Nonlinear stage of modulation instability in the scalar and vector nonlinear Schrodinger equations.

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SOLITONS, COLLAPSES AND TURBULENCE, 2014
in honor of Vladimir Zakharov's 75 birthday.

Problem statement

We study solutions of the following NLSE with nonvanishing boundary conditions:

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - (|\varphi|^2 - |A|^2)\varphi = 0,$$
$$|\varphi|^2 \to |A|^2 \text{ when } x \to \pm\infty \qquad (1)$$

A simple NLSE solution $\varphi = \varphi_0 = A$ - condensate. The condensate is unstable with respect to small perturbations - modulation instability.

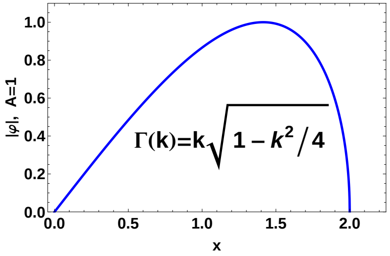


Figure 1: Increment of modulation instability.

Problem statement

What is the nonlinear stage of modulation instability of the small localized condensate perturbation?

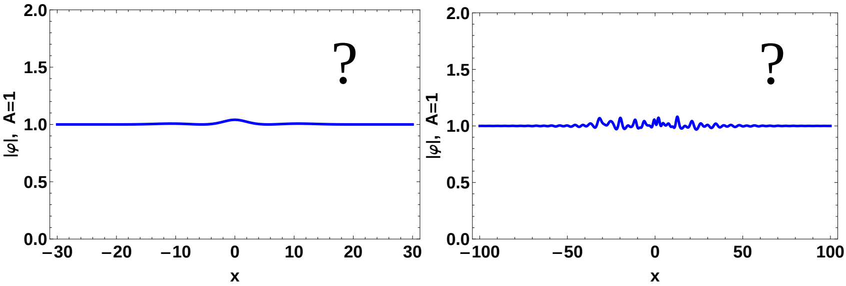


Figure 2: Examples of initial perturbations which will be discussed.

Problem statement

NLSE is the compatibility condition for the following overdetermined linear system for a matrix function Ψ:

$$\mathbf{\Psi}_x = \widehat{\mathbf{U}}\mathbf{\Psi}, \tag{2}$$

$$\mathrm{i}\mathbf{\Psi}_t = (\lambda\widehat{\mathbf{U}} + \widehat{\mathbf{W}})\mathbf{\Psi}. \tag{3}$$

$$\widehat{\mathbf{U}} = \mathbf{I}\lambda + \mathbf{u}, \qquad \widehat{\mathbf{W}} = \frac{1}{2}\begin{pmatrix} |\varphi|^2 - A^2 & \varphi_x \\ \varphi_x^* & -|\varphi|^2 + A^2 \end{pmatrix},$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{pmatrix}. \tag{4}$$

Here λ - spectral parameter. Note, that L - operator for the spectral problem $L\mathbf{\Psi} = \lambda\mathbf{\Psi}$ has the following form:

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - \begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix} \tag{5}$$

N-solitonic solution on the condensate background

When $\varphi_0 = A$, the matrix solution $\mathbf{\Psi}_0$ can be found in the following form:

$$\Psi_0(x,t,\lambda) = \begin{pmatrix} \exp(\phi(x,t,\lambda)) & S(\lambda)\exp(-\phi(x,t,\lambda)) \\ S(\lambda)\exp(\phi(x,t,\lambda)) & \exp(-\phi(x,t,\lambda)) \end{pmatrix} \quad (6)$$

Where

$$\phi = px + \Omega t, \;\; p^2 = \lambda^2 - A^2, \;\; \Omega = -\mathrm{i}\lambda p, \;\; S = -\frac{A}{\lambda + p}$$

Then one can construct a new solution of Eq. 1 using the following recipe. Choose N complex numbers λ_n ($n = 1, ..., N$; $Re\lambda_n > 0$) and another set of N arbitrary complex numbers C_n. Denote

$$\mathbf{F}_n = \mathbf{\Psi}_0(-\lambda_n^*) = \left(\begin{array}{cc} \exp(-\phi_n^*) & -S_n^* \exp(\phi_n^*) \\ -S_n^* \exp(-\phi_n^*) & \exp(\phi_n^*) \end{array} \right), \;\; \mathbf{q}_n^* = \mathbf{F}_n \left(\begin{array}{c} 1 \\ C_n \end{array} \right). \quad (7)$$

and define N vectors $\mathbf{q}_n$ by relation

$$\begin{aligned} q_{n1} &= \exp(-\phi_n) - C_n^* S_n \exp(\phi_n), \\ q_{n2} &= -S_n \exp(-\phi_n) + C_n^* \exp(\phi_n). \end{aligned} \quad (8)$$

N-solitonic solution on the condensate background

Then a new solution is given by expression

$$\varphi = \varphi_0 + 2\widetilde{M}_{12}/M. \tag{9}$$

Here $\widetilde{M}_{\alpha\beta}$ $(\alpha = 1, 2)$ is the following determinant

$$\widetilde{M}_{\alpha\beta} = \begin{vmatrix} 0 & q_{1,\beta} & \cdots & q_{n,\beta} \\ q^*_{1,\alpha} & & & \\ \vdots & & M^T_{nm} & \\ q^*_{n,\alpha} & & & \end{vmatrix}. \tag{10}$$

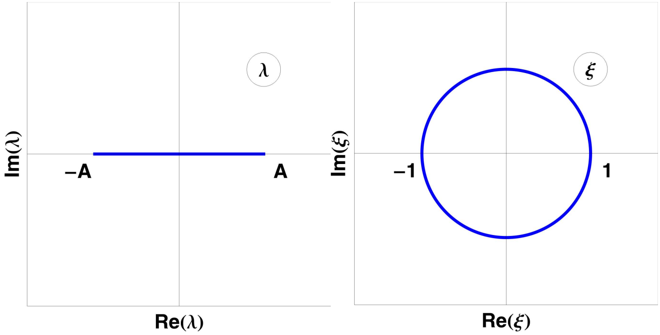
Where M_{nm} is a Hermitian matrix:

$$M_{nm} = \frac{(\mathbf{q}_n \cdot \mathbf{q}^*_m)}{\lambda_n + \lambda^*_m}, \qquad M = det(M_{nm}). \tag{11}$$

Uniformization

Ψ_0 has a cut $(-A, A)$. We apply the Jukowsky transform:

$$\lambda = \frac{A}{2}(\xi + \frac{1}{\xi}) \tag{12}$$



and use the following parametrization:

$$\xi_n = R_n e^{i\alpha_n} = e^{z_n} e^{i\alpha_n}, \qquad C_n = e^{i\theta_n + \mu_n} \tag{13}$$

Asymptotic properties of the N-solitonic solution

Asymptotic of the one-solitonic solution:

$$\varphi \to -A\exp(\pm 2\mathrm{i}\alpha), \qquad x \to \pm\infty. \tag{14}$$

Asymptotic of the N-solitonic solution:

$$\varphi \to -A\exp(\pm 2\mathrm{i}(\alpha_1 + \cdots \alpha_n)), \qquad x \to \pm\infty. \tag{15}$$

If we assume that the modulation instability develops from a localized perturbation, we have to demand that $\varphi(x \to +\infty) = \varphi(x \to -\infty)$. That is

$$\alpha_1 + \cdots + \alpha_n = 0;\ \pm\frac{\pi}{2}, \pm\pi ... \tag{16}$$

We call such solutions as regular solitonic solutions of the first and second type.

One-solitonic solutions

Classification of one-solitonic solutions.

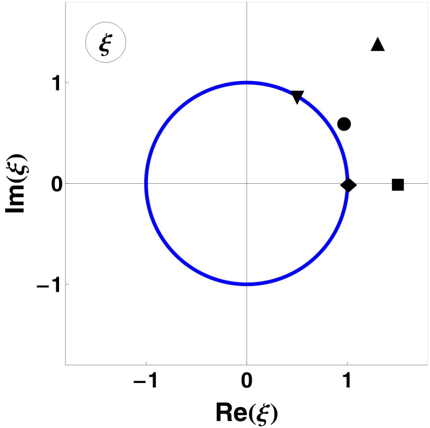


Figure 3: ■ - Kuznetsov soliton, ▼ - Akhmediev breather ▲ - general solution, ● -quasi-Akhmediev breather, ◆ - Peregrine breather.

Kuznetsov soliton $R > 1, \;\; \alpha = 0$

$$\varphi = -A\frac{\cosh z\cosh 2u + \cosh 2z\cos 2v + \mathrm{i}\sinh 2z\sin 2v}{\cosh z\cosh 2u + \cos 2v}.$$

$$u = A\sinh(z)x, \qquad v = -(A^2/2)\sinh(2z)t. \tag{17}$$

This solution is periodic in time. Period of its oscillations:

$$T = 4\pi/A^2\sinh 2z, \quad R \to 1, \quad T \to \infty; \quad R \to \infty, \quad T \to 0. \tag{18}$$

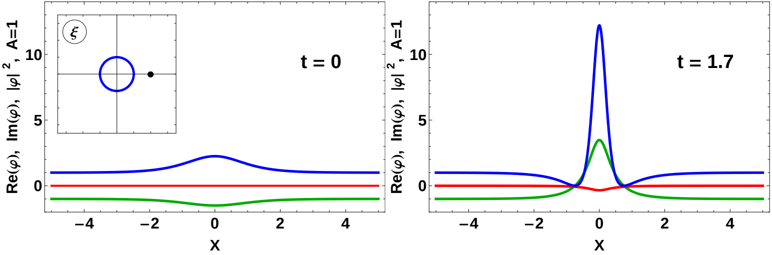


Figure 4: Kuznetsov soliton φ at the moment of minimum(left) and maximum(right) of its amplitude. $R = 2, \; \alpha = 0$. Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

Peregrine breather $R = 1, \ \ \alpha = 0$

The Peregrine breather can be obtained by resolving indeterminacy $\frac{0}{0}$ in Kuznetsov or Akhmediev solution:

$$\varphi = -A + 4A\frac{1 - 2iA^2t}{1 + 4A^2x^2 + 4A^4t^2}. \tag{19}$$

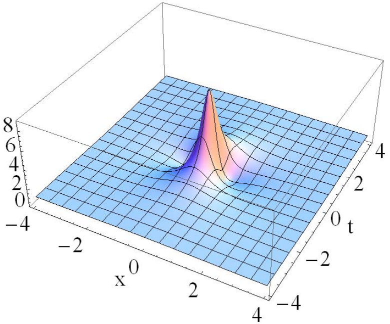


Figure 5: $|\varphi(x,t)|^2$. The solution appear once in space and time and reaches high amplitude $|\varphi(0,0)| = 3$, what makes the Peregrine breather an important model of freak wave.

Akhmediev breather, $R = 1, \quad \alpha \neq 0$

$$\varphi = -A\frac{\cos 2\alpha \cosh 2u + \cos\alpha \cos 2v + \mathrm{i}\sin 2\alpha \sinh 2u}{\cosh 2u + \cos\alpha\cos 2v},$$
$$u = (A^2/2)\sin(2\alpha)t, \qquad v = A\sin(\alpha)x. \tag{20}$$

Asymptotics:

$$\varphi \to -A\exp(\pm 2\mathrm{i}|\alpha|), \qquad t \to \pm\infty \tag{21}$$

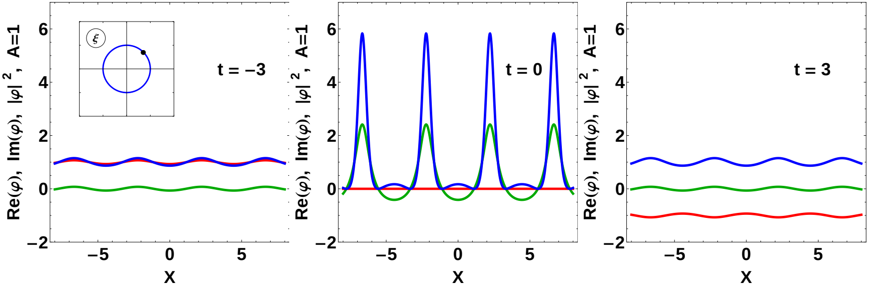


Figure 6: Akhmediev breather φ at different moments of time. $R = 1, \ \alpha = \pi/4$. Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

General one-solitonic solution on the condensate background

In general case $N = 1$ we obtain one-solitonic solution, characterized by four parameters R, α, θ, μ:

$$\varphi = \frac{N}{\cosh z \cosh 2u + \cos\alpha \cos 2v},$$

$$N = \big[\cosh z \cos 2\alpha \cosh 2u + \cosh 2z \cos\alpha \cos 2v + \mathrm{i}\big(\cosh z \sin 2\alpha \sinh 2u + \sinh 2z \cos\alpha \sin 2v\big)\big],$$

$$\begin{aligned} u &= æx - \gamma t + \mu/2, & v &= kx - \omega t - \theta/2, \\ æ &= A \sinh z \cos\alpha, & \gamma &= -\frac{A^2}{2}\cosh 2z \sin 2\alpha, \\ k &= A \cosh z \sin\alpha, & \omega &= \frac{A^2}{2}\sinh 2z \cos 2\alpha. \end{aligned} \tag{22}$$

Asymptotics:

$$\begin{aligned} \varphi &\to -A \exp(\pm 2\mathrm{i}\alpha), & x &\to \pm\infty, \\ |\varphi|^2 &= A^2, & x &\to \pm\infty. \end{aligned} \tag{23}$$

General one-solitonic solution $R > 1, \ \ \alpha \neq 0$

$$V_{gr} = \frac{\gamma}{æ} = -\frac{A \cosh 2z \sin \alpha}{\sinh z}, \qquad V_{ph} = \frac{\omega}{k} = \frac{A \sinh z \cos 2\alpha}{\sin \alpha}. \qquad (24)$$

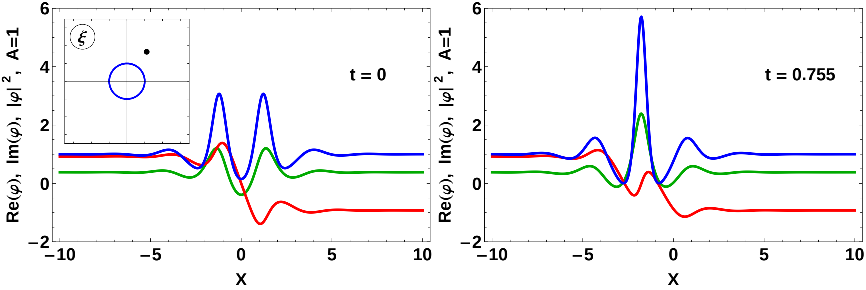


Figure 7: General one-solitonic solution φ at the moment of minimum(left) and maximum(right) of its amplitude. $R = 2, \ \alpha = 5\pi/16, \ \mu = 0, \ \theta = 0.$ Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

quasi-Akhmediev breather $R \to 1$

Very large size:

$$L \approx \frac{1}{Az\cos\alpha}, \tag{25}$$

high group velocity:

$$V_{group} \approx -\frac{A\sin\alpha}{z}, \qquad V_{ph} \approx \frac{Az\cos 2\alpha}{\sin\alpha}. \tag{26}$$

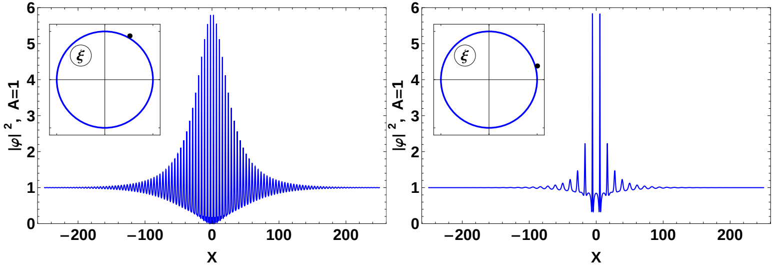


Figure 8: $|\varphi|^2$. quasi-Akhmediev breather at $t = 0$ with different α. Left: $R = 1.02,\ \alpha = \pi/3, \mu = 0,\ \theta = 0$, right: $R = 1.02,\ \alpha = \pi/11, \mu = 0,\ \theta = 0$.

Two-solitonic solution

General formula

$$\varphi = A - 2\frac{N}{\Delta}$$

$$N = \frac{|q_1|^2 q_{21}^* q_{22}}{\lambda_1 + \lambda_1^*} - \frac{(q_1^* q_2) q_{21}^* q_{12}}{\lambda_1^* + \lambda_2} - \frac{(q_1 q_2^*) q_{11}^* q_{22}}{\lambda_2^* + \lambda_1} + \frac{|q_2|^2 q_{11}^* q_{12}}{\lambda_2 + \lambda_2^*}$$

$$\Delta = \frac{|q_1|^2 |q_2|^2}{(\lambda_1 + \lambda_1^*)(\lambda_2 + \lambda_2^*)} - \frac{(\overrightarrow{q}_1 \overrightarrow{q}_2^*)(\overrightarrow{q}_1^* \overrightarrow{q}_2)}{(\lambda_1^* + \lambda_2)(\lambda_2^* + \lambda_1)} \quad (27)$$

Asymptotics:

$$\varphi \to -Ae^{\pm 2\mathrm{i}(\alpha_1 + \alpha_2)} \qquad \text{when} \quad x \to \pm\infty \quad (28)$$

Solution is regular when:

$$\alpha_1 + \alpha_2 = 0$$
$$\alpha_1 + \alpha_2 = \frac{\pi}{2} \quad (29)$$

General two-solitonic solution

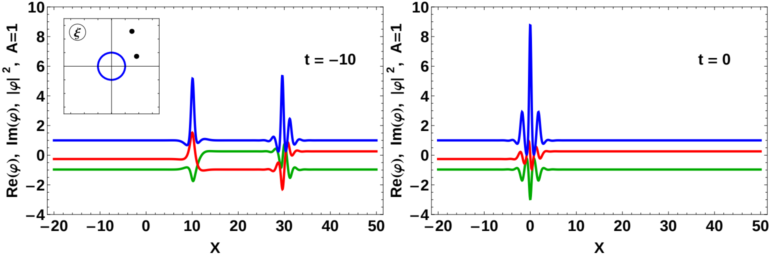


Figure 9: General two-solitonic solution φ at different moments of time. $R_1 = 2,\ \alpha_1 = \pi/8,\ R_2 = 3,\ \alpha_2 = \pi/3.$ Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

Regular two-solitonic solution

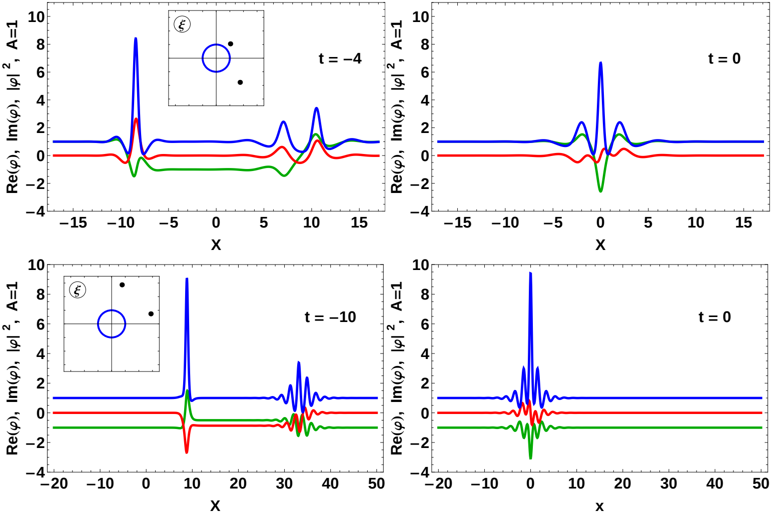


Figure 10: The first type (up): $R_1 = 1.5,\ \alpha_1 = \pi/4,\ R_2 = 2.5,\ \alpha_2 = -\pi/4$. the second type (down): $R_1 = 3,\ \alpha_1 = \pi/12,\ R_2 = 3,\ \alpha_2 = 5\pi/12$.

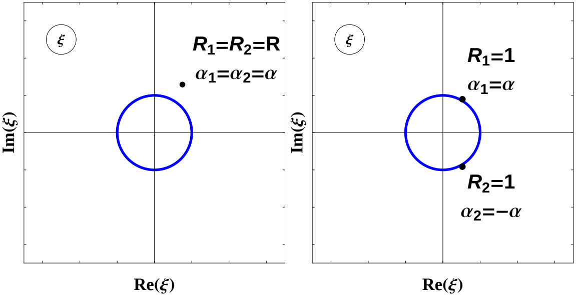
$\lambda_1 = \lambda_2 = \eta.$

Let us consider the case, when both poles are located in the same point: $\lambda_1 = \lambda_2 = \eta$. Then:

$$N = \frac{|q_1|^2 q_{21}^* q_{22} - (q_1^* q_2) q_{21}^* q_{12} - (q_1 q_2^*) q_{11}^* q_{22} + |q_2|^2 q_{11}^* q_{12}}{\eta + \eta^*} \equiv 0.$$

$$\Delta = \frac{|q_1|^2 |q_2|^2 - (\vec{q}_1 \vec{q}_2^*)(\vec{q}_1^* \vec{q}_2)}{(\eta + \eta^*)^2} \equiv \frac{|q_{11} q_{22} - q_{12} q_{21}|^2}{(\eta + \eta^*)^2} \qquad (30)$$

Two different cases:



$\lambda_1 = \lambda_2 = \eta.$

In the first case:

$$R_1 = R_2 = R, \quad \alpha_1 = \alpha_2 = \alpha, \quad \lambda_1 = \lambda_2 = \frac{A}{2}\left(R\mathrm{e}^{\mathrm{i}\alpha} + \frac{1}{R}\mathrm{e}^{-\mathrm{i}\alpha}\right)$$

$$\Delta = \frac{4\sin^2\alpha}{A^2\cos^2\alpha}\sin^2\frac{\theta_1 - \theta_2}{2}. \tag{31}$$

That is $\Delta \neq 0$ when $\theta_1 \neq \theta_2$. $\theta_1 = \theta_2 = \theta$ - degenerate case.
In the second case:

$$R_1 = R_2 = 1, \quad \alpha_1 = -\alpha_2 = \alpha, \quad \lambda_1 = \lambda_2 = A\cos\alpha$$

$$\Delta = \frac{4\sin^2\alpha}{A^2\cos^2\alpha}\sin^2\frac{\theta_1 + \theta_2}{2}. \tag{32}$$

Here $\Delta \neq 0$ when $\theta_1 + \theta_2 \neq 0$. $\theta_1 = -\theta_2 = \theta$ - degenerate case.
Conclusion: $\varphi \to A$ **when** $\lambda_1 \to \lambda_2$ **except degenerate cases.**

Passage to the limit

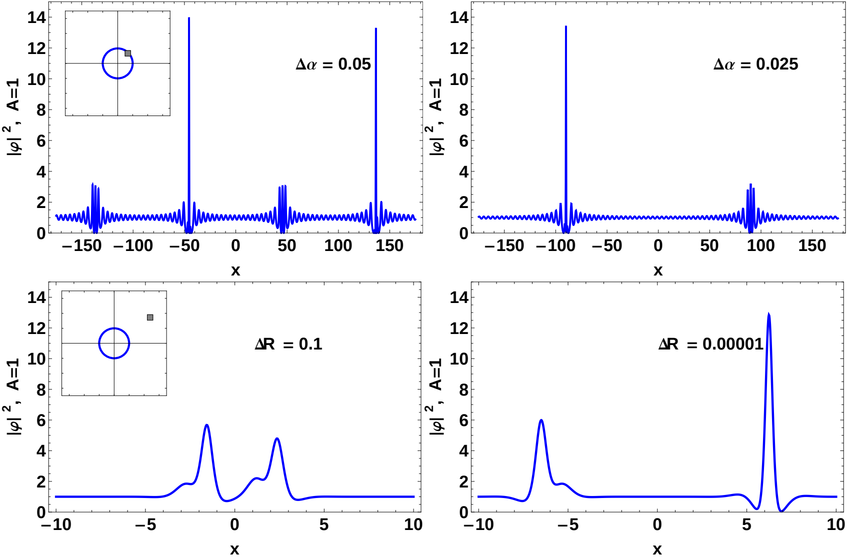


Figure 11: Absolute squared value of two-solitonic solution φ with close poles located at the cut (up) and in an arbitrary point (down) at the moment of time $t = 0$

Passage to the limit

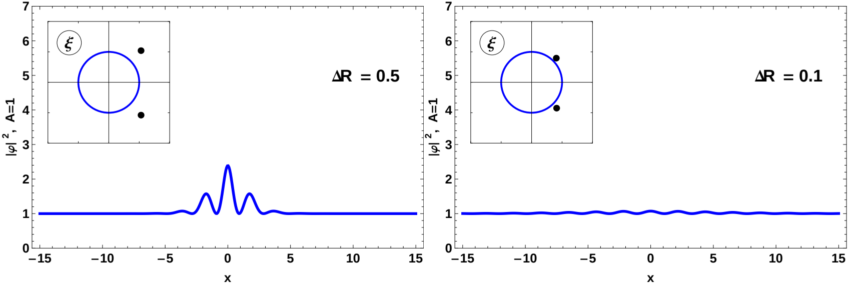


Figure 12: Absolute squared value of two-solitonic solution φ with close poles located at different sides of the cut.

Superregular solitonic solutions

Superregular pair

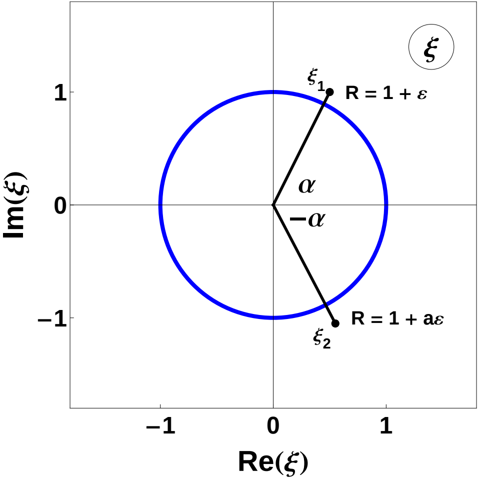


Figure 13: Superregular pair of poles corresponds to a small perturbation of the condensate.

Superregular two-solitonic solutions

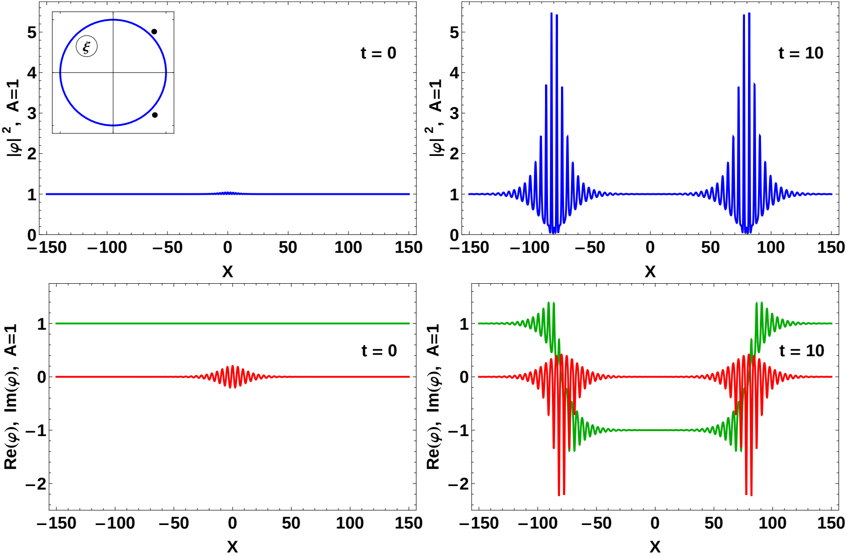


Figure 14: $R_1 = R_2 = 1.075,\ \alpha_1 = \pi/4,\ \alpha_2 = -\pi/4,\ \mu_1 = \mu_2 = 0,\ \theta_1 = \theta_2 = \pi/2$
Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

Superregular two-solitonic solutions

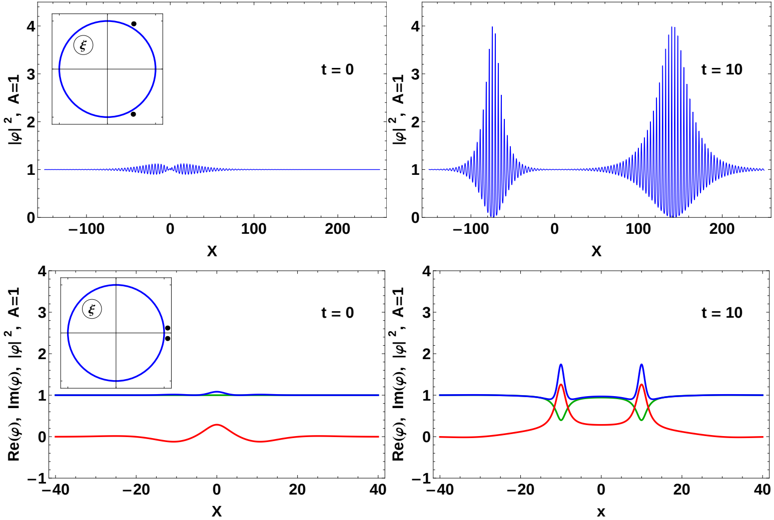
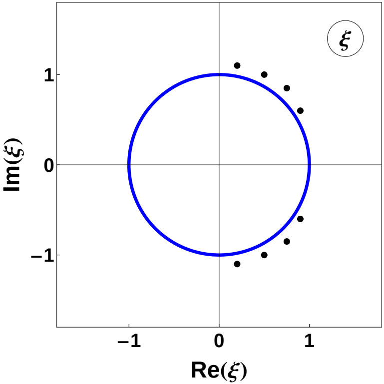


Figure 15: $R_1 = R_2 = 1.075,\ \alpha_1 = \pi/4,\ \alpha_2 = -\pi/4,\ \mu_1 = \mu_2 = 0,\ \theta_1 = \theta_2 = \pi/2$
Green lines - $Re(\varphi)$, red - $Im(\varphi)$ blue - $|\varphi|^2$.

N superregular pairs

$$\alpha_n = -\alpha_{n+N}, \qquad R_n = 1 + \varepsilon, \qquad R_{n+N} = 1 + a_n\varepsilon. \tag{33}$$



Superregular four-solitonic solution

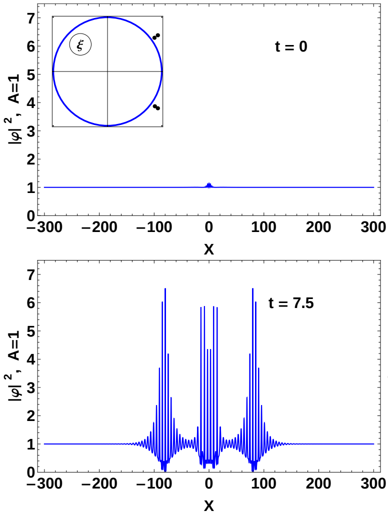


Figure 16:
$|\varphi|^2 . R_1 = 1.05,\ R_3 = 1.05,\ \alpha_1 = \pi/5,\ \alpha_3 = -\pi/5,\ \mu_1 = \mu_3 = 0,\ \theta_1 = \theta_3 = \pi/2;$
$R_2 = 1.075,\ R_4 = 1.075,\ \alpha_2 = \pi/5,\ \alpha_4 = -\pi/5,\ \mu_2 = \mu_4 = 0,\ \theta_2 = \theta_4 = \pi/2.$

Superregular six-solitonic solution

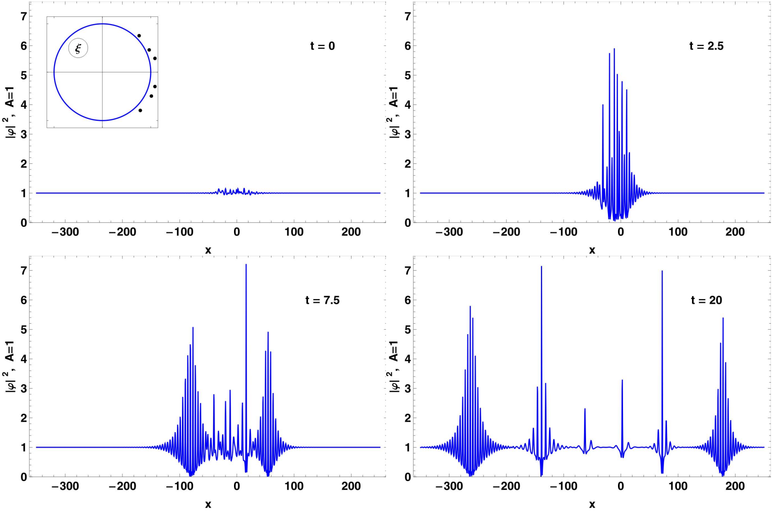


Figure 17: $R_1 = 1.05,\ R_4 = 1.075,\ \alpha_1 = \pi/4,\ \alpha_4 = -\pi/4,\ \mu_1 = \mu_4 = 0,\ \theta_1 = \theta_4 = \pi/2;$
$R_2 = 1.05,\ R_5 = 1.1,\ \alpha_2 = \pi/7,\ \alpha_5 = -\pi/7,\ \mu_1 = \mu_5 = 0,\ \theta_1 = \theta_5 = \pi/2;$
$R_3 = 1.1,\ R_6 = 1.1,\ \alpha_3 = \pi/12,\ \alpha_6 = -\pi/12,\ \mu_3 = \mu_6 = 5,\ \theta_3 = \theta_6 = \pi/2$

Perturbations in details.

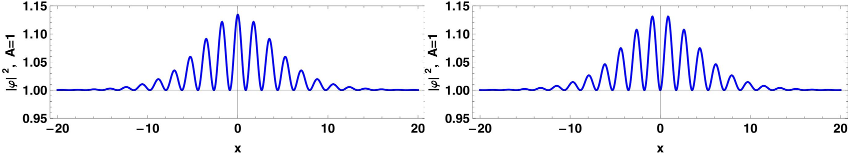


Figure 18: $\theta_1 = \pi/2,\ \theta_2 = \pi/2$ (left), $\theta_1 = 0,\ \theta_2 = \pi$ (right)

approximately: $$\delta\varphi \approx 4\mathrm{i}\varepsilon A \frac{\cos\alpha \cos(2A\sin\alpha x - \frac{\theta_1-\theta_2}{2})}{\cosh(4\varepsilon A\cos\alpha x)}. \quad (34)$$

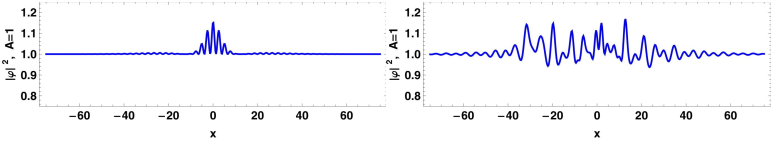


Figure 19: four-solitonic (left) and six-solitonic (right) solutions.

$$\varphi_n = A + \sum_{m=1}^{N} \delta\varphi_n. \quad (35)$$

Experimental observation of superregular solitonic solutions

Experiment. Hydrodynamics.

NLSE in dimensional variables (V.E. Zakharov, 1968):

$$\mathrm{i}\left(\frac{\partial a}{\partial t} + c_g \frac{a}{\partial x}\right) - \frac{\omega_0}{8k_0^2}\frac{\partial^2 a}{\partial x^2} - \frac{\omega_0 k_0^2}{2}|a|^2 a = 0 \qquad (36)$$

it can be obtained from dimensionless NLSE by the following transform:

$$t \to -\frac{\omega_0}{8k_0^2}t, \qquad x \to x - c_g t = x - \frac{\omega_0}{2k_0}t, \qquad \varphi \to \sqrt{2}k_0^2 a. \qquad (37)$$

experimentally observable value is the wave amplitude η:

$$\eta = Re[a(x,t)\exp(\mathrm{i}(k_0 x - \omega_0 t))] \qquad (38)$$

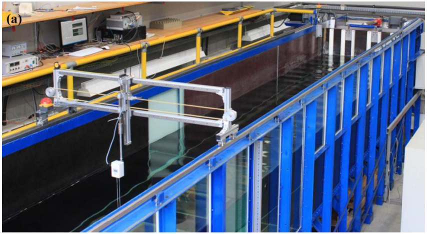
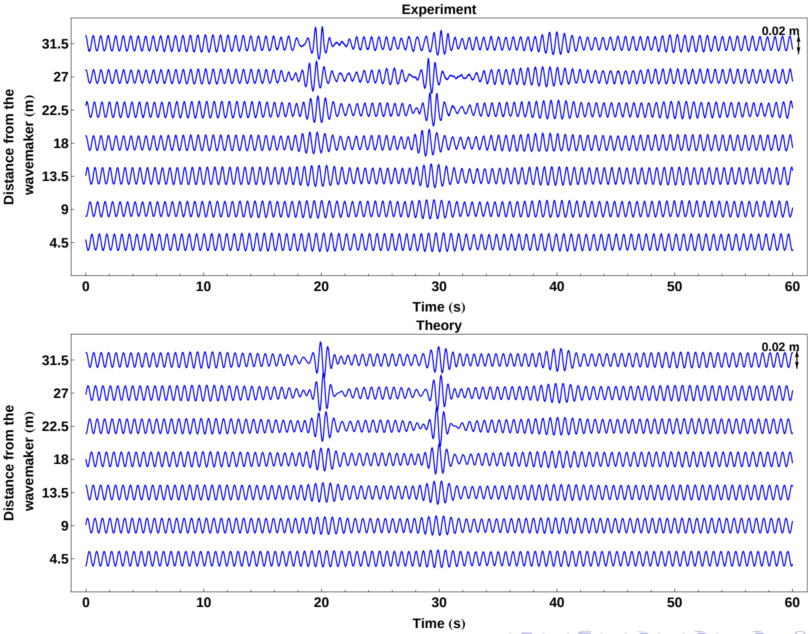
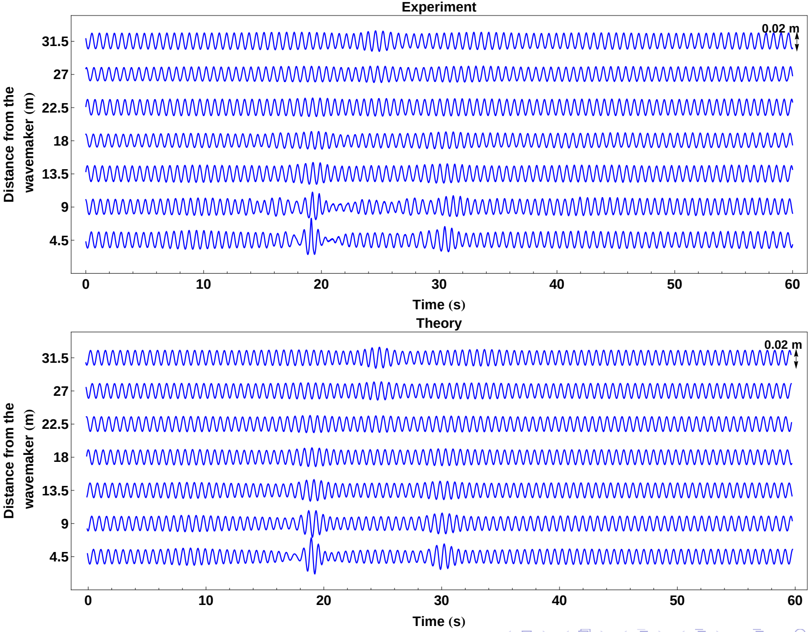


Figure 20: Photo of the water-wave tank.

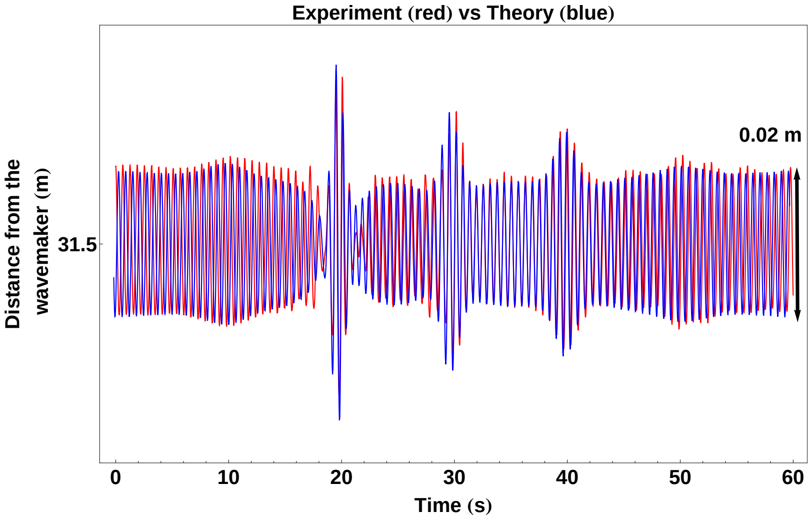
Growth of the perturbation.



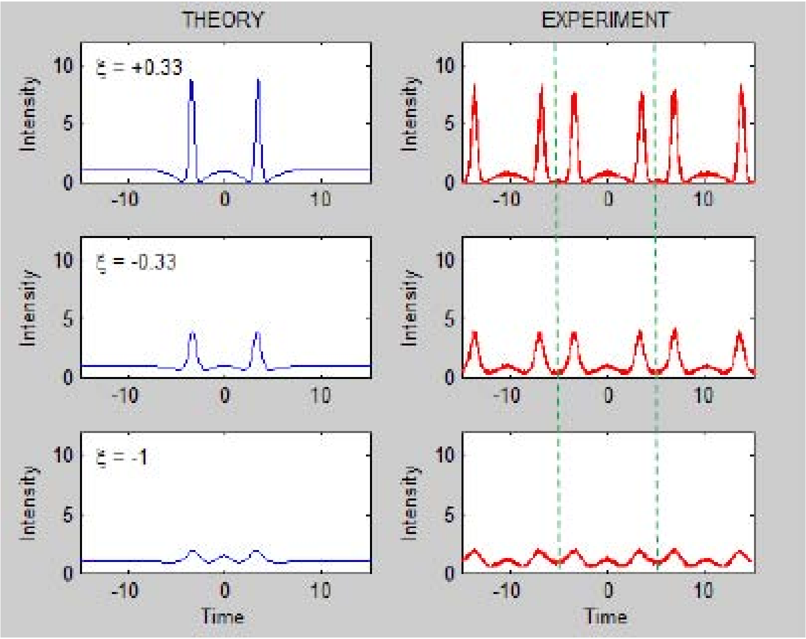
Annihilation of the perturbation.



Theory vs Experiment



Experiment. Optics. Preliminary results.



Vector Nonlinear Schrodinger Equation on the condensate background

Manakov system. The dressing method.

$$i\varphi_{1t} - \frac{1}{2}\varphi_{1xx} - (|\varphi_1|^2 - |a_1|^2 + |\varphi_2|^2 - |a_2|^2)\varphi_1 = 0,$$
$$i\varphi_{2t} - \frac{1}{2}\varphi_{2xx} - (|\varphi_1|^2 - |a_1|^2 + |\varphi_2|^2 - |a_2|^2)\varphi_2 = 0. \qquad (39)$$

Where $a = \sqrt{a_1^2 + a_2^2}$. $|\varphi_1| \to |a_1|$, $|\varphi_2| \to |a_2|$. Condensate:

$$\varphi_0 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad (40)$$

The equation (39) is the compatibility condition for the following overdetermined linear system for a matrix function $\mathbf{\Psi}$:

$$\mathbf{\Psi}_x = \widehat{\mathbf{U}}\mathbf{\Psi}, , \qquad i\mathbf{\Psi}_t = (\lambda\widehat{\mathbf{U}} + \widehat{\mathbf{W}})\mathbf{\Psi}.$$

$$\widehat{\mathbf{U}} = \mathbf{I}\lambda + \mathbf{u}, \qquad \mathbf{u} = \begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ -\varphi_1^* & 0 & 0 \\ -\varphi_2^* & 0 & 0 \end{pmatrix},$$

$$\widehat{\mathbf{W}} = \begin{pmatrix} |\varphi_1|^2 + |\varphi_2|^2 - a^2 & \varphi_{1x} & \varphi_{2x} \\ \varphi_{1x}^* & -|\varphi_1|^2 + a^2 & -\varphi_1^*\varphi_2 \\ \varphi_{2x}^* & -\varphi_1\varphi_2^* & -|\varphi_2|^2 + a^2 \end{pmatrix}. \qquad (41)$$

Manakov system. The dressing method.

Now $\mathbf{q}_\alpha$, $\alpha = (0,1,2)$ and:

$$\begin{aligned} \varphi_1 &= \varphi_{01} + 2\widetilde{M}_{01}/M. \\ \varphi_2 &= \varphi_{02} + 2\widetilde{M}_{02}/M. \end{aligned} \tag{42}$$

For the condensate:

$$\mathbf{\Psi_0}(x,t,\xi) = \begin{pmatrix} 0 & \mathrm{e}^{\phi} & S\mathrm{e}^{-\phi} \\ -\frac{a_2}{a}\mathrm{e}^{-\phi_0} & \frac{a_1}{a}S\mathrm{e}^{\phi} & \frac{a_1}{a}\mathrm{e}^{-\phi} \\ \frac{a_1}{a}\mathrm{e}^{-\phi_0} & \frac{a_2}{a}S\mathrm{e}^{\phi} & \frac{a_2}{a}\mathrm{e}^{-\phi} \end{pmatrix} \tag{43}$$

$$\phi_0 = \lambda x - \frac{\mathrm{i}}{2}(\lambda^2 + K^2)t, \quad \phi = Kx + \Omega t, \quad K^2 = \lambda^2 - a^2,$$

$$\Omega = -\mathrm{i}\lambda K, \quad S = -\frac{a}{\lambda + k}$$

$$\begin{aligned} \varphi_1 &= \varphi_{10} - \frac{2(\eta+\eta^*)q_0^* q_1}{|q_0|^2 + |q_1|^2 + |q_2|^2}, \\ \varphi_2 &= \varphi_{20} - \frac{2(\eta+\eta^*)q_0^* q_2}{|q_0|^2 + |q_1|^2 + |q_2|^2}. \end{aligned} \tag{44}$$

Manakov system. In uniformizing variables.

$$\lambda = \frac{a}{2}\left(\xi + \frac{1}{\xi}\right).$$

$$\mathbf{\Psi_0}(x,t,\xi) = \begin{pmatrix} 0 & \mathrm{e}^{\phi} & -\frac{1}{\xi}\mathrm{e}^{-\phi} \\ -\frac{a_2}{a}\mathrm{e}^{-\phi_0} & -\frac{a_1}{a\xi}\mathrm{e}^{\phi} & \frac{a_1}{a}\mathrm{e}^{-\phi} \\ \frac{a_1}{a}\mathrm{e}^{-\phi_0} & -\frac{a_2}{a\xi}\mathrm{e}^{\phi} & \frac{a_2}{a}\mathrm{e}^{-\phi} \end{pmatrix}. \qquad (45)$$

$$\phi_0 = \frac{a}{2}\left(\xi + \frac{1}{\xi}\right)x - \frac{\mathrm{i}a^2}{4}\left(\xi^2 + \frac{1}{\xi^2}\right)t, \qquad \phi = \frac{a}{2}\left(\xi - \frac{1}{\xi}\right)x - \frac{\mathrm{i}a^2}{4}\left(\xi^2 - \frac{1}{\xi^2}\right)t.$$

$$\begin{aligned} q_{n0} &= \mathrm{e}^{-\phi_n} + \mathrm{e}^{-z_n - \mathrm{i}\alpha_n}\mathrm{e}^{\phi_n}, \\ q_{n1} &= \frac{1}{a}[-a_2\mathrm{e}^{\phi_{0n}} + a_1(\mathrm{e}^{-\phi_n}\mathrm{e}^{-z_n - \mathrm{i}\alpha_n} + \mathrm{e}^{\phi_n})], \\ q_{n2} &= \frac{1}{a}[a_1\mathrm{e}^{\phi_{0n}} + a_2(\mathrm{e}^{-\phi_n}\mathrm{e}^{-z_n - \mathrm{i}\alpha_n} + \mathrm{e}^{\phi_n})]. \end{aligned} \qquad (46)$$

Manakov system. General one-solitonic solution.

$$\varphi_1 = a_1 - 8\cos\alpha\cosh z\frac{-a_2N_1 + a_1N_2}{\exp(2u_0+z) + 4\cosh 2u\cosh z + 4\cos 2v\cos\alpha},$$

$$\varphi_2 = a_2 - 8\cos\alpha\cosh z\frac{a_1N_1 + a_2N_2}{\exp(2u_0+z) + 4\cosh 2u\cosh z + 4\cos 2v\cos\alpha}.$$

$$N_1 = \exp(u_0 + z/2 + \mathrm{i}v_0 + \mathrm{i}\alpha/2)[\cos(v-\alpha/2)\cosh(u-z/2)$$
$$-\mathrm{i}\sin(v-\alpha/2)\sinh(u-z/2)],$$
$$N_2 = \cos\alpha\cosh 2u + \mathrm{i}\sin\alpha\sinh 2u + \cos 2v\cosh z + \mathrm{i}\sin 2v\sinh z, \quad (47)$$

$$u_0 = æ_0 x - \gamma_0 t + \mu_0/2, \qquad v_0 = k_0 x - \omega_0 t - \theta_0/2,$$
$$æ_0 = a\cosh z\cos\alpha, \qquad \gamma_0 = -(a^2/2)\sinh 2z\sin 2\alpha,$$
$$k_{0n} = a\sinh z\sin\alpha, \qquad \omega_{0n} = (a^2/2)\cosh 2z\cos 2\alpha,$$
$$u = æx - \gamma t + \mu/2, \qquad v = kx - \omega t - \theta/2,$$
$$æ = a\sinh z\cos\alpha, \qquad \gamma = -(a^2/2)\cosh 2z\sin 2\alpha,$$
$$k = a\cosh z\sin\alpha, \qquad \omega = (a^2/2)\sinh 2z\cos 2\alpha. \quad (48)$$

asymptotics:

$$\varphi_{1,2} \to a_{1,2} \quad \text{when} \quad x \to \infty$$
$$\varphi_{1,2} \to -a_{1,2}\exp(-2\mathrm{i}\alpha) \quad \text{when} \quad x \to -\infty. \quad (49)$$

Manakov system. Vector Akhmediev breather.

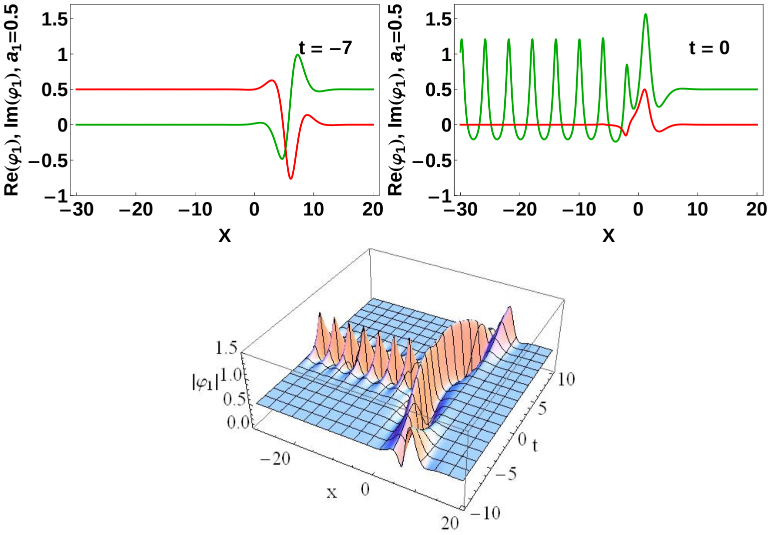


Figure 21: $a_1 = 0.5,\ a_2 = 1,\ R = 1,\ \alpha = \pi/4$. Green lines - $Re(\varphi)$, red - $Im(\varphi)$, 3D picture - $|\varphi|$.

Manakov system. Vector Kuznetsov soliton.

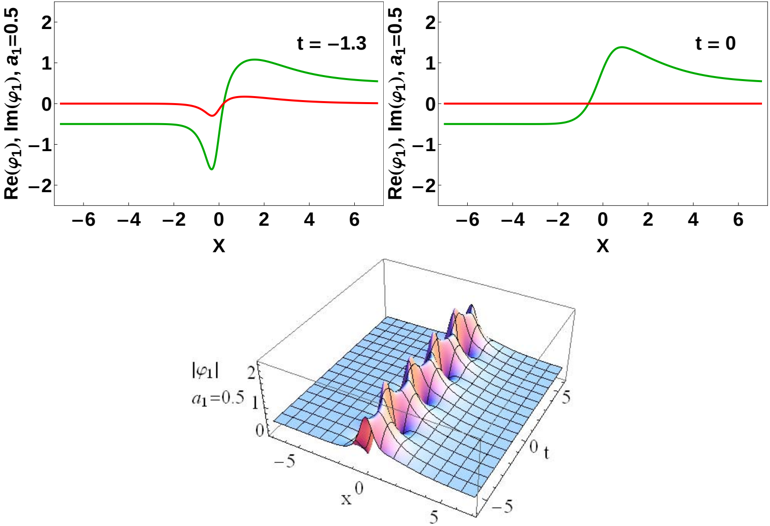


Figure 22: $a_1 = 0.5,\ a_2 = 1,\ R = 2,\ \alpha = 0.$ Green lines - $Re(\varphi)$, red - $Im(\varphi)$, 3D picture - $|\varphi|$.

Manakov system. Vector general soliton.

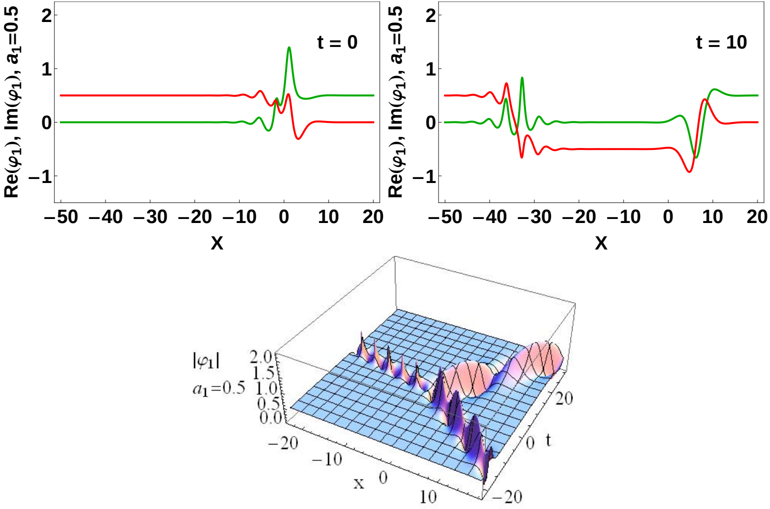


Figure 23: $a_1 = 0.5,\ a_2 = 1,\ R = 1.3,\ \alpha = \pi/4.$ Green lines - $Re(\varphi)$, red - $Im(\varphi)$, 3D picture - $|\varphi|$.

When the numerator vanishes?

$$\begin{aligned} \varphi_1 &= \varphi_{01} + 2\widetilde{M}_{01}/M. \\ \varphi_2 &= \varphi_{02} + 2\widetilde{M}_{02}/M. \end{aligned} \tag{50}$$

Only when N=3, that is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$:

$$\widetilde{M}_{01} = \begin{vmatrix} 0 & q_{1,1} & q_{2,1} & q_{3,1} \\ q^*_{1,0} & \frac{(q^*_1 \cdot q_1)}{\lambda+\lambda^*} & \frac{(q^*_1 \cdot q_2)}{\lambda+\lambda^*} & \frac{(q^*_1 \cdot q_3)}{\lambda+\lambda^*} \\ q^*_{2,0} & \frac{(q^*_2 \cdot q_1)}{\lambda+\lambda^*} & \frac{(q^*_2 \cdot q_2)}{\lambda+\lambda^*} & \frac{(q^*_2 \cdot q_3)}{\lambda+\lambda^*} \\ q^*_{3,0} & \frac{(q^*_3 \cdot q_1)}{\lambda+\lambda^*} & \frac{(q^*_3 \cdot q_2)}{\lambda+\lambda^*} & \frac{(q^*_3 \cdot q_3)}{\lambda+\lambda^*} \end{vmatrix} \equiv 0. \tag{51}$$

analogically $\widetilde{M}_{02} \equiv 0$. That is superregular solutions appear only in the trivial case:

$$\begin{aligned} \varphi_1(x,t) &= a_1 \varphi(a^2 t, ax), \\ \varphi_2(x,t) &= a_2 \varphi(a^2 t, ax). \end{aligned} \tag{52}$$

where φ - is the superregular solution of the scalar NLSE (1).

List of publications

1. V.E. Zakharov and A.A. Gelash. Nonlinear Stage of Modulation Instability, Phys. Rev. Lett. 111, 054101 (2013)
2. V. E. Zakharov and A. A. Gelash. Freak waves as a result of modulation instability, Procedia IUTAM 9C, 5, 165-175 (2013)
3. A.A. Gelash and V.E. Zakharov. Superregular solitonic solutions: a novel scenario of the nonlinear stage of Modulation Instability, Nonlinearity 27 (2014) R1-R39.
4. V.E. Zakharov, A. A. Gelash, A. Chabchoub and B. Kibler. Experimental observation of superregular solitonic solutions, In preparation.

Thank you for your attention!