SOLITONS, COLLAPSES AND TURBULENCE
Zakharov 75
Chernogolovka, August 06, 2014

On the averaged multi-dimensional Poisson brackets

A.Ya. Maltsev

L.D. Landau Institute for Theoretical Physics

We will consider here the Poisson brackets obtained by the "averaging" of local multi-dimensional Poisson brackets

$$\{\varphi^{i}(\mathbf{x}), \varphi^{j}(\mathbf{y})\} = \sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \; \delta_{l_1 x^1 \dots l_d x^d}(\mathbf{x}-\mathbf{y}) \qquad (1)$$

on the families of m-phase quasiperiodic solutions of local Hamiltonian systems

$$\varphi^{i}_{t} = F^{i}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) \equiv F^{i}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x^1}, \dots, \boldsymbol{\varphi}_{x^d}, \dots) \quad , \qquad i = 1, \dots, n \qquad (2)$$

represented in the following general form

$$\varphi^{i}(\mathbf{x}, t) = \varphi^{i}_{[\mathbf{a},\boldsymbol{\theta}_0]}(\mathbf{x}, t) = $$
$$= \Phi^{i}\left(\mathbf{k}_1(\mathbf{a})\,x^1 + \dots + \mathbf{k}_d(\mathbf{a})\,x^d + \boldsymbol{\omega}(\mathbf{a})\,t + \boldsymbol{\theta}_0, \; \mathbf{a}\right) \qquad (3)$$

with some smooth 2π-periodic in each θ^{α} functions $\Phi^{i}(\boldsymbol{\theta}, \mathbf{a})$.

Thus, we assume that $\mathbf{x} = (x^1, \dots, x^d)$, $\mathbf{y} = (y^1, \dots, y^d)$ represent points of the Euclidean space $\mathbb{R}^d$ and the expression (1) defines a skew-symmetric Hamiltonian operator on the space of smooth functions

$$\boldsymbol{\varphi}(\mathbf{x}) \;=\; \left(\varphi^1(\mathbf{x}), \dots, \varphi^n(\mathbf{x})\right)$$

satisfying the Jacobi identity.

We will call brackets (1) general local field-theoretic Poisson brackets in $\mathbb{R}^d$ and assume that system (2) represents a Hamiltonian system generated by a local Hamiltonian functional

$$H \;=\; \int P_H\,(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots \,)\;\, d^d x \tag{4}$$

according to bracket (1).

We assume that the family (3) is defined with the aid of a smooth finite-parametric set $\hat{\Lambda}$ of 2π-periodic in each θ^{α} functions

$$\Phi^{i}\left(\boldsymbol{\theta}+\boldsymbol{\theta}_{0},\,\mathbf{a}\right) \;=\; \Phi^{i}\left(\theta^{1}+\theta_{0}^{1},\ldots,\theta^{m}+\theta_{0}^{m},\; a^{1},\ldots,a^{N}\right)$$

with a smooth dependence of the wave numbers $\mathbf{k}_{q}(\mathbf{a}) = (k_{q}^{1}(\mathbf{a}),\ldots,k_{q}^{m}(\mathbf{a}))$ and frequencies $\boldsymbol{\omega}(\mathbf{a}) = (\omega^{1}(\mathbf{a}),\ldots,\omega^{m}(\mathbf{a}))$ on the parameters $\mathbf{a} = (a^{1},\ldots,a^{N})$.
All the functions $\Phi^{i}(\boldsymbol{\theta},\,\mathbf{a})$ should satisfy the system

$$\omega^{\alpha}\,\Phi^{i}_{\theta^{\alpha}} \;-\; F^{i}\left(\boldsymbol{\Phi},\; k_{1}^{\beta_{1}}\,\boldsymbol{\Phi}_{\theta^{\beta_{1}}},\ldots,k_{d}^{\beta_{d}}\,\boldsymbol{\Phi}_{\theta^{\beta_{d}}},\ldots\right) \;=\; 0 \qquad (5)$$

The parameters θ_{0}^{α} represent the initial phase shifts of solutions (3) and take by definition all possible real values on the family $\hat{\Lambda}$. We assume also that the values of the parameters $\mathbf{a}$ do not change under the initial phase shifts. Let us denote by Λ the family (3) of the functions $\varphi^{i}(\mathbf{x},\,t)$ corresponding to the family $\hat{\Lambda}$.

The procedure of averaging of a Poisson bracket is closely connected with the Whitham averaging method ([26, 27, 28]). For this reason we will put here additional requirements of regularity and completeness on the family Λ which we formulate below.

Let us say first that we will everywhere consider here the generic situation where the values $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega})$ represent independent parameters on the full family of m-phase solutions of system (2). Thus, we assume that the number of real parameters $(a^1, \dots, a^N)$ is equal to $md + m + s$, $s \geq 0$. In particular, the parameters $(a^1, \dots, a^N)$ can be locally chosen in the form $\mathbf{a} = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ where $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega})$ represent the wave numbers and the frequencies of the m-phase solutions and $\mathbf{n} = (n^1, \dots, n^s)$ are some additional parameters (if any).

Let us consider now linear operators $\hat{L}^i_{j[\mathbf{a},\boldsymbol{\theta}_0]} = \hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ given by the linearization of system (5) on the corresponding solutions $\boldsymbol{\Phi}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_1,\dots,\mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n})$. It's not difficult to see that the functions $\boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_1,\dots,\mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n})$, $\alpha = 1,\dots,m$, and $\boldsymbol{\Phi}_{n^l}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_1,\dots,\mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n})$, $l = 1,\dots,s$, represent kernel vectors of the operators $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ on the space of 2π-periodic in each θ^α functions which depend smoothly on all the parameters $(\mathbf{k}_1,\dots,\mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n},\, \boldsymbol{\theta}_0)$. Let us put now the following requirements on the operators $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ on the family $\hat{\Lambda}$:

1) We require that the vectors $\mathbf{\Phi}_{\theta^{\alpha}}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$, $\mathbf{\Phi}_{n^l}(\boldsymbol{\theta}+\boldsymbol{\theta}_0,\, \mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ are linearly independent and represent the maximal linearly independent set among the kernel vectors of the operator $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ on the space of 2π-periodic in each θ^{α} functions smoothly depending on the parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n})$.

2) The operators $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$ have exactly $m+s$ linearly independent regular left eigen-vectors $\boldsymbol{\kappa}^{(q)}_{[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n}]}(\boldsymbol{\theta}+\boldsymbol{\theta}_0)$, $\;q = 1, \dots, m+s$, on the space of 2π-periodic in each θ^{α} functions, corresponding to the zero eigenvalue and depending smoothly on the parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d,\, \boldsymbol{\omega},\, \mathbf{n})$.

Under all the requirements formulated above we will call the corresponding family Λ a *complete regular family* of m-phase solutions of system (2).

It is well known that the Whitham approach gives a description of the slowly modulated m-phase solutions of nonlinear PDE's. The Whitham solutions represent asymptotic solutions of nonlinear systems with the main part having the form

$$\varphi_{(0)}(\mathbf{x},\, t,\, \boldsymbol{\theta}) \;=\; = \; \Phi\left(\frac{\mathbf{S}(\mathbf{X},\, T)}{\epsilon} + \boldsymbol{\theta}_{(0)}(\mathbf{X},\, T) + \boldsymbol{\theta},\; \mathbf{S}_{X^1}, \ldots, \mathbf{S}_{X^d},\, \mathbf{S}_T,\, \mathbf{n}(\mathbf{X},\, T)\right) \quad (6)$$

where $\mathbf{X} = \epsilon\, \mathbf{x}$, $T = \epsilon\, t$, $\epsilon \to 0$, are the slow spatial and time variables and the function

$$\mathbf{S}(\mathbf{X},\, T) \;=\; \left(S^1(\mathbf{X},\, T), \ldots, S^m(\mathbf{X},\, T)\right)$$

represents the "modulated phase" of the solution. Thus, the main part of the Whitham solution represents an m-phase solution of the nonlinear system with the slow modulated parameters $\mathbf{a}(\mathbf{X},\, T)$ and a rapidly changing phase.

We have also the natural connection

$$S^{\alpha}_{T} \; = \; \omega^{\alpha}(\mathbf{X}, T) \;\; , \qquad S^{\alpha}_{X^q} \; = \; k^{\alpha}_{q}(\mathbf{X}, T) \tag{7}$$

between the derivatives of the modulated phase and the parameters $\boldsymbol{\omega}(\mathbf{X}, T)$ and $\mathbf{k}_q(\mathbf{X}, T)$.
Relations (7) give the natural constraints

$$k^{\alpha}_{qT} \; = \; \omega^{\alpha}_{X^q} \;\; , \qquad k^{\alpha}_{qX^p} \; = \; k^{\alpha}_{pX^q}$$

on the functions $\boldsymbol{\omega}(\mathbf{X}, T)$ and $\mathbf{k}_q(\mathbf{X}, T)$, which can be considered as the first part of the Whitham system on the parameters $\mathbf{a}(\mathbf{X}, T)$.

The second part of the Whitham system is defined usually by the requirement of existence of a bounded next correction to the initial approximation (6) and can be defined in different ways which are usually equivalent to each other (see e.g. [26, 27, 28, 17, 13, 1, 14, 15, 7, 8, 16]). In our scheme we will define the second part of the Whitham system for a complete regular family Λ of m-phase solutions of (2) as the orthogonality at every $\mathbf{X}$ and T of all the regular eigen-vectors

$$\boldsymbol{\kappa}^{(q)}_{[\mathbf{S}_{X^1},\dots,\mathbf{S}_{X^d},\mathbf{S}_T,\mathbf{n}(\mathbf{X},T)]}\left(\frac{\mathbf{S}(\mathbf{X},T)}{\epsilon}+\boldsymbol{\theta}_{(0)}(\mathbf{X},T)+\boldsymbol{\theta}\right) \quad , \quad q=1,\dots,m+s$$

to the first ϵ-discrepancy $\mathbf{f}_1(\boldsymbol{\theta},\mathbf{X},T)$, obtained after the substitution of the main approximation (6) into the system

$$\epsilon\,\varphi^i_T \;=\; F^i\left(\boldsymbol{\varphi},\,\epsilon\,\boldsymbol{\varphi}_{\mathbf{X}},\,\epsilon^2\,\boldsymbol{\varphi}_{\mathbf{XX}},\,\dots\right)$$

It is well known that the full Whitham system, defined in one of the standard ways, does not put any restrictions on the variables $\theta_0(\mathbf{X}, T)$ and represents a system of PDE's just on the parameters $\mathbf{a}(\mathbf{X}, T)$ (see e.g. [26, 27, 28, 17]). In particular, it is also not difficult to show that the orthogonality conditions

$$\int_0^{2\pi} \dots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{S}_{X^1},\dots,\mathbf{S}_{X^d},\mathbf{S}_T,\mathbf{n}(\mathbf{X},T)]\, i} \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_{(0)}(\mathbf{X}, T) + \boldsymbol{\theta} \right) \times \\ \times \;\; f^i_1(\boldsymbol{\theta}, \mathbf{X}, T) \; \frac{d^m\theta}{(2\pi)^m} \;\; = \;\; 0 \quad (8)$$

defined for any complete regular family Λ, possesses the same property.

In general, relations (8) can be written as a system of $m + s$ quasilinear equations

$$P^{(q)}_{\alpha}(\mathbf{S_X}, \mathbf{S}_T, \mathbf{n}) \, S^{\alpha}_{TT} \; + \; Q^{(q)p}_{\alpha}(\mathbf{S_X}, \mathbf{S}_T, \mathbf{n}) \, S^{\alpha}_{X^p T} \; +$$

$$+ \; R^{(q)pk}_{\alpha}(\mathbf{S_X}, \mathbf{S}_T, \mathbf{n}) \, S^{\alpha}_{X^p X^k} \; + \; V^{(q)}_{l}(\mathbf{S_X}, \mathbf{S}_T, \mathbf{n}) \, n^{l}_{T} \; +$$

$$+ \; W^{(q)p}_{l}(\mathbf{S_X}, \mathbf{S}_T, \mathbf{n}) \, n^{l}_{X^p} \;\; = \;\; 0 \quad , \qquad q \, = \, 1, \dots, m + s$$

with some smooth functions $P^{(q)}_{\alpha}$, $Q^{(q)p}_{\alpha}$, $R^{(q)pk}_{\alpha}$, $V^{(q)}_{l}$, $W^{(q)p}_{l}$.

Let us say here that for the single-phase case ($m = 1$) the set of the "regular" left eigen-vectors $\boldsymbol{\kappa}^{(q)}_{[k_1,\dots,k_d,\omega,\mathbf{n}]}(\theta + \theta_0)$, $q = 1, \dots, s+1$, represents usually the full set of linearly independent left eigen-vectors of the operators $\hat{L}^i_{j[k_1,\dots,k_d,\omega,\mathbf{n},\theta_0]}$, corresponding to the zero eigen-value, for all the values of $(k_1, \dots, k_d, \omega, \mathbf{n}, \theta_0)$ on a complete regular family Λ. However, for the multi-phase case ($m > 1$) the situation is usually more complicated and "irregular" left eigen-vectors of $\hat{L}^i_{j[\mathbf{k}_1,\dots,\mathbf{k}_d,\boldsymbol{\omega},\mathbf{n},\boldsymbol{\theta}_0]}$, corresponding to the zero eigen-value, also arise for special values of parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d,\ \boldsymbol{\omega},\ \mathbf{n})$. As a result, the corrections to the main approximation (6) of the Whitham solution for the multi-phase case have usually rather different form in comparison with the case $m = 1$ (see e.g. [2, 3, 4]). Let us say, however, that the regular Whitham system still plays the central role in the description of the slow-modulated m-phase solutions both in the cases $m = 1$ and $m > 1$.

One of the most elegant ways of constructing the Whitham system was suggested by Whitham and is connected with the averaging of the Lagrangian function of the initial system. This method is applicable to any system having a local Lagrangian structure and gives a local Lagrangian structure for the corresponding Whitham system (see e.g. [28]). Let us say, that the Lagrangian approach gives usually essential advantages both in constructing and investigation of the Whitham equations.

The class of local Lagrangian systems can be significantly expanded being included into a larger class of systems having local field-theoretic Hamiltonian structure. In general, the systems of this kind can be considered as the evolution systems (2) which can be represented in the form

$$\varphi^i_t \;=\; \hat{J}^{ij} \, \frac{\delta H}{\delta \varphi^j(\mathbf{x})}$$

where $\hat{J}^{ij}$ is the Hamiltonian operator

$$\hat{J}^{ij} \;=\; \sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \, \left(\frac{d}{dx^1} \right)^{l_1} \dots \left(\frac{d}{dx^d} \right)^{l_d} \quad ,$$

defined by the Poisson bracket (1), and H is the Hamiltonian functional having the form (4).

The Hamiltonian theory of the Whitham equations was started by B.A. Dubrovin and S.P. Novikov, who introduced the concept of the Hamiltonian structure of Hydrodynamic Type. In general, the Dubrovin-Novikov bracket in $\mathbb{R}^d$ can be written in the following local form

$$\{U^{\nu}(\mathbf{X}), U^{\mu}(\mathbf{Y})\} \; = \; g^{\nu\mu\, l}(\mathbf{U}(\mathbf{X}))\, \delta_{X^l}(\mathbf{X}-\mathbf{Y}) + b^{\nu\mu\, l}_{\lambda}(\mathbf{U}(\mathbf{X}))\; U^{\lambda}_{X^l}\, \delta(\mathbf{X}-\mathbf{Y}) \tag{9}$$

(summation over repeated indices).

The general theory of the brackets (9) is rather nontrivial. Rather deep results on the classification of brackets (9) were obtained in [6, 22, 23] where the full description of brackets (9), satisfying special non-degeneracy conditions, was presented. However, there are many interesting examples where a nontrivial structure of a system is defined by a non-generic bracket (9) (see e.g. [11, 12]). In general, we can say that the full theory of the brackets (9) represents rather interesting branch of the theory of the Poisson brackets and is still waiting for its final completion.

A special class of the Dubrovin-Novikov brackets (9) is given by the one-dimensional brackets of Hydrodynamic Type. The brackets (9) have in this case the following general form

$$\{U^{\nu}(X)\,,\,U^{\mu}(Y)\} \;=\; g^{\nu\mu}(\mathbf{U}(X))\,\delta'(X-Y) \,+\, b^{\nu\mu}_{\lambda}(\mathbf{U}(X))\,U^{\lambda}_{X}\,\delta(X-Y) \quad , \tag{10}$$

and are closely connected with Differential Geometry. Thus, it can be proved ([5, 6, 7, 8]) that the expression (10) with non-degenerate tensor $g^{\nu\mu}(\mathbf{U})$ defines a Poisson bracket on the space of fields $\mathbf{U}(X)$ if and only if the tensor $g^{\nu\mu}(\mathbf{U})$ defines a flat pseudo-Riemannian metric with upper indices on the space of $\mathbf{U}$ while the values $\Gamma^{\nu}_{\mu\gamma} = -\,g_{\mu\lambda}\,b^{\lambda\nu}_{\gamma}$ represent the corresponding Christoffel symbols ($g_{\nu\tau}(\mathbf{U})\,g^{\tau\mu}(\mathbf{U}) = \delta^{\mu}_{\nu}$).

The theory of the Poisson brackets of Hydrodynamic Type provides the basement for the theory of integrability of multi-component one-dimensional Hydrodynamic Type systems

$$U^{\nu}_{T} \;=\; V^{\nu}_{\mu}(\mathbf{U})\;U^{\mu}_{X} \quad , \qquad \nu = 1, \dots, N \tag{11}$$

Thus, according to conjecture of S.P. Novikov, every diagonalizable system (11) which is Hamiltonian with respect to some bracket (10) with the Hamiltonian of Hydrodynamic Type

$$H \; = \; \int_{-\infty}^{+\infty} h(\mathbf{U}) \; dX$$

can be integrated.

The conjecture of S.P. Novikov was proved by S.P. Tsarev ([24, 25]) who suggested a method for solving of diagonal Hamiltonian systems

$$U^{\nu}_{T} \;\; = \;\; V^{\nu}(\mathbf{U}) \; U^{\nu}_{X} \quad , \qquad \nu = 1, \ldots, N \tag{12}$$

The method of Tsarev can be applied in fact to a wider class of systems (12) which were called by S.P. Tsarev semi-Hamiltonian. In particular, the class of the semi-Hamiltonian systems contains the diagonal systems Hamiltonian with respect to the weakly nonlocal Poisson brackets of Hydrodynamic Type - the Mokhov-Ferapontov bracket ([21]) and more general Ferapontov brackets ([9, 10]), which appeared as generalizations of the brackets of B.A. Dubrovin and S.P. Novikov. The diagonal semi-Hamiltonian systems represent the widest class of integrable one-dimensional systems of Hydrodynamic Type.

B.A. Dubrovin and S.P. Novikov suggested also a method of averaging of local field-theoretic Hamiltonian structures for the case of one spatial dimension.

The Dubrovin-Novikov procedure is based on the existence of N local integrals of system (2)

$$I^{\nu} \; = \; \int P^{\nu}(\varphi, \varphi_{x}, \dots)\, dx$$

which commute with the Hamiltonian H and with each other

$$\{I^{\nu}\, ,\, H\} \, = \, 0 \quad , \quad \{I^{\nu}\, ,\, I^{\mu}\} \, = \, 0 \tag{13}$$

according to the bracket (1) $(d = 1)$. It is supposed also that the set of parameters $\mathbf{a}$ on the family Λ can be chosen in the form $(a^{1}, \dots, a^{N}) \, = \, (U^{1}, \dots, U^{N})$, where

$$U^{\nu} \; = \; \langle P^{\nu} \rangle \; \equiv \; \int_{0}^{2\pi} \dots \int_{0}^{2\pi} \, P^{\nu} \left(\mathbf{\Phi}, \, k^{\alpha} \mathbf{\Phi}_{\theta^{\alpha}}, \dots \right) \, \frac{d^{m}\theta}{(2\pi)^{m}}$$

represent the values of the densities $P^{\nu}(\varphi, \varphi_{x}, \dots)$ on Λ, averaged over the angle (phase) variables.

We can write for the time evolution of the densities $P^{\nu}(\varphi, \varphi_x, \dots)$ according to system (2):

$$P^{\nu}_t(\varphi, \varphi_x, \dots) \;\equiv\; Q^{\nu}_x(\varphi, \varphi_x, \dots) \;,$$

where $Q^{\nu}(\varphi, \varphi_x, \dots)$ are some smooth functions of φ and its spatial derivatives. It is convenient to write also the Whitham system as a system of conservation laws

$$\langle P^{\nu} \rangle_T \;=\; \langle Q^{\nu} \rangle_X \quad , \qquad \nu = 1, \dots, N \;, \tag{14}$$

using the functions $P^{\nu}(\varphi, \varphi_x, \dots)$ and $Q^{\nu}(\varphi, \varphi_x, \dots)$.

The procedure of construction of the Dubrovin-Novikov bracket for system (14) can be described in the following way:
Let us calculate the pairwise Poisson brackets of the densities $P^{\nu}(x)$, $P^{\mu}(y)$, which can be represented in the form:

$$\{P^{\nu}(x)\,,\,P^{\mu}(y)\} \;=\; \sum_{k\geq 0} A^{\nu\mu}_{k}(\varphi,\varphi_{x},\dots)\;\delta^{(k)}(x-y)$$

which some smooth functions $A^{\nu\mu}_{k}(\varphi,\varphi_{x},\dots)$.
According to conditions (13) we can write the relations

$$A^{\nu\mu}_{0}(\varphi,\varphi_{x},\dots) \;\equiv\; \partial_{x} Q^{\nu\mu}(\varphi,\varphi_{x},\dots)$$

for some functions $Q^{\nu\mu}(\varphi,\varphi_{x},\dots)$.

Let us put now $U^{\nu} = \langle P^{\nu} \rangle$ and define the Poisson bracket

$$\{U^{\nu}(X), U^{\mu}(Y)\} = \langle A_1^{\nu\mu} \rangle(\mathbf{U}) \; \delta'(X-Y) + \frac{\partial \langle Q^{\nu\mu} \rangle}{\partial U^{\gamma}} \; U^{\gamma}_X \; \delta(X-Y) \tag{15}$$

on the space of functions $\mathbf{U}(\mathbf{X})$.
System (14) can be defined now as a Hamiltonian system with respect to the bracket (15) with the Hamiltonian functional

$$H_{av} = \int_{-\infty}^{+\infty} \langle P_H \rangle \left(\mathbf{U}(X) \right) \; dX$$

Let us say that the complete justification of the Dubrovin-Novikov procedure represents in fact a nontrivial question. Let us give here the reference on paper [18] where some review of this question and the most detailed consideration of the justification problem were presented. In particular, we can state that the Dubrovin-Novikov procedure is well justified for a complete regular family Λ having certain regular Hamiltonian properties ([18]).

In the case of several spatial dimensions ($d > 1$) the procedure of bracket averaging should be actually modified, which is connected mostly with a special role of the variables $\mathbf{S}(\mathbf{X})$ revealed in this situation. Let us formulate here the corresponding procedure and the conditions of its applicability according to the scheme proposed in [19, 20].

Let us consider a complete regular family Λ of m-phase solutions of system (2) parametrized by the $m(d+1)+s$ parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ and m initial phase shifts $\boldsymbol{\theta}_0$. We will call the complete regular family Λ a *complete Hamiltonian family* of m-phase solutions of (2) if it satisfies the following requirements:

1) The bracket (1) has at every point $(\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n}, \boldsymbol{\theta}_0)$ of Λ the same number s' of "annihilators" defined by linearly independent solutions $\mathbf{v}^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]}(\mathbf{x})$ of the equation

$$\sum_{l_1,\ldots,l_d} B^{ij}_{(l_1,\ldots,l_d)}(\varphi_{[\mathbf{a},\boldsymbol{\theta}_0]}, \varphi_{[\mathbf{a},\boldsymbol{\theta}_0]\,\mathbf{x}}, \ldots)\; v^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]\,j\,,\,l_1x^1\ldots\,l_dx^d}(\mathbf{x}) \quad = \quad 0 \;, \quad (16)$$

such that all the functions $v^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]\,i}(\mathbf{x})$ can be represented in the form

$$v^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]\,i}(\mathbf{x}) \;=\; v^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]\,i}\left(\mathbf{k}_1 x^1 + \ldots + \mathbf{k}_d x^d\right)$$

for some smooth 2π-periodic in each θ^α functions $v^{(k)}_{[\mathbf{a},\boldsymbol{\theta}_0]\,i}(\boldsymbol{\theta})$.

2) For the derivatives φ_{ω^α}, φ_{n^l} of the functions $\varphi_{[\mathbf{a},\theta_0]}(\mathbf{x}) = \varphi_{[\mathbf{k}_1,\dots,\mathbf{k}_d,\,\omega,\,\mathbf{n},\,\theta_0]}(\mathbf{x})$ we have the relations

$$\mathrm{rank} \left\| \begin{matrix} (\varphi_{\omega^\alpha} \cdot \mathbf{v}^{(k)}) \\ (\varphi_{n^l} \cdot \mathbf{v}^{(k)}) \end{matrix} \right\| = s'$$

$(\alpha = 1, \dots, m,\ l = 1, \dots, s,\ k = 1, \dots, s')$, where the expressions

$$\left(\varphi_{\omega^\alpha} \cdot \mathbf{v}^{(k)}\right) \equiv \lim_{K \to \infty} \frac{1}{(2K)^d} \int_{-K}^{K} \dots \int_{-K}^{K} \varphi^i_{\omega^\alpha}(\mathbf{x})\ v_i^{(k)}(\mathbf{x})\ d^d x$$

$$\left(\varphi_{n^l} \cdot \mathbf{v}^{(k)}\right) \equiv \lim_{K \to \infty} \frac{1}{(2K)^d} \int_{-K}^{K} \dots \int_{-K}^{K} \varphi^i_{n^l}(\mathbf{x})\ v_i^{(k)}(\mathbf{x})\ d^d x$$

represent the convolutions of the variation derivatives of annihilators with the tangent vectors φ_{ω^α}, φ_{n^l}.

It is convenient to introduce here also the families $\Lambda_{\mathbf{k}_1,\dots,\mathbf{k}_d}$ representing the functions $\varphi_{[\mathbf{k}_1,\dots,\mathbf{k}_d,\,\boldsymbol{\omega},\,\mathbf{n},\,\boldsymbol{\theta}_0]}$ with the fixed parameters $(\mathbf{k}_1,\dots,\mathbf{k}_d)$. Let us give here the following definition:

We say that a complete Hamiltonian family Λ is *equipped with a minimal set of commuting integrals* if there exist $m+s$ functionals I^γ, $\gamma = 1,\dots,m+s$, having the form

$$I^\gamma \;=\; \int P^\gamma\,(\boldsymbol{\varphi},\,\boldsymbol{\varphi}_{\mathbf{x}},\,\boldsymbol{\varphi}_{\mathbf{xx}},\dots)\;d^d x \qquad (17)$$

such that:

1) The functionals I^γ commute with the Hamiltonian functional (4) and with each other according to the bracket (1):

$$\{I^\gamma\,,\,H\} \;=\; 0 \quad , \qquad \{I^\gamma\,,\,I^\rho\} \;=\; 0 \;, \qquad (18)$$

2) The values U^{γ}:

$$U^{\gamma} \quad = \quad \lim_{K \to \infty} \, \frac{1}{(2K)^{d}} \, \int_{-K}^{K} \dots \int_{-K}^{K} \, P^{\gamma} \left(\varphi_{[\mathbf{a},\boldsymbol{\theta}_{0}]}, \, \varphi_{[\mathbf{a},\boldsymbol{\theta}_{0}]\,\mathbf{x}}, \, \dots \right) \, d^{d}x$$

of the functionals I^{γ} on Λ represent independent parameters on every family $\Lambda_{\mathbf{k}_{1},\dots,\mathbf{k}_{d}}$, such that the total set of parameters on Λ can be represented in the form $(\mathbf{k}_{1}, \dots, \mathbf{k}_{d}, \, U^{1}, \dots, U^{m+s}, \, \boldsymbol{\theta}_{0})$;

3) The Hamiltonian flows, generated by the functionals I^{γ}, leave invariant the family Λ and the values of all the parameters $(\mathbf{k}_{1}, \dots, \mathbf{k}_{d}, \, \mathbf{U})$ of the functions $\varphi_{[\mathbf{k}_{1},\dots,\mathbf{k}_{d}, \, \mathbf{U}, \, \boldsymbol{\theta}_{0}]}(\mathbf{x})$ and generate the linear time evolution of the phase shifts $\boldsymbol{\theta}_{0}$ with constant frequencies $\boldsymbol{\omega}^{\gamma} = (\omega^{1\gamma}, \dots, \omega^{m\gamma})$, such that

$$\mathrm{rk} \,\, || \, \omega^{\alpha\gamma} \, (\mathbf{k}_{1}, \dots, \mathbf{k}_{d}, \, \mathbf{U}) \, || \,\, = \,\, m$$

everywhere on Λ;

4) At every point $(\mathbf{k}_1, \ldots, \mathbf{k}_d,\, \mathbf{U},\, \theta_0)$ of Λ the linear space, generated by the variation derivatives $\delta I^{\gamma} / \delta \varphi^{i}(\mathbf{x})$, contains the variation derivatives $\mathbf{v}^{(k)}_{[\mathbf{k}_1,\ldots,\mathbf{k}_d,\mathbf{U},\theta_0]}(\mathbf{x})$ of all the annihilators of bracket (1) introduced above. In other words, at every point $(\mathbf{k}_1, \ldots, \mathbf{k}_d,\, \mathbf{U},\, \theta_0)$ we can write for a complete set $\{\mathbf{v}^{(k)}_{[\mathbf{k}_1,\ldots,\mathbf{k}_d,\mathbf{U},\theta_0]}(\mathbf{x})\}$ of linearly independent quasiperiodic solutions of (16) the relations:

$$v^{(k)}_{[\mathbf{k}_1,\ldots,\mathbf{k}_d,\mathbf{U},\theta_0]\, i}(\mathbf{x}) \;=\; \sum_{\gamma=1}^{m+s} \gamma^{k}_{\gamma}(\mathbf{k}_1, \ldots, \mathbf{k}_d,\, \mathbf{U}) \;\; \frac{\delta I^{\gamma}}{\delta \varphi^{i}(\mathbf{x})}\bigg|_{\Lambda}$$

with some functions $\gamma^{k}_{\gamma}(\mathbf{k}_1, \ldots, \mathbf{k}_d,\, \mathbf{U})$ on Λ.

Like in the one-dimensional case, we can write the following relations for the time evolution of the densities $P^{\gamma}(\varphi, \varphi_{\mathbf{x}}, \dots)$:

$$P^{\gamma}_{t}(\varphi, \varphi_{\mathbf{x}}, \dots) \;\; = \;\; Q^{\gamma 1}_{x^1}(\varphi, \varphi_{\mathbf{x}}, \dots) + \dots + Q^{\gamma d}_{x^d}(\varphi, \varphi_{\mathbf{x}}, \dots)$$

Let us consider now the modulation equations for a complete Hamiltonian family Λ equipped with a minimal set of commuting integrals $\{I^1, \dots, I^{m+s}\}$. It is convenient to choose now the parameters of the slowly modulated solutions of (2) in the form

$$(\mathbf{S}(\mathbf{X}, T), \mathbf{U}(\mathbf{X}, T)) \;\; =$$

$$= \;\; \left(S^1(\mathbf{X}, T), \dots, S^m(\mathbf{X}, T), U^1(\mathbf{X}, T), \dots, U^{m+s}(\mathbf{X}, T) \right) \;\; ,$$

such that the parameters $\mathbf{k}_q(\mathbf{X}, T)$ are defined by the relations $\mathbf{k}_q = \mathbf{S}_{X^q} \quad (\mathbf{X} = \epsilon\, \mathbf{x}, \;\; T = \epsilon\, t)$.

The regular Whitham system can be written here in the following form

$$\begin{aligned} S^{\alpha}_{T} &= \omega^{\alpha}\left(\mathbf{S}_{X^1}, \dots, \mathbf{S}_{X^d}, \mathbf{U}\right) \;, \qquad \alpha = 1, \dots, m \;, \\ U^{\gamma}_{T} &= \langle Q^{\gamma 1} \rangle_{X^1} + \dots + \langle Q^{\gamma d} \rangle_{X^d} \;, \qquad \gamma = m + s \;, \end{aligned} \tag{19}$$

which is equivalent to the system defined by (7)-(8).

The procedure of averaging of the Poisson bracket (1) represents a modification of the Dubrovin - Novikov procedure and can be formulated in the following way ([19, 20]):
Like in the one-dimensional case, let us calculate the pairwise Poisson brackets of the densities $P^{\gamma}(\mathbf{x})$, $P^{\rho}(\mathbf{y})$, which can be represented now in the form

$$\{P^{\gamma}(\mathbf{x}), P^{\rho}(\mathbf{y})\} \;=\; \sum_{l_1,\dots,l_d} A^{\gamma\rho}_{l_1\dots l_d}(\varphi, \varphi_{\mathbf{x}}, \dots)\ \delta^{(l_1)}(x^1-y^1)\ \dots\ \delta^{(l_d)}(x^d-y^d)$$

$(l_1, \dots, l_d \geq 0)$.
In the same way, we can write here the relations

$$A^{\gamma\rho}_{0\dots0}(\varphi, \varphi_{\mathbf{x}}, \dots) \;\;\equiv\;\; \partial_{x^1}\, Q^{\gamma\rho 1}(\varphi, \varphi_{\mathbf{x}}, \dots) + \dots + \partial_{x^d}\, Q^{\gamma\rho d}(\varphi, \varphi_{\mathbf{x}}, \dots)$$

for some functions $(Q^{\gamma\rho 1}(\varphi, \varphi_{\mathbf{x}}, \dots), \dots, Q^{\gamma\rho d}(\varphi, \varphi_{\mathbf{x}}, \dots))$.

We define the averaged Poisson bracket $\{\ldots,\ldots\}_{\rm AV}$ on the space of fields $(\mathbf{S}(\mathbf{X}), \mathbf{U}(\mathbf{X}))$ by the following equalities:

$$\left\{S^{\alpha}(\mathbf{X})\,,\, S^{\beta}(\mathbf{Y})\right\}_{\rm AV} \;=\; 0$$

$$\{S^{\alpha}(\mathbf{X})\,,\, U^{\gamma}(\mathbf{Y})\}_{\rm AV} \;=\; \omega^{\alpha\gamma}\left(\mathbf{S}_{X^1},\ldots,\mathbf{S}_{X^d},\mathbf{U}(\mathbf{X})\right)\,\delta(\mathbf{X}-\mathbf{Y}) \quad ,$$

$$\{U^{\gamma}(\mathbf{X})\,,\, U^{\rho}(\mathbf{Y})\}_{\rm AV} \;=\; \langle A^{\gamma\rho}_{10\ldots0}\rangle\left(\mathbf{S}_{X^1},\ldots,\mathbf{S}_{X^d},\mathbf{U}(\mathbf{X})\right)\;\delta_{X^1}(\mathbf{X}-\mathbf{Y}) \;+$$

$$+\;\ldots\;+\;\langle A^{\gamma\rho}_{0\ldots01}\rangle\left(\mathbf{S}_{X^1},\ldots,\mathbf{S}_{X^d},\mathbf{U}(\mathbf{X})\right)\;\delta_{X^d}(\mathbf{X}-\mathbf{Y}) \;+$$

$$+\;\left[\langle Q^{\gamma\rho\,p}\rangle\left(\mathbf{S}_{X^1},\ldots,\mathbf{S}_{X^d},\mathbf{U}(\mathbf{X})\right)\right]_{X^p}\;\delta(\mathbf{X}-\mathbf{Y}) \tag{20}$$

System (19) can be written now as a Hamiltonian system with the bracket (20) and the Hamiltonian functional

$$H_{av} \;=\; \int \langle P_H\rangle\,\left(\mathbf{S}_{X^1},\ldots,\mathbf{S}_{X^d},\,\mathbf{U}(\mathbf{X})\right)\; d^d X$$

[1] S.Yu. Dobrokhotov and V.P.Maslov., Finite-Gap Almost Periodic Solutions in the WKB Approximation. *J. Soviet. Math.*, 1980, V. 15, 1433-1487.

[2] S. Yu. Dobrokhotov., Resonances in asymptotic solutions of the Cauchy problem for the Schrodinger equation with rapidly oscillating finite-zone potential., *Mathematical Notes*, **44**:3 (1988), 656-668.

[3] S. Yu. Dobrokhotov., Resonance correction to the adiabatically perturbed finite-zone almost periodic solution of the Korteweg - de Vries equation., *Mathematical Notes*, **44**:4 (1988), 551-555.

[4] S.Yu. Dobrokhotov, I.M. Krichever., Multi-phase solutions of the Benjamin-Ono equation and their averaging., *Math. Notes*, **49** (6) (1991), 583-594.

[5] B.A.Dubrovin and S.P. Novikov., Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov - Whitham averaging method., *Soviet Math. Dokl.*, Vol. 27, (1983) No. 3, 665-669.

[6] B.A.Dubrovin and S.P. Novikov., On Poisson brackets of hydrodynamic type., *Soviet Math. Dokl.*, Vol. 30, (1984) 651-654.

[7] B.A. Dubrovin and S.P. Novikov., Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory., *Russian Math. Survey*, **44** : 6 (1989), 35-124.

[8] B.A.Dubrovin and S.P. Novikov., Hydrodynamics of soliton lattices, *Sov. Sci. Rev. C, Math. Phys.*, 1993, V.9. part 4. P. 1-136.

[9] E.V. Ferapontov., Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type, *Functional Analysis and Its Applications*, Vol. 25, No. 3 (1991), 195-204.

[10] E.V. Ferapontov., Dirac reduction of the Hamiltonian operator $\delta^{ij}\frac{d}{dx}$ to a submanifold of the Euclidean space with flat normal connection, *Functional Analysis and Its Applications*, Vol. 26, No. 4 (1992), 298-300.

[11] E.V. Ferapontov, A.V. Odesskii, N.M. Stoilov., Classification of integrable two-component Hamiltonian systems of hydrodynamic type in 2+1 dimensions, *J. Math. Phys.*, **52**:7, 073505 (2011), arXiv:1007.3782 .

[12] E.V. Ferapontov, P. Lorenzoni, A. Savoldi., Hamiltonian operators of Dubrovin-Novikov type in 2D, arXiv:1312.0475 .

[13] Flaschka H., Forest M.G., McLaughlin D.W., Multiphase averaging and the inverse spectral solution of the Korteweg - de Vries equation, *Comm. Pure Appl. Math.*, - 1980.- Vol. 33, no. 6, 739-784.

[14] R. Haberman., The Modulated Phase shift for Weakly Dissipated Nonlinear Oscillatory Waves of the Korteweg-deVries Type., *Studies in applied mathematics*, **78** (1) (1988), 73-90.

[15] R. Haberman., Standard Form and a Method of Averaging for Strongly Nonlinear Oscillatory Dispersive Traveling Waves., *SIAM Journal on Applied Mathematics* **51** (6) (1991), 1489-1798.

[16] I.M. Krichever., The averaging method for two-dimensional integrable equations, *Functional Analysis and Its Applications* **22**(3) (1988), 200-213.

[17] Luke J.C., A perturbation method for nonlinear dispersive wave problems, *Proc. Roy. Soc. London Ser. A*, **292**, No. 1430, 403-412 (1966).

[18] A.Ya.Maltsev., Whitham's method and Dubrovin - Novikov bracket in single-phase and multiphase cases., SIGMA **8** (2012), 103, arXiv:1203.5732 .

[19] A.Ya. Maltsev., The multi-dimensional Hamiltonian Structures in the Whitham method., Journ. of Math. Phys. **54** : 5 (2013), 053507, arXiv:1211.5756 .

[20] A.Ya. Maltsev., On the minimal set of conservation laws and the Hamiltonian structure of the Whitham equations., arXiv:1403.3935 .

[21] O.I. Mokhov and E.V. Ferapontov., Nonlocal Hamiltonian operators of hydrodynamic type associated with constant curvature metrics, *Russian Math. Surveys*, **45**:3 (1990), 218-219.

[22] O.I. Mokhov., Poisson brackets of Dubrovin - Novikov type (DN-brackets)., *Functional Analysis and Its Applications*, **22** (4) (1988), 336-338.

[23] O.I. Mokhov., The classification of nonsingular multidimensional Dubrovin-Novikov brackets., *Functional Analysis and Its Applications*, **42** (1) (2008), 33-44.

[24] S.P. Tsarev., On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.*, Vol. 31 (1985), No. 3, 488-491.

[25] S.P. Tsarev., The geometry of Hamiltonian systems of Hydrodynamic Type. The Generalized Hodograph method., *Mathematics of the USSR-Izvestiya* **37** (2) (1991), 397.

[26] G. Whitham, A general approach to linear and non-linear dispersive waves using a Lagrangian, *J. Fluid Mech.* **22** (1965), 273-283.

[27] G. Whitham, Non-linear dispersive waves, *Proc. Royal Soc. London Ser. A* **283** no. 1393 (1965), 238-261.

[28] G. Whitham, Linear and Nonlinear Waves. Wiley, New York (1974).