Inverse and direct cascades in 2D Nonlinear Shrödinger (Gross-Pitaevskii) equation

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Gross-Pitaevskii (nonlinear Schrödinger) equation

describes the evolution of a temporal envelope of a spectrally narrow wave packet, independent of the origin of the waves and the nature of the nonlinearity

$$\boxed{i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0}$$

Conserved quantities

Wave action:

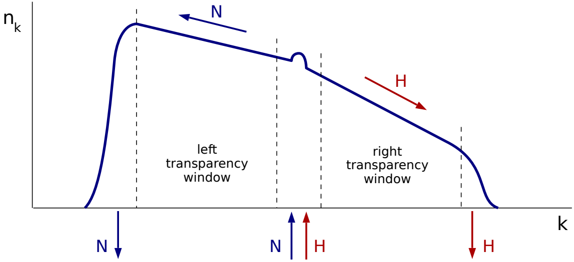
$$\mathcal{N} = \int |\psi|^2 d\mathbf{r} = \int \underbrace{|\psi_k|^2}_{n_k} d\mathbf{k}$$

Measure of nonlinearity: H_p/H_k.

Hamiltonian:

$$\mathcal{H} = \int \underbrace{|\nabla\psi|^2}_{H_k} \mp \underbrace{\tfrac{1}{4}|\psi|^4}_{H_p} d\mathbf{r}$$

Cascades of turbulence



Weak turbulence theories are based on kinetic equation

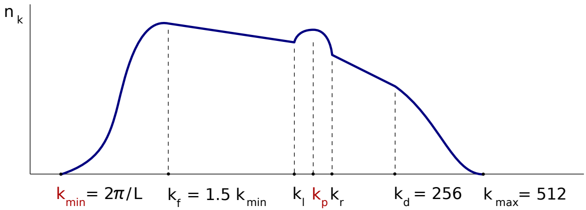
$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{2}{(2\pi)^3} \int \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(k^2 + k_1^2 - k_2^2 - k_3^2) \times$$
$$(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2}) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

Theoretical framework: Zakharov, Lvov, Falkovich (1992)
Inverse cascade: Dyachenko, Newell, Pushkarev, Zakharov (1992)
Direct cascade: Malkin (1996)

Goals of our numerical simulations

1. Compare turbulent spectra to weakly-nonlinear theories
2. Push beyond weakly-nonlinear regime
3. Evaluate the flux of wave action and energy in inverse and direct cascades

Numerical setup



Defocusing nonlinearity, forcing in **k**-space:

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}_k\psi + i\hat{g}_k .$$

Simulation size: up to 16386^2 grid points ($k_{\min} \geq \frac{1}{16}$).

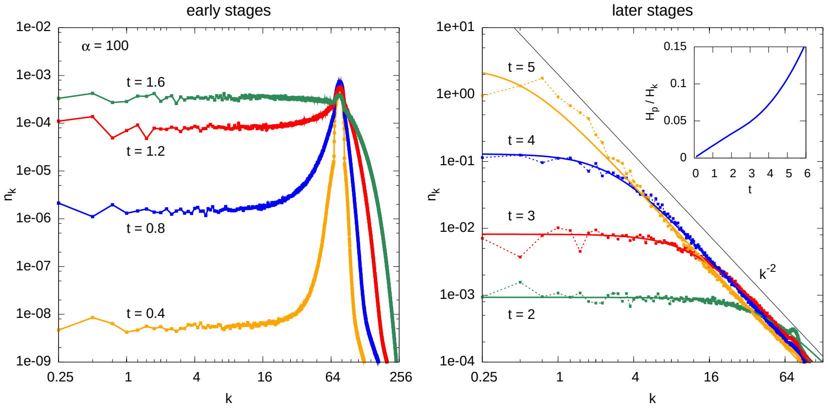
Pumping: $g_k = |g_k|e^{i\phi_k}$, $|g_k| \propto \sqrt{(k^2 - k_l^2)(k_r^2 - k^2)}$, random ϕ_k, $k_l < k < k_r$. Deposition rate $\alpha = \dot{N} \equiv \overline{|\psi|^2}$.

Small-scale damping: $f_k = -\beta(k/k_d)^4(k/k_d - 1)^2$, $k > k_d$.

Large-scale friction: $f_k = -(1, 1, \frac{1}{\sqrt{2}})\,\gamma$ for $k = (0, 1, \sqrt{2})k_{\min}$.

Evolution of Inverse Cascade Spectra in Simulations Without Friction

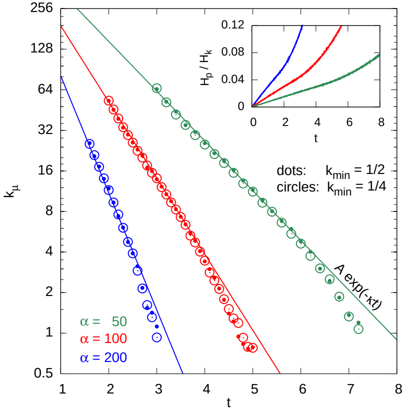
Inverse cascade: time evolution of non-stabilized spectra

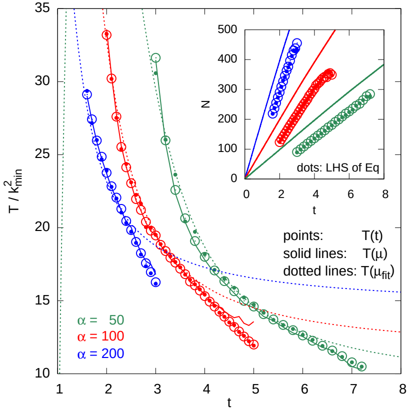


Early stage: Equipartitioned distribution of wave action.
Later stage: Thermal quazi-equilibrium with chemical potential $\mu = k_\mu^2$,

$$n_k = \frac{T(t)}{k_\mu^2(t) + k^2}.$$

Relation between $T(t)$ and $k_\mu(t)$



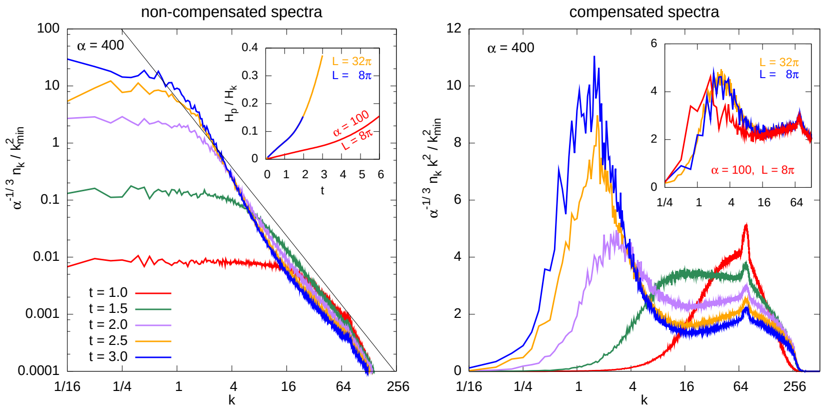


Balance of wave action:

$$\pi \frac{T}{k_{\min}^2} \ln \left[1 + \left(\frac{k_p}{k_\mu} \right)^2 \right] = \tilde{\alpha} t - N_d .$$

Assumption of $T(t) \to const$ leads to $k_\mu = A e^{-\varkappa t}$.
Deviation is due to non-linear effects, not due to limited domain size.

Non-linear effects in large boxes

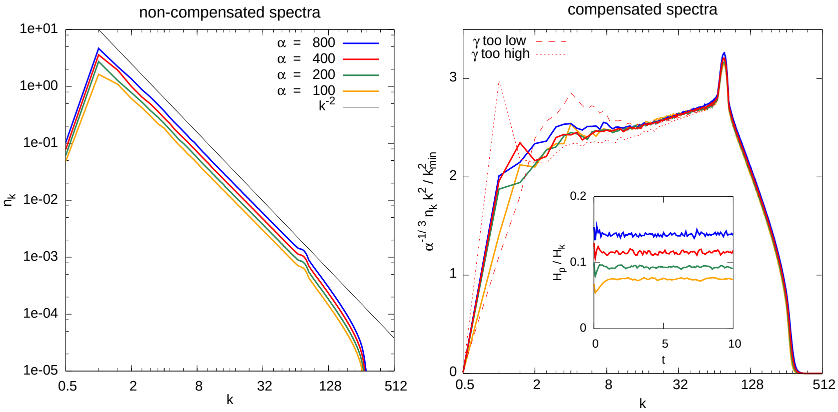


- At higher nonlinearities, n_k cannot be described by equilibrium fit.
- Spectra flatten at a lower k and pile-up at intermediate k.
- Flattening occurs even in large boxes.
- Pumping at lower rate α reduces piling-up and extends the spectrum.

Stabilized Spectra of Inverse Cascade

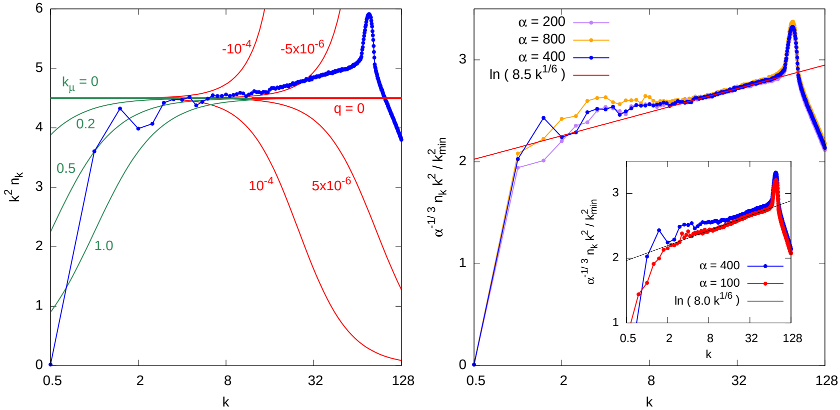
Comparison to Weakly-Nonlinear Theory

Inverse cascade: Effect of forcing and friction



- Deviation from $n_k \sim k^{-2}$ is small.
- Weak turbulence, four-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/3}$ scaling.
- Too high or too low γ leads to the distortion of spectrum at small k.

Comparison to weakly-nonlinear theory

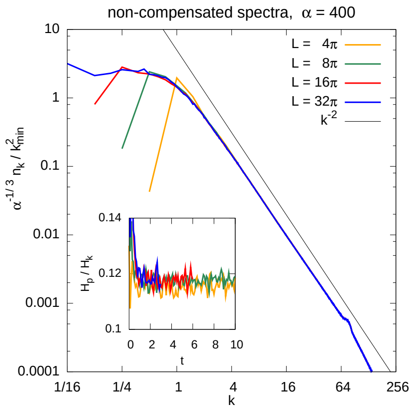


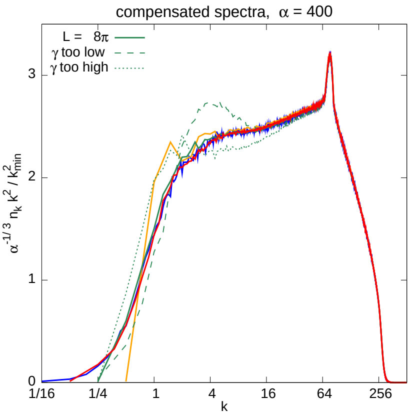
Dyachenko, Newell, Pushkarev, Zakharov (1992):

$$k^2 n_k = \frac{T}{1 + \left(\frac{k_\mu}{k}\right)^2 + qk^2 \left[\ln \frac{k}{k_{\min}}\right]^2}, \qquad q \equiv 4aQT^{-3}$$

Analytical correction does not agree with data.

Effect of domain size

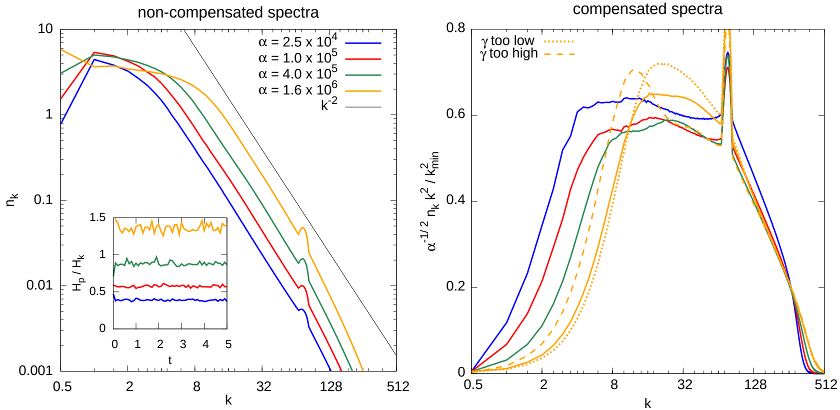




Can we extend the universal part of the spectrum by reducing k_{min}?

- For given α, domain size does not affect k^{-2} part of the spectrum.
- Pushing $k_{\text{min}} \to 0$ widens equipartitioned part, with $k_\mu = const$.
- Adjustment of friction does not extend universal part.
- Longer spectrum is expected for lower pumping rate α.

Toward higher nonlinearity

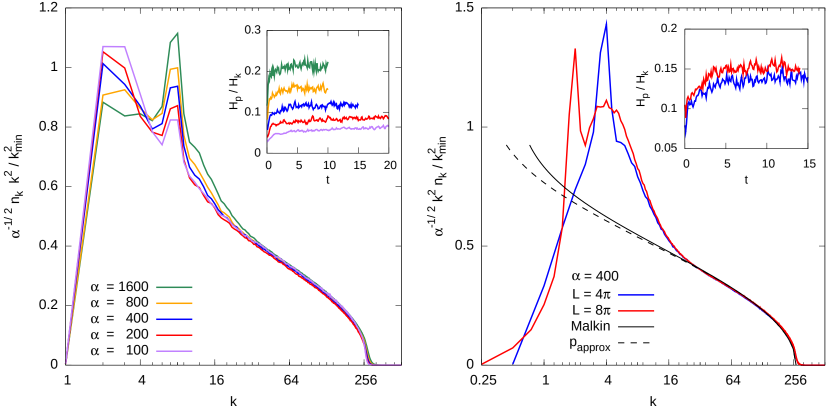


- At large k, deviation from $n_k \sim k^{-2}$ is small; unlike at weak nonlinearity, compensated spectra have negative slopes.
- Strong turbulence, three-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/2}$ scaling.
- Nonlinearity makes equipartitioned part of the spectrum wider.

Stabilized Spectra of Direct Cascade

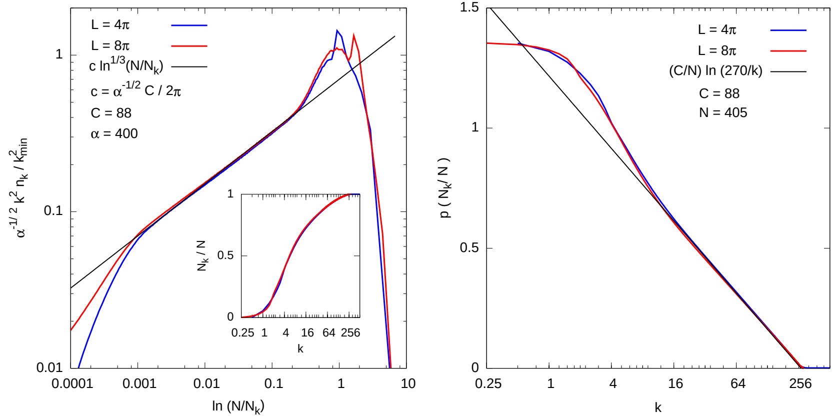
Comparison to Weakly-Nonlinear Theory

Direct cascade: compensated spectra



- Three-wave interactions are dominant, $n_k \sim \alpha^{1/2}$.
- Spectra at larger scales are distorted due to nonlinearity and sensitive to friction, γ.
- Spectra at small scales are universal and well-described by Malkin's theory (1996).

Comparison to weakly-nonlinear theory (Malkin, 1996)

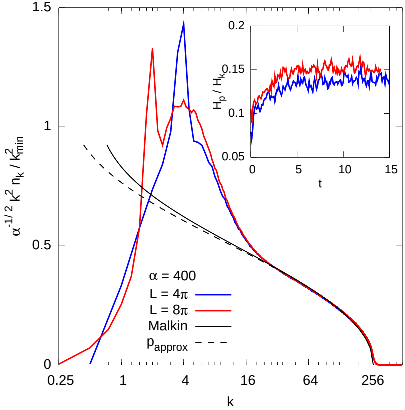


Implicit description in terms of the fraction of wave action contained within a sphere of radius k, N_k/N, and energy flux P,

$$\frac{n_k k^2}{k_{\rm min}^2} = \frac{C}{2\pi}\left[\ln\frac{N}{N_k}\right]^{\frac{1}{3}}, \qquad \frac{C}{N}\ln\frac{k_d}{k} = p\left(\frac{N_k}{N}\right).$$

Here, $p(m) = \int_m^1 \left[\ln y^{-1}\right]^{-\frac{1}{3}} dy$ and $C \propto P^{\frac{1}{3}}$. We show that $C \propto \alpha^{\frac{1}{2}}$.

Comparison to weakly-nonlinear theory (Malkin, 1996)



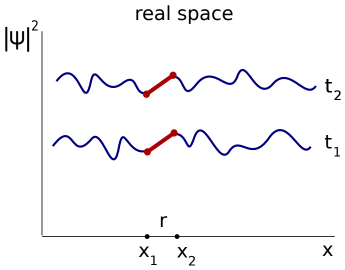
The parametric representation does not provide explicit expression for $n_k(k)$.
Using approximation $p_{\text{approx}}(m) = \frac{3}{2}(1-m)^{\frac{2}{3}}$, we obtain

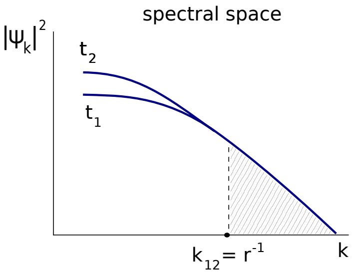
$$\frac{n_k k^2}{k_{\text{min}}^2} = \frac{C}{2\pi} \ln^{\frac{1}{3}} \left[1 - \left(\frac{2C}{3N} \ln \frac{k_d}{k} \right)^{\frac{3}{2}} \right].$$

Low pumping rates (smaller nonlinearity) might extent the range of appicability.

Fluxes of Wave Action and Energy

Flux of wave action





$\langle|\psi|^2\rangle$ grows in time and long modes appear, but

$$\langle|\psi_1 - \psi_2|^2\rangle = const$$

only $k = 1/r$ contribute to $\langle|\psi_1 - \psi_2|^2\rangle = \int |\psi_k|^2(1 - \cos kr)dk$

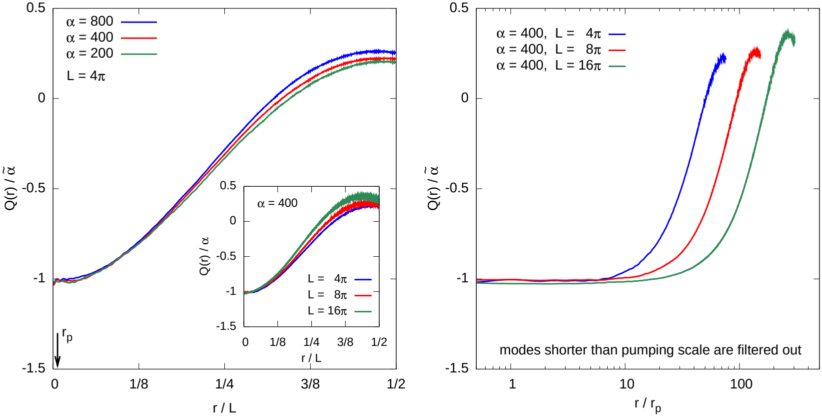
$$\langle|\psi_1 - \psi_2|^2\rangle \sim \int_{1/r}^{\infty} |\psi_k|^2 dk = const$$

Take time derivative of $\langle|\psi_1 - \psi_2|^2\rangle = 2N - \langle\psi_1\psi_2^* + \psi_1^*\psi_2\rangle$ to obtain,

$$Q(r) \equiv 2\,\mathrm{Im}\langle\psi_1^*|\psi_2|^2\psi_2\rangle = -\dot{N}$$

$Q(r)$ does not depend on distance between two points, r.
Analog of Kolmogorov's 4/5-law!

Flux of wave action in inverse cascade, $r \gg r_p$



$$Q(r) \equiv 2\,\mathrm{Im}\langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N}$$

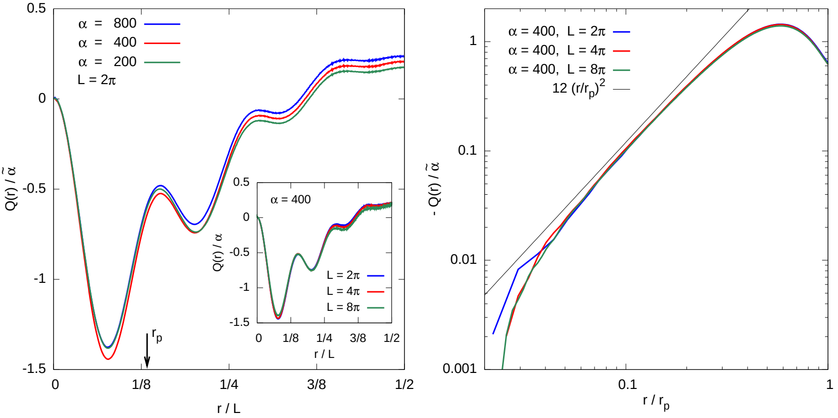
Simulations confirm:

$-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha$ for all scales.

$-Q(r) = \dot{N}$ for $r_p \lesssim r \lesssim L/16$.

$Q(r)$ is constant across the scales in inverse cascade.

Flux of energy in direct cascade, $r \ll r_p$



Simulations show:

$$-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha \quad \text{for all scales.}$$

$$-Q''(r) = const, \quad \text{therefore} \quad P \sim Qr^{-2} = const \quad \text{for} \quad r \ll r_p.$$

$P(r)$ is constant across the scales in direct cascade.

Conclusions

- To the first order, weak turbulence spectra can be described by thermal quazi-equilibrium with chemical potential.
- Correction by Dyachenko, Newell, Pushkarev, Zakharov (1992) for inverse cascade spectra does not work.
- Correction by Malkin (1996) for direct cascade spectra works well.
- High nonlinearities distort spectra from thermal equilibrium.
- Analog of Kolmogorov's 4/5 law:

$$Q(r) \equiv 2\,\mathrm{Im}\langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N} \qquad \text{for} \quad r > r_p;$$

  the flux of wave action is independent of scale in inverse cascade, while the flux of energy is independent of scale in direct cascade.