- A pulse is an extremum of energy for a given momentum and waveaction
- Balance of energy and momentum of the pulse and the radiation yields

$$\left(\frac{dE^{(f)}(N^{(f)}, P^{(f)}(N^{(f)}))}{dN^{(f)}}\right)^2 = -\frac{dP^{(f)}(N^{(f)})}{dN^{(f)}}$$

- Approximation $E^{(f)} \approx \sqrt{N^{(f)} P^{(f)}}$
- $N^{(f)}$ decays in time
- $E^{(f)} \sim {N^{(f)}}^{\sqrt{2}-1}$ decays
- $P^{(f)} \sim {N^{(f)}}^{\sqrt{8}-3}$ increases
- width of pulse
  $q \sim k_m^{(3\sqrt{2}-5)/(2\sqrt{2}-2)}$
  (k_m: maximum pulse-amplitude)



- Radiation driven by a pulse:

$$i\dot{a}_k - \omega_k a_k = |T_k| \exp(-i\Lambda t)$$

- Time-dependent pulse frequency with linear chirp approximation $\Lambda_k(t) \approx \omega_k + \dot{\Lambda}_k t$
- Amplitude of radiation

$$|a_k|^2 \sim T_k^2 \sqrt{|k_m|}/\dot{k}_m$$

after the driving frequency $\Lambda_k(t)$ has moved through resonance

The spectrum of the pulses

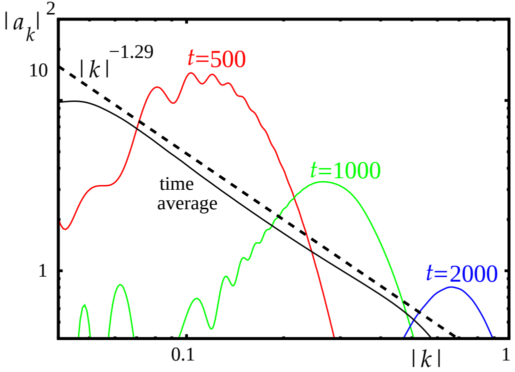
- Driving force $T_k \sim q^2 k_m^{-9/4}$
- Speed of a pulse in k-space: $\dot{k}_m \sim q^3 k_m^{-3/2}$
- Wave action of a pulse $N^{(f)} \sim q k_m^{-3/2}$
- Spectrum:

$$\begin{aligned} \langle |a^{(f)}_{k=k_m}|^2 \rangle &\sim N^{(f)}/\dot{k}_m \\ &\sim q^{-2} \\ &\sim \sim k_m^{(3\sqrt{2}-5)/(2-\sqrt{2})} \sim k_m^{-1.29} \end{aligned}$$

Analytic solution of the MMT spectrum

- Solve coupled equations for pulse and radiation
- Time-average of the pulse yields the spectrum

$$\langle |a_k^{(f)}|^2 \rangle_{time} \sim k^{-2+1/\sqrt{2}} \sim k^{-1.29}$$



Conclusions

- Spatial homogeneity of wave turbulence spontaneously broken
- Transfer of energy by radiating pulses

B.R., A.C. Newell, V.E. Zakharov, PRL 103, 074502 (2009);
A.C. Newell, B.R., V.E. Zakharov, PRL 108, 194502 (2012);
B.R., A.C. Newell, PLA 377, 1260 (2013)