Modelling transient sea states with the generalized kinetic equation

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motivation

All the existing approaches to the modelling of longtime evolution of wind waves are based on the key concept of wave turbulence. The statistical description is provided by the kinetic equation derived by K. Hasselmann (1962)

$$\frac{dn(\mathbf{k},\mathbf{x},t)}{dt} = S_{input} + S_{diss} + S_{nl}$$

The interaction term S_{nl}, dominant for energy carrying waves, is derived from first principles employing an asymptotic procedure based upon smallness of nonlinearity parameter ε and a number of additional assumptions:

$$S_{nl} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\mathbf{k}_{123}, \quad (1)$$

where $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, $n_i \equiv n(\mathbf{k}_i)$, $\delta_{0+1-2-3} \equiv \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$ and T_{0123} is given by an explicit but page long formula.

- The expression for S_{nl} is derived under the assumption of quasi-stationarity of the random wave field, and the resulting equation has a $O(\varepsilon^{-4})$ timescale of evolution
- Therefore, strictly speaking, the Hasselmann equation is not applicable to the situations with rapid changes of the environment, such as wind gusts
- Due to the lack of alternatives, this fact is usually ignored, and the Hasselmann theory is used to model the response to an instant and sharp increase or decrease of wind (e.g. Young & van Agthoven 1997). It is not clear to what extent these results can be trusted
- There is experimental evidence that after a sharp change of wind random wind waves appear to evolve faster than $O(\varepsilon^{-4})$ (Vledder & Holthuijsen 1993; Waseda, Toba & Tullin 2001; Autard & Caulliez 1995; Caulliez 2013)
- An example of phenomenon that cannot be legitimately described on the basis of the Hasselmann equation is squall

statistical description

Let us derive the wave kinetic equation. The starting point is the Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} \quad = \quad \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123} + \ldots$$

Notation: $\delta_{0+1-2-3} = \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $\mathrm{d}\mathbf{k}_{123} = \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3$. Here, $b(\mathbf{k})$ is complex amplitude in Fourier space, linked to the Fourier-transformed primitive physical variables $\zeta(\mathbf{k}, t)$ and $\varphi(\mathbf{k}, t)$ (position of the free surface and the velocity potential at the surface respectively) through an integral-power series

$$b(\mathbf{k}) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{\omega(\mathbf{k})}{k}} \zeta(\mathbf{k}) + \mathrm{i}\sqrt{\frac{k}{\omega(\mathbf{k})}} \varphi(\mathbf{k}) \right\} + O(\varepsilon). \qquad (2)$$

Derivation of the Zakharov equation assumes that wave slopes are $O(\varepsilon)$ small, and includes expansion in powers of ε

Now we consider ensembles of random wave fields (each governed by the deterministic Zakharov equation). We are interested in the ensemble averaged characteristics of the wave field.

Assumption of spatial homogeneity gives

$$\langle b_0^* b_1 \rangle = n_0 \delta_{0-1}$$

the brackets mean ensemble averaging, the second-order correlator n_0 is the spectral density of wave action at wavevector $\mathbf{k} = \mathbf{k}_0$.
The classical problem is to find and solve a closed equation in terms of $n(k)$, i.e. to find evolution of wave action spectral density $n(k)$ with time.

Multiplying the Zakharov equation by b_0^* and its c.c. by b_0, upon ensemble averaging we find

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im} \int T_{0123} J_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

$$J_{0123}^{(0)} \delta_{0+1-2-3} = < b_0^* b_1^* b_2 b_3 >$$

Assumption of Gaussianity gives

$$< b_0^* b_1^* b_2 b_3 > = n_0 n_1 \left(\delta_{0-2} \delta_{1-3} + \delta_{0-3} \delta_{1-2} \right).$$

which is real and, since T_{0123} is also real, does not contribute to evolution of n_0.

Non-gaussian correction $J^{(1)}_{0123}$ is specified by an evolution equation containing on the right-hand-side the sixth-order correlator I_{012345}. By assuming quasi-Gaussianity $I^{(0)}_{012345}$ is expressed in terms of the products of pair correlators. As a result we have

$$\left(\mathrm{i}\frac{\partial}{\partial t}+\Delta\omega\right)J^{(1)}_{0123}=2T_{0123}f_{0123}, \tag{3}$$

where $\Delta\omega=\omega_0+\omega_1-\omega_2-\omega_3$, and
$f_{0123}=n_2n_3(n_0+n_1)-n_0n_1(n_2+n_3)$

It is usually assumed that n_0 and, hence, f_{0123} depends on slow time μt, such that $\mu/\Delta\omega \ll 1$.

Then neglecting $\frac{\partial}{\partial t}$ in $\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = 2T_{0123} f_{0123} \qquad \Rightarrow$

$$J^{(1)}_{0123}(t) \simeq \frac{2T_{0123}}{\Delta\omega} f_{0123}.$$

This solution represents a large t asymptotics and is understood in terms of generalized functions

$$J^{(1)}_{0123}(t) = 2T_{0123}\left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)\right] f_{0123}(t), \qquad \text{(P is ‘principal value’)}$$

This asymptotic derivation yields the classic kinetic (Hasselmann) equation and is valid as long as the interest is confined to slow $O(\varepsilon^{-4})$ evolution.

The generalised kinetic equation

If we allow for faster variability of statistical moments of wave field, we can use the exact solution for $J^{(1)}$ in the form

$$J^{(1)}_{0123}(t) = -2\mathrm{i}T_{0123} \int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123}\,\mathrm{d}\tau + J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}.$$

$J^{(1)}_{0123}(0)$ is specified by initial conditions. The resulting generalized kinetic equation (GKE) is

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re} \int T^2_{0123} \left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123}\,\mathrm{d}\tau \right] \delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}$$

$$+2\mathrm{Im} \int \left[\mathrm{i}T_{0123} J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t} \right] \delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123}.$$

The GKE aims to capture ε^{-2} evolution and tends to the Hasselmann equation at large times. The stationary solutions of the Hasselmann equation are also equilibrium solutions of the GKE.

Initial stages

In general setting the evolution of spectral density n depends not only on the initial distribution of n, but also on the initial distribution of $J^{(1)}_{0123}(0)$.

"Cold start". Zero value of $J^{(1)}_{0123}(0)$ corresponds to the situations where the wave field is initially free, so that the wave components are not correlated, and waves begin to interact only after $t = 0$. Then the GKE has the form

$$\frac{\partial n_0}{\partial t} = 2 \int T^2_{0123} \left[\int_0^t \cos[\Delta\omega(\tau - t)] f_{0123} \, \mathrm{d}\tau \right] \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

$$\frac{\partial n_0}{\partial t}|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}|_{t=0} = 2 \int T^2_{0123} f_{0123} \, \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

All odd derivatives are zero, all even derivatives are known.

Timescales

$$\frac{\partial n_0}{\partial t}|_{t=0} = 0, \qquad \frac{\partial^2 n_0}{\partial t^2}|_{t=0} = 2\int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \, \mathrm{d}\mathbf{k}_{123}$$

Since $n \sim \varepsilon^2$ and the RHS is $\sim n^3 \sim \varepsilon^6$, then the timescale of initial evolution is $O(\varepsilon^{-2})$.

Higher moments

Kurtosis due to wave interactions

$$C_4^{(d)} = m_4/m_2^2 - 3$$

where

$$m_4 = \frac{3}{2}\mathrm{Re}\int (\omega_0\omega_1\omega_2\omega_3)^{1/2} J_{0123}^{(1)}\,\mathrm{d}\mathbf{k}_{0123}$$

$$m_2 = \int \omega_0 n_0\,\mathrm{d}\mathbf{k}_0$$

Kurtosis depends on $\mathrm{Re}J_{0123}^{(1)}$, while the spectral evolution involves $\mathrm{Im}J_{0123}^{(1)}$. All resonant and non-resonant interactions contribute to kurtosis, while the spectral evolution depends only on the interactions close to resonance.

Janssen's equation

Pulling f_{0123} out of the integral

$$\int_0^t e^{-\mathrm{i}\Delta\omega(\tau - t)} f_{0123}\,\mathrm{d}\tau$$

and setting $J^{(1)}_{0123}(0) = 0$, we get

$$\frac{\partial n_0}{\partial t} = 4 \int T^2_{0123} f_{0123} \delta_{0+1-2-3} \frac{\sin(\Delta\omega t)}{\Delta\omega}\,\mathrm{d}\mathbf{k}_{123}$$

This is Janssen's kinetic equation (Janssen 2003). Its derivation is based on the assumption that $\mu/\Delta\omega \ll 1$, and f_{0123} depends on slow time μt. In the limit $t \to \infty$ we get

$$\lim_{t\to\infty} \frac{\sin(\Delta\omega t)}{\Delta\omega} = \pi\delta(\Delta\omega),$$

and the equation tends to the Hasselmann equation.

Simulations of the GKE

Grid: 101×31 (ω, θ), $0.5 \leq \omega \leq 3$ with logarithmic spacing, $-2\pi/3 \leq \theta \leq 2\pi/3$,

Initial conditions: Donelan et al (1985) with $2 \leq U/c \leq 7.5$, $\omega_p = 1$.

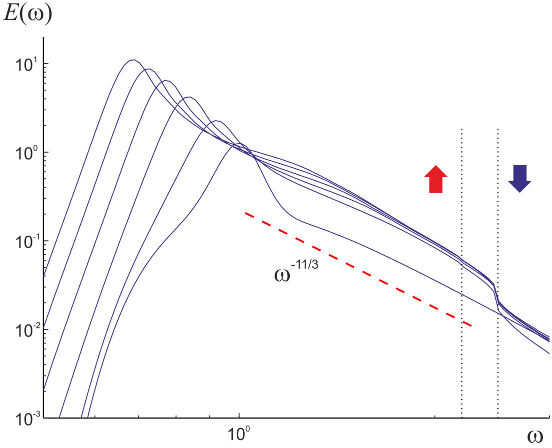
Algorithm: Runge-Kutta-Fehlberg with automatic step choice (step $\leq$ 1/3 of the period). After each step, all the previous history is stored in $J^{(1)}$, so there is no integration over the past.

The Hasselmann collision integral is computed using the code kindly provided by Gerbrant van Vledder (Delft University of Technology).

In the GKE, all resonant and non-resonant interactions should be taken into account, but actually only nearly resonant interactions contribute to the spectral evolution. We use a fairly large cutoff $\Delta\omega/\omega_{min} = 0.25$, the total number of interactions is $3 \cdot 10^9$.

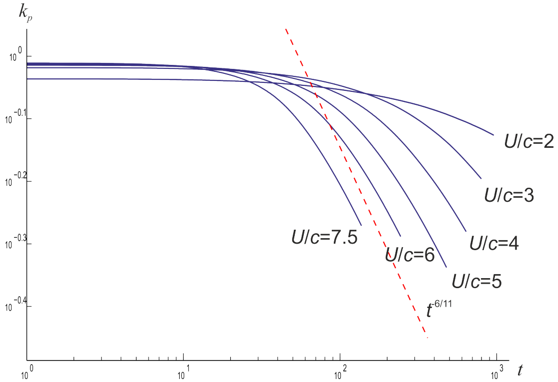
A typical computation takes 1-3 days on 64 processors.

Simulations of the GKE



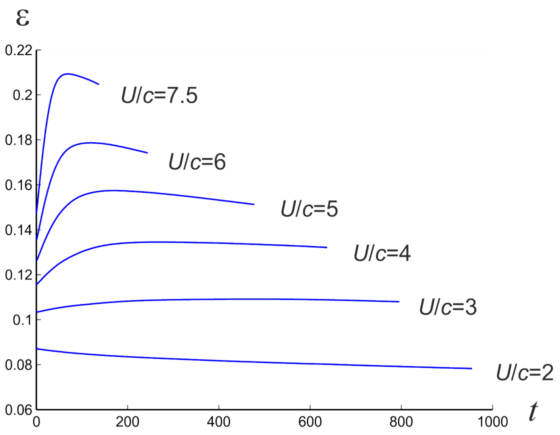
Evolution of energy spectrum $E(\omega)$ under constant wind. Initial condition is taken as an empirical spectrum by Donelan et al (1985) for $U/c = 5$, wind forcing corresponds to $U/c = 5$ at the initial moment, $\omega_p = 1$. Spectra are plotted approximately every 100 characteristic periods

Self-similarity

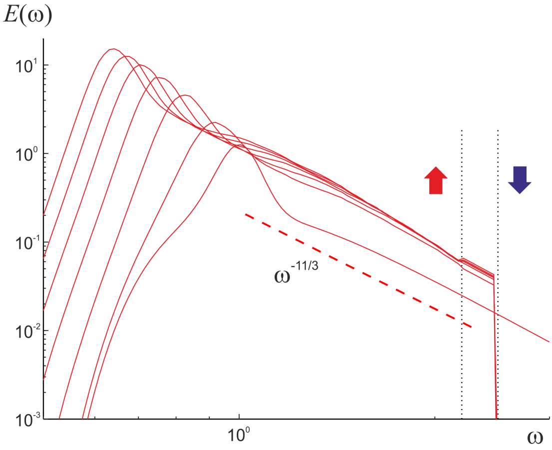


Evolution of the spectral peak for various U/c, and the theoretical downshift rate $t^{-6/11}$

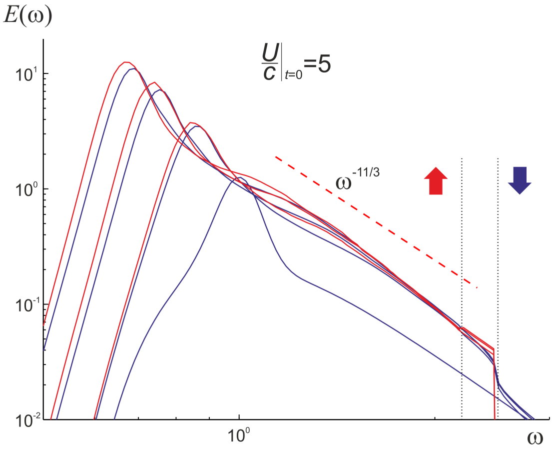
Steepness



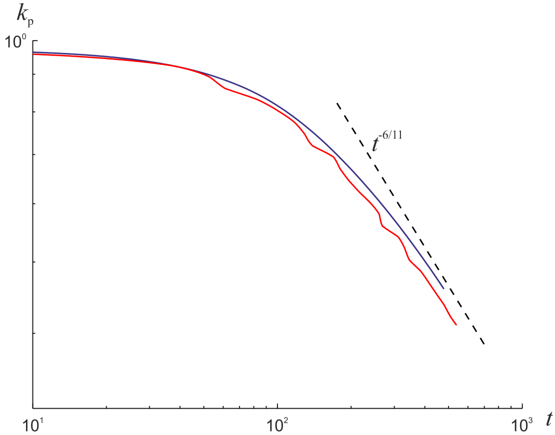
ε
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0
200
400
600
800
1000
t
U/c=7.5
U/c=6
U/c=5
U/c=4
U/c=3
U/c=2



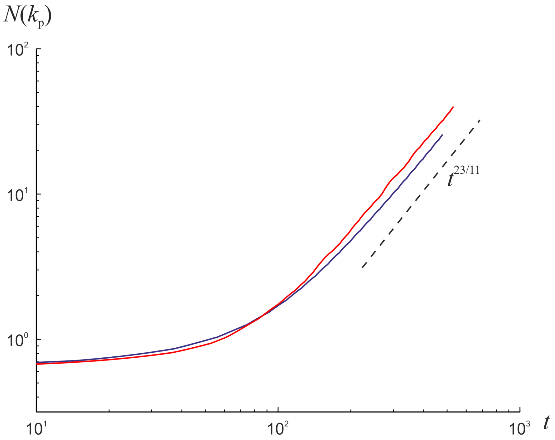
Same evolution as in figure 1, obtained by the numerical solution of the Hasselmann equation



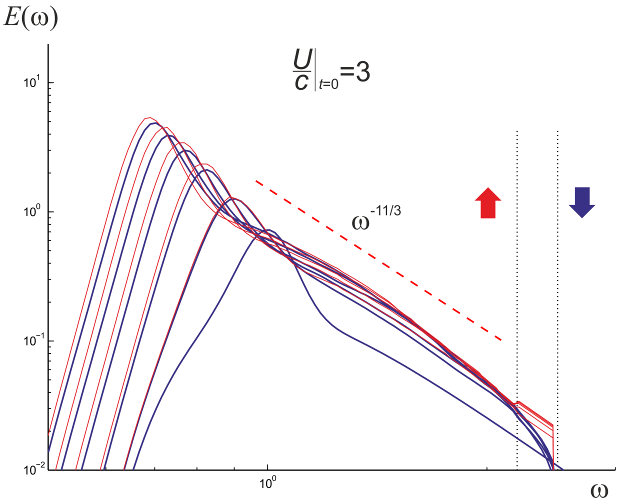
GKE vs Hasselmann. Spectra are plotted every 160 characteristic periods



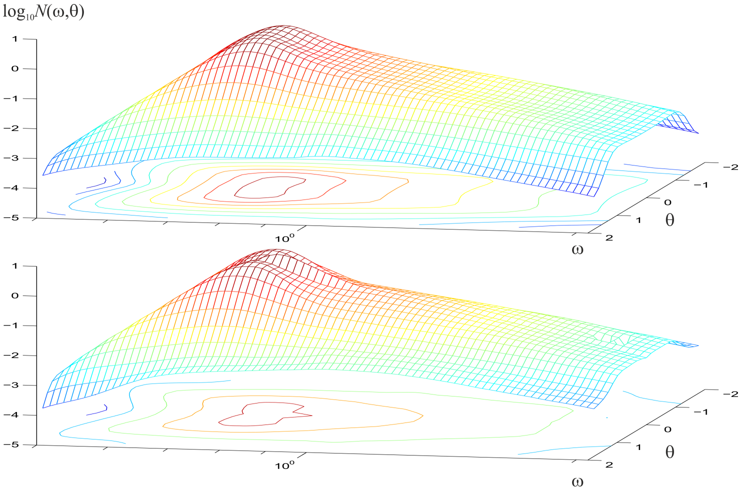
Wavenumber of the spectral peak for constant wind forcing ($U/c = 5$ at $t = 0$), GKE vs Hasselmann



Wave action of the spectral peak $n(k_p)$ for constant wind forcing ($U/c = 5$ at $t = 0$), GKE vs Hasselmann



GKE vs Hasselmann, for initial $U/c = 3$. Spectra are plotted every 350 characteristic periods

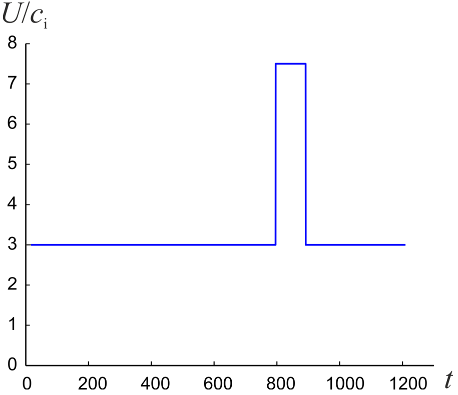


Comparison of 2D spectra: GKE (top) and Hasselmann (bottom), for initial $U/c = 3$, after evolution for 1000 periods

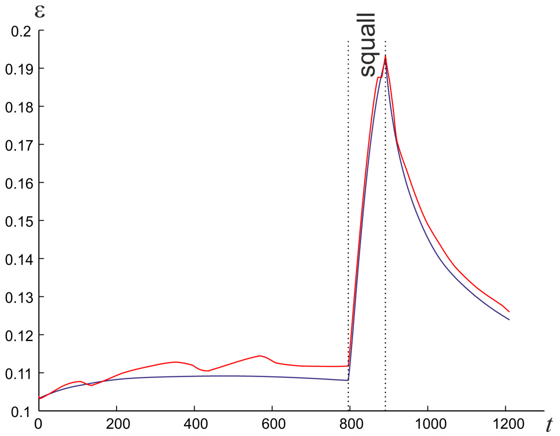
Squall

A squall is a sudden, sharp increase in the speed of the sustained winds over a short time interval.

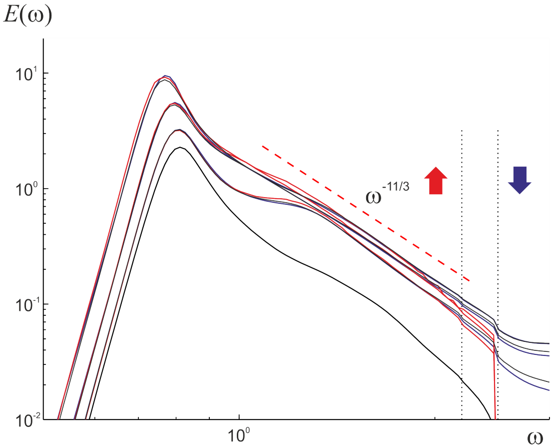
"A furious squall came up, and the waves broke over the boat, so that it was nearly swamped" (Mark 4:37)



Wind speed as function of time for the squall. Wind is normalised by the phase speed of the spectral peak of the initial condition, time is measured in periods of the spectral peak of the initial condition.

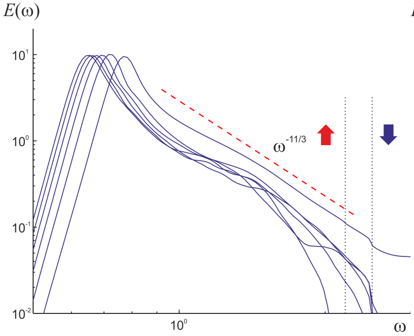


Wave steepness before, during and after the squall, GKE and Hasselmann

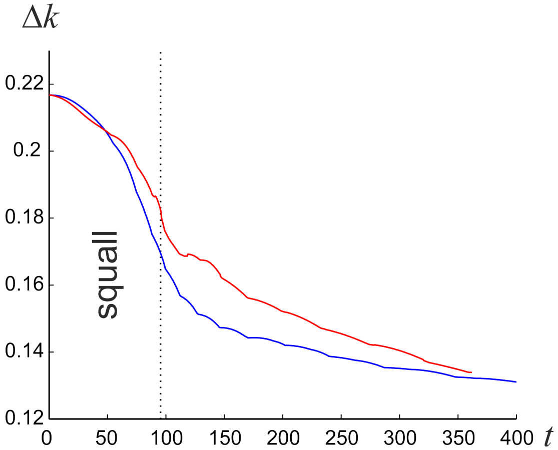


Comparison of the GKE, Janssen and Hasselmann equations solutions during the squall. For a direct comparison, initial conditions are taken as the GKE spectrum at the beginning of the squall

After the squall



Spectral width after the squall



Evolution of spectral width (taken as the width of the one-dimensional spectrum at half amplitude of the peak), GKE vs Hasselmann

JONSWAP fit

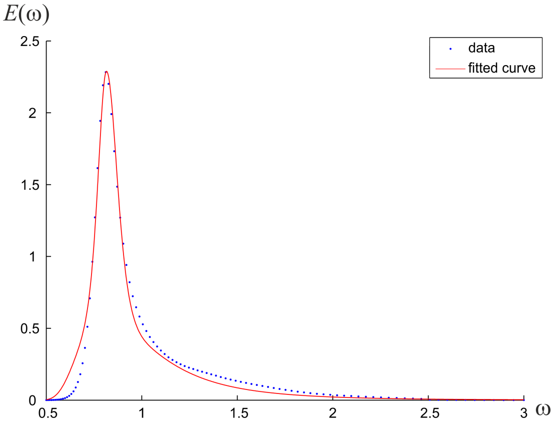
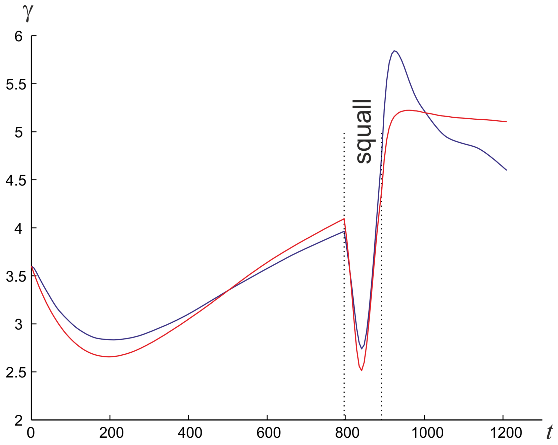


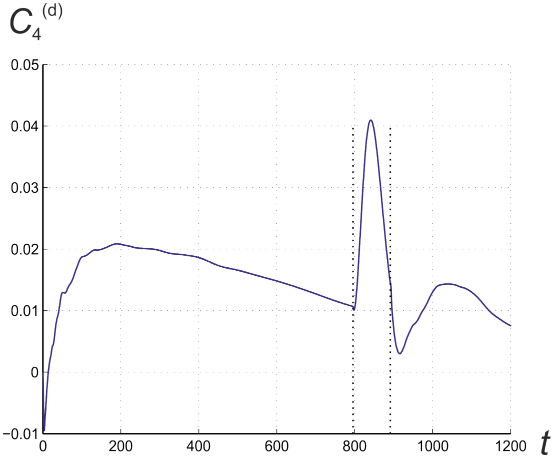
Illustration of JONSWAP fitting to the numerical spectra

Evolution of peakedness γ



Evolution of the fitted JONSWAP parameter γ, GKE vs Hasselmann

Higher moments



Evolution of dynamic kurtosis $C_4^{(d)}$ during and after the squall

Discussion

- Under steady wind GKE and Hasselmann equations are, as expected, in perfect agreement
- During the squall, when the spectrum grows rapidly, they are in perfect agreement too, which is unexpected
- The evolution with both equations has the $O(\varepsilon^{-4})$ timescale
- After the squall, both equations show a peculiar shape of the spectra with a dip on the spectral slope
- But it is after the squall that the GKE and Hasselmann solutions diverge: GKE spectra are more narrow with higher peakedness
- No $O(\varepsilon^{-2})$ evolution was found
- It appears that the Hasselmann equation has a much wider range of applicability than it follows from the used assumptions
- The GKE can be efficiently computed with a fast highly parallel algorithm, and has a potential to replace the Hasselmann equation as the basis of wave modelling