Formation of singularities at the interface of two fluids as a result of the Kelvin-Helmholtz instability

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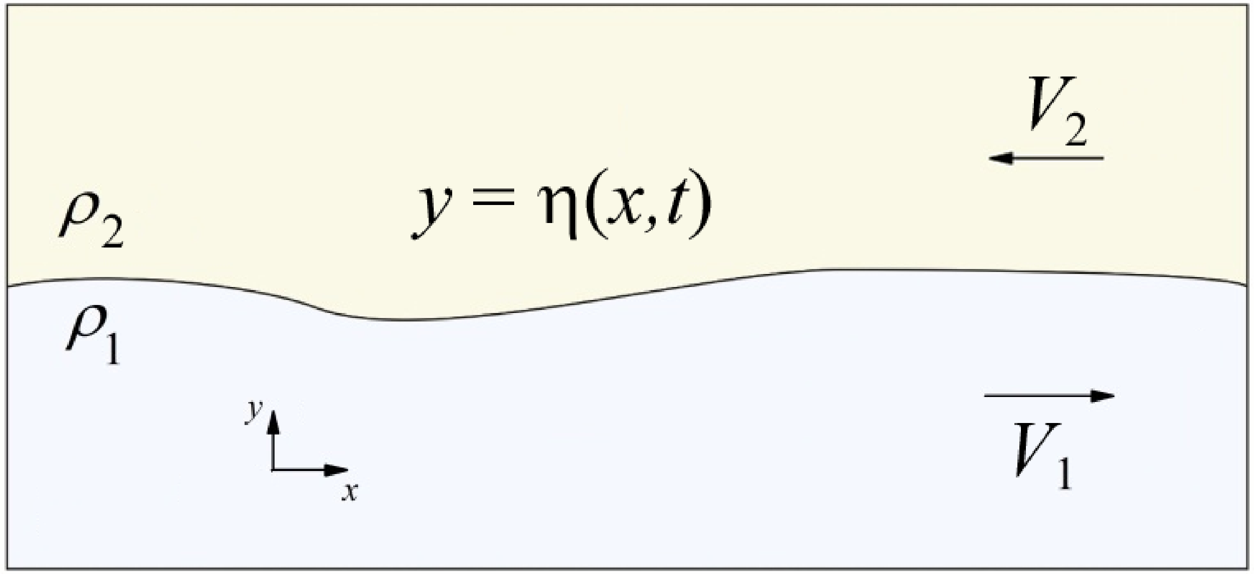
Geometry of the problem



We put $\eta = 0$ in the unperturbed state.

It is convenient to consider fluid motions in the center-of-mass frame, i.e., when the following condition is satisfied:

$$\rho_1 V_1 + \rho_2 V_2 = 0.$$

Let us assume that both liquids are inviscid and incompressible, and the flow is irrotational (potential); the functions $\Phi_{1,2}$ are the velocity potentials.

Preliminary remarks

In the work by Moore [1], for the case of one fluid, $\rho_1 = \rho_2$, it was established that the nonlinear stage of the Kelvin-Helmholtz instability is accompanied by developing singularities on the surface of the tangential discontinuity or, as saying, in the presence of a vortex sheet. In this case the vortex sheet motion can be described in the frame of the integral Birkhoff-Rott equation. Its analysis showed that weak singularities arise in a finite time. For them, the surface itself remains smooth, but its curvature becomes singular. Later on, it was established that these singularities become seeds for the vortex spirals centers.

In Refs. [2,3] an attempt was made to generalize the Moore's results to the case of fluids with different densities (i.e., $\rho_1 \neq \rho_2$). There the situation was considered where a tangential velocity discontinuity emerged as a result of the Rayleigh-Taylor instability development, namely, when the Kelvin-Helmholtz instability was secondary. In this case, a tendency to the formation of singularities of the Moore's type was also demonstrated, however, in the analysis of the motion equations there was used a number of approximations which, in our opinion, require an additional investigation. The most essential assumption was the so-called "localized approximation": the idea was to neglect the nonlinear cross-terms in the equations written in terms of analytic continuations into the upper and lower half-planes of the corresponding complex variable. This procedure allowed to reduce the equations to the local form, which does not contain nonlocal integro-differential operators. However, the neglected nonlinear interaction, generally speaking, is not small compared to the local one.

In the present work, we will study this point in more detail. On the basis of the Hamiltonian formalism, it will be shown that, for the proper choice of variables, cross-terms disappear in a natural way, and the formation of singularities can be described analytically.

[1]. D.W. Moore, Proc. R. Soc. Lond. A **365**, 105 (1979).

[2]. G. Baker, R.E. Caflisch, and M. Siegel, J. Fluid Mech. **252**, 51 (1993).

[3]. R.E. Caflisch, O.F. Orellana, and M. Siegel, SIAM J. Appl. Math. **50**, 1517 (1990).

Initial equations

$$\Delta\Phi_1 = 0, \qquad y < \eta(x,t),$$

$$\Delta\Phi_2 = 0, \qquad y > \eta(x,t),$$

$$\rho_1\left(\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla\Phi_1)^2}{2}\right) - \rho_1\left(\frac{\partial\Phi_2}{\partial t} + \frac{(\nabla\Phi_2)^2}{2}\right) = \frac{\rho_1 V_1^2 - \rho_2 V_2^2}{2}, \qquad y = \eta(x,t),$$

$$\eta_t = \partial_n\Phi_1\sqrt{1+\eta_x^2} = \partial_n\Phi_2\sqrt{1+\eta_x^2}, \qquad y = \eta(x,t),$$

$$\partial_n\Phi_1 = \partial_n\Phi_2, \qquad y = \eta(x,t),$$

$$\Phi_{1,2} \to V_{1,2}x, \qquad y \to \mp\infty.$$

Here ∂_n denotes the derivative along the boundary normal:

$$\partial_n = \frac{\partial_y - \eta_x \cdot \partial_x}{\sqrt{1+\eta_x^2}}.$$

Hamiltonian formalism

The equations of motion can be written in the Hamiltonian form [4,5]:

$$\boxed{\psi_t = -\frac{\delta H}{\delta \eta}, \qquad \eta_t = \frac{\delta H}{\delta \psi},}$$

where the Hamiltonian coincides with the total energy of the system,

$$\boxed{H = \rho_1 \iint\limits_{y\le\eta} \frac{(\nabla\Phi_1)^2 - V_1^2}{2}\,dxdy + \rho_2 \iint\limits_{y\ge\eta} \frac{(\nabla\Phi_2)^2 - V_2^2}{2}\,dxdy.}$$

Here $\psi(x,t) \equiv \rho_1\psi_1 - \rho_2\psi_2,$ where $\psi_{1,2} \equiv \tilde{\Phi}_{1,2}\big|_{y=\eta}$ and $\tilde{\Phi}_{1,2} \equiv \Phi_{1,2} - V_{1,2}x.$

The variables ψ and η generalize the canonical variables introduced by V.E. Zakharov for the surface waves [6].

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[5]. E. A. Kuznetsov and M. D. Spektor,Sov. Phys. JETP **44** (1), 136 (1976).
[6]. V. E. Zakharov, Prikl. Mekh. Tekh. Fiz. 2, 86 (1968).

Weakly nonlinear equations of motion

Consider behavior of the system in the approximation of small angles of inclination of the boundary, when $|\eta_x| << 1$. For this case it is convenient to represent Hamiltonian as an integral over the interface,

$$H = \int_S \left(\frac{\psi \, \partial_n \Phi_1}{2} - \frac{\rho_1 V_1 (\psi_1 + \psi_2) \eta_x}{2\sqrt{1+\eta_x^2}} \right) dS.$$

Expanding the integrand in the Hamiltonian in powers of the canonical variables up to the second- and third-order terms, we get

$$\boxed{H = \frac{1}{2}\int \left[\psi \hat{k} \psi - \eta \hat{k} \eta \right] dx + \frac{A}{2}\int \eta \left[(\psi_x)^2 - (\hat{k}\psi)^2 + (\eta_x)^2 - (\hat{k}\eta)^2 \right] dx - \sqrt{1-A^2} \int \eta \left[\eta_x \hat{k} \psi + \psi_x \hat{k} \eta \right] dx.}$$

Here $\hat{k} \equiv -\partial_x \hat{H}$ is the integral operator with the Fourier transform equal to $|k|$; $\hat{H}$ is the Hilbert transform, and $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ the Atwood number.

Let us switch to dimensionless variables:

$$\psi \to \psi \cdot c\lambda(\rho_1 + \rho_2), \quad \eta \to \eta \cdot \lambda, \quad t \to t \cdot \lambda / c, \quad x \to x \cdot \lambda,$$

where λ is the characteristic spatial scale and $c = V_1 \sqrt{\rho_1/\rho_2}$.

Equations of motion, corresponding to the Hamiltonian, have the form

$$\boxed{\begin{aligned}&\psi_t-\hat{k}\eta=-\sqrt{1-A^2}\left[\eta\hat{k}\psi_x-\psi_x\hat{k}\eta-\hat{k}(\eta\psi_x)\right]+(A/2)\left[(\hat{k}\psi)^2-(\psi_x)^2+(\hat{k}\eta)^2-(\eta_x)^2+2(\eta\eta_x)_x+2\hat{k}(\eta\hat{k}\eta)\right],\\&\eta_t-\hat{k}\psi=-\sqrt{1-A^2}\left[\hat{k}(\eta\eta_x)-(\eta\hat{k}\eta)_x\right]-A\left[(\eta\psi_x)_x+\hat{k}(\eta\hat{k}\psi)\right].\end{aligned}}$$

Dispersion relation: $\boxed{\omega^2=-k^2<0.}$

Let us perform a canonical transformation from the variables ψ and η to new ones

$$\boxed{f=(\psi+\eta)/2,\qquad g=(\psi-\eta)/2.}$$

The equations of motion preserves the Hamiltonian form:

$$\boxed{f_t=\frac{\delta H}{\delta g},\qquad g_t=-\frac{\delta H}{\delta f},}$$

and the Hamiltonian transforms as $H\to 2H.$ It can be written as follows:

$$\boxed{H=\int f\hat{k}g\,dx+(A/2)\int(f-g)\left[(f_x)^2-(\hat{k}f)^2+(g_x)^2-(\hat{k}g)^2\right]dx-\sqrt{1-A^2}\int(f-g)\left[f_x\hat{k}f-g_x\hat{k}g\right]dx.}$$

Reduced equations of motion

The linearized equations of motion are separated into two independent equations:

$$\boxed{f_t - \hat{k}f = 0, \qquad g_t + \hat{k}g = 0.}$$

The equation for f describes exponential growth of perturbations, while the equation for g describes their damping. Hence, at times of order of the inverse growth rate, function g can be considered small in comparison with f, and the quadratic and cubic terms with respect to g can be neglected in the Hamiltonian. Then

$$\boxed{H = \int f\,\hat{k}\,g\,dx + (A/2)\int (f-g)\left[(f_x)^2 - (\hat{k}f)^2\right]dx - \sqrt{1-A^2}\int (f-g)\left[f_x\hat{k}f\right]dx.}$$

The corresponding equations of motion have the form:

$$f_t - \hat{k}f = (A/2)\left[(\hat{k}f)^2 - (f_x)^2\right] + \sqrt{1-A^2}\left[f_x\hat{k}f\right],$$

$$g_t + \hat{k}g = (A/2)\left[(\hat{k}f)^2 - (f_x)^2 + 2(f\,f_x)_x + 2\hat{k}(f\,\hat{k}f)\right] + \sqrt{1-A^2}\left[\hat{k}(f\,f_x) - f\,\hat{k}f_x\right]$$

$$\boxed{-A\left[(gf_x)_x + \hat{k}(g\hat{k}f)\right] - \sqrt{1-A^2}\left[\hat{k}(gf_x) - (g\hat{k}f)_x\right]}.$$

Solution of equations of motion

Lets us introduce the analytic continuations of the functions f and g into the upper half-plane of the complex variable x : $F = \hat{P}f$, $G = \hat{P}g$. Here $\hat{P} = (1 - i\hat{H})/2$ is the projection operator.

The equations take the following form:

$$\boxed{\begin{aligned} F_t + iF_x &= -e^{i\gamma}F_x^2, \\ G_t - iG_x &= -e^{i\gamma}F_x^2 + 2e^{-i\gamma}\hat{P}(F\bar{F}_x)_x, \end{aligned}}$$

where $\gamma = \arccos A$.

Differentiating the equation for F with respect to x leads to the equation of the Hopf-type:

$$\boxed{V_t + iV_x = -2e^{i\gamma}VV_x, \qquad V = F_x.}$$

The solution to this equation can be found by means of the method of characteristics,

$$\boxed{V = V_0(\tilde{x}), \qquad x = \tilde{x} + it + 2e^{i\gamma}V_0(\tilde{x})t, \qquad V_0(x) = V|_{t=0}\ .}$$

The analysis of these expressions (see, also, Refs. [7,8] and [9]) shows that, in the general case, singularities appear in the solutions. The following expansion is valid in the neighborhood of the singular point $t = t_c$, $x = x_c$, $\tilde{x} = \tilde{x}_c$:

$$V(x,t) = V_0(\tilde{x}_c) + V_0'(\tilde{x}_c)\left[\frac{\delta x - \left(i + 2e^{i\gamma}V_0(\tilde{x}_c)\right)\delta t}{t_c e^{i\gamma} V_0''(\tilde{x}_c)}\right]^{1/2} + \ldots,$$

where $\delta t = t - t_c$, $\delta x = x - x_c$.

It can be seen that the derivatives V_x, V_t, or, which is the same thing, the derivatives F_{xx}, F_{xt} become singular. In particular, we have

$$V_x(x,t) \approx \frac{V_0'(\tilde{x}_c)}{2\sqrt{t_c e^{i\gamma} V_0''(\tilde{x}_c)}}\left[\delta x - \left(i + 2e^{i\gamma}V_0(\tilde{x}_c)\right)\delta t\right]^{-1/2},$$

that is, in the general case, $V_x(x_c,t) \sim |\delta t|^{-1/2}$.

[7]. E.A. Kuznetsov, M.D. Spector, V.E. Zakharov, Phys. Rev. E **49**, 1283 (1994).
[8]. E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov, Phys. Lett. A **182**, 387 (1993).
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For small angles of the surface inclination, the value of G will be of the order of $O(F^2)$ and, therefore, its role will be insignificant for weakly non-linear evolution of this system (recall that $\eta = 2\,\mathrm{Re}(F-G)$). However, this statement requires to be verified in the vicinity of the point $x = x_c$ where the second derivative of the function F becomes singular.

$$\boxed{G_t - iG_x = -e^{i\gamma}F_x^2 + 2e^{-i\gamma}\hat{P}(F\bar{F}_x)_x.}$$

One can see that if this equation does not contain the projection operator, the second term in the right-hand side would be singular due to the second derivative $\bar{F}$; however the action of $\hat{P}$ suppresses the appearance of a singularity at $\mathrm{Im}\, x > 0.$ Then in the neighborhood of the touching point, the function G has to follow the asymptotics of the function F, i.e., G should be sought in the form

$$G = G\left(\delta x - \left(i + 2e^{i\gamma}V_0(\tilde{x}_c)\right)\delta t\right).$$

In this case, we obtain $-2\left[i + e^{i\gamma}V_0(\tilde{x}_c)\right]G_x = -e^{i\gamma}F_x^2 + 2e^{-i\gamma}\hat{P}(F\bar{F}_x)_x.$ Hence the real part of G_x is small compared to the real part of F_x, i.e., the surface shape near the singularity is determined by the function F in the small-angle approximation:

$$\boxed{\eta \simeq 2\,\mathrm{Re}F.}$$

Dynamics of the interface

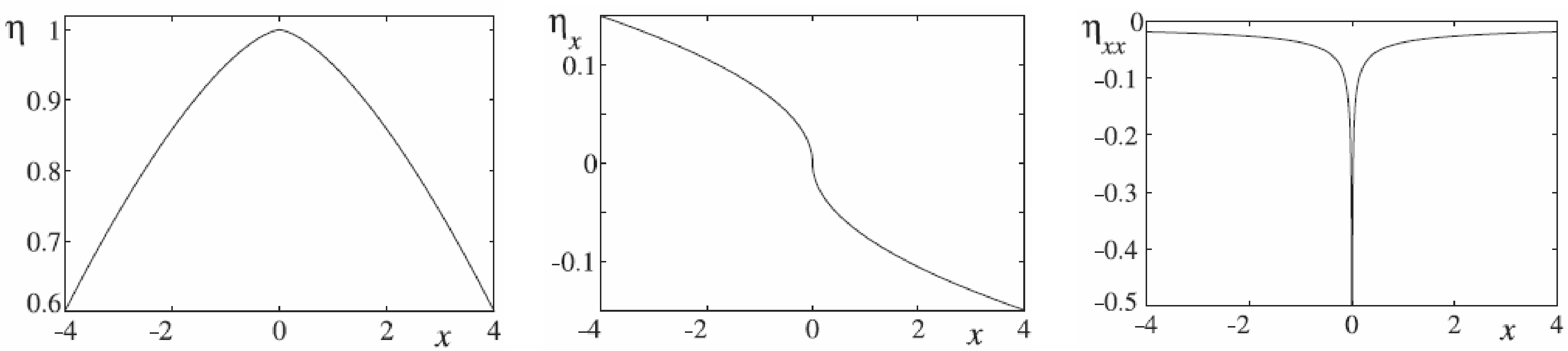
The curvature in the leading order behaves as $\eta_{xx} \approx 2\operatorname{Re} F_{xx} = 2\operatorname{Re} V_x$. We get in the neighborhood of the singular point:

$$\eta_{xx} \approx \operatorname{Re}\left\{ \frac{V_0'(\tilde{x}_c)}{\sqrt{t_c e^{i\gamma} V_0''(\tilde{x}_c)\left(\delta x - i\delta t\right)}} \right\}.$$

This expression describes the formation of weak root singularities on the interface in a finite time for which the curvature becomes infinite, while the slope angles remain small.

So, in the limit $A \to 1$ (i.e., $\rho_2/\rho_1 \to 0$), we get for the symmetric perturbation of the interface:

$$\boxed{\begin{array}{ll} \eta(x,t_c) - \eta(x_c,t_c) \sim -|x-x_c|^{3/2}, & \eta_{xx}(x,t_c) \sim -|x-x_c|^{-1/2}, \\ \eta_x(x,t_c) \sim -(x-x_c)\cdot|x-x_c|^{-1/2}, & \eta_{xx}(x_c,t) \sim -(t_c-t)^{-1/2}. \end{array}}$$



In the general case, at the moment of singularity formation, the interfacial curvature in the vicinity of the singular point is defined by

$$\boxed{\eta_{xx}(x,t_c) \approx |x-x_c|^{-1/2}\left(c_1 + c_2 \operatorname{sgn}(x-x_c)\right),}$$

where the constants $c_{1,2}$ are defined by the initial perturbation shape of the interface and also by Atwood number A. If $|c_1|<|c_2|$, then the curvature changes its sign at the singular point. Otherwise, it has a definite sign.

For fluids with comparable densities (Atwood number is small in absolute value, $|A| \ll 1$), the curvature always changes its sign at the singular point. The classic situation when the fluids are identical ($A=0$) corresponds to this case.

If the density of one fluid considerably exceeds the density of another fluid (i.e., for Atwood number close to unity, $|A| \approx 1$) the interfacial curvature has a definite sign near the singularity: it is negative for $A \approx -1$ and positive for $A \approx 1$.

It is important that the slope angles remain small at the moment of singularity formation. Let $\alpha_0 << 1$ be the characteristic initial angle. Then, at the moment t_c, we have $\alpha_c \sim |1/\ln \alpha_0|$ for periodic perturbations of the interface, and $\alpha_c \sim \alpha_0^{1/3}$ for localized perturbations.

Conclusion

The dynamics of singularity formation on the interface between two ideal fluids is studied for the Kelvin-Helmholtz instability development within the Hamiltonian formalism. It is shown [10] that the equations of motion derived in the small interface angle approximation (gravity and capillary forces are neglected) admit exact solutions in the implicit form. The analysis of these solutions shows that, in the general case, weak root singularities are formed on the interface in a finite time. For them the curvature becomes infinite, while the slope angles remain small. For Atwood numbers close to unity in absolute values, the surface curvature has a definite sign correlated with the boundary deformation directed towards the light fluid. For the fluids with comparable densities, the curvature changes its sign in a singular point. In the particular case of the fluids with equal densities, the obtained results are consistent with those obtained by Moore based on the Birkhoff-Rott equation analysis.

[10]. N.M. Zubarev and E.A. Kuznetsov, JETP **119**, 169–178 (2014).

Thank you for attention!