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Ideal hydrodynamics inside as well as outside non-rotating black hole: Hamiltonian description in the Painlevé-Gullstrand coordinates

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It is demonstrated that with using Painlevé-Gullstrand coordinates in their quasi-Cartesian variant, the Hamiltonian functional for relativistic perfect fluid hydrodynamics near a non-rotating black hole differs from the corresponding flat-spacetime Hamiltonian just by a simple term. Moreover, the internal region of the black hole is then described uniformly together with the external region, because in Painlevé-Gullstrand coordinates there is no singularity at the event horizon. An exact solution is presented which describes stationary accretion of an ultra-hard matter ($\varepsilon \propto n^2$) onto a moving black hole until reaching the central singularity. Equation of motion for a thin vortex filament on such accretion background is derived in the local induction approximation. The Hamiltonian for a fluid having ultra-relativistic equation of state $\varepsilon \propto n^{4/3}$ is calculated in explicit form, and the problem of centrally-symmetric stationary flow of such matter is solved analytically.

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1. Painlevé-Gullstrand metrics

The change of time coordinate

$$t_{\mathrm{S}} = t - 2\sqrt{2Mr} - 2M \ln \frac{1 - \sqrt{2M/r}}{1 + \sqrt{2M/r}} \tag{1}$$

in Schwarzschild metrics (describing a non-rotating black hole of mass M) reduces it to the Painlevé-Gullstrand metrics:

$$ds^2 = dt^2 - (dr + \sqrt{\frac{2M}{r}}\, dt)^2 - r^2(d\theta^2 + \sin^2\theta \; d\phi^2). \tag{2}$$

Now singularity at the event horizon $r_g = 2M$ is absent. If instead of the "spherical" spatial coordinates one introduces in a standard way the "Cartesian" coordinates $\mathbf{r} = (x, y, z)$, then the given stationary metrics is rewritten in a compact form:

$$ds^2 = dt^2 - (d\mathbf{r} - \mathbf{U} dt)^2, \qquad \mathbf{U}(\mathbf{r}) = -\frac{\mathbf{r}}{r}\sqrt{\frac{2M}{r}}, \tag{3}$$

and the scalar square here is in the simplest sense, that is the sum of squares of three components. As we shall see later, another important advantage of metrics (3) is its constant determinant $g = \det\|g_{ik}\| = -1$.

2. General structure of equations of motion for an ideal fluid

We consider a non-magnetized isentropic perfect fluid. Basic variables are the "coordinate" density $\rho(t, \mathbf{r})$ and "coordinate" velocity $\mathbf{v}(t, \mathbf{r})$. The continuity equation:

$$\rho_t + \nabla \cdot \mathbf{j} = 0, \qquad \mathbf{j} = \rho \mathbf{v}. \tag{4}$$

In the general relativity it corresponds to the condition of zero divergence for the current 4-vector $n^i = n dx^i/ds$, where scalar n is the (physical) density of number of conserved particles in the proper frame of reference, that is

$$n^i_{;i} \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} n \frac{dx^i}{ds} \right) = 0. \tag{5}$$

The relation between field n and the dynamical variables ρ and $\mathbf{j}$:

$$n = \sqrt{g_{00}\rho^2 + 2g_{0\alpha}\rho j^\alpha + g_{\alpha\beta} j^\alpha j^\beta} / \sqrt{-g}. \tag{6}$$

Besides the kinematic equation (4), there is the second, dynamic, equation of motion. It is determined by a Lagrangian functional $\mathcal{L}\{\rho, \mathbf{j}\}$, and has the following structure:

$$\frac{\partial}{\partial t}\left(\frac{\delta \mathcal{L}}{\delta \mathbf{j}}\right) = \left[\frac{\mathbf{j}}{\rho} \times \text{curl}\left(\frac{\delta \mathcal{L}}{\delta \mathbf{j}}\right)\right] + \nabla\left(\frac{\delta \mathcal{L}}{\delta \rho}\right), \tag{7}$$

The Lagrangian of relativistic hydrodynamics is determined by an equation of state $\varepsilon(n)$ which is the relation between n and the proper mass-energy density ε. A general form of the Lagrangian $\mathcal{L}\{\rho, \mathbf{j}\}$ is

$$\mathcal{L} = -\int \sqrt{-g}\varepsilon(\sqrt{g_{00}\rho^2 + 2g_{0\alpha}\rho j^\alpha + g_{\alpha\beta}j^\alpha j^\beta}/\sqrt{-g})d\mathbf{r}. \tag{8}$$

Here, instead of n, the expression (6) has been substituted into equation of state.

Eq.(7) is the expressed in terms of ρ and $\mathbf{j}$ variational Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta\mathbf{x}(\mathbf{a})}-\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}(\mathbf{a})}=0 \tag{9}$$

for a mapping $\mathbf{r}=\mathbf{x}(t,\mathbf{a})$ which describes trajectory of each element of the liquid medium, labeled by a label $\mathbf{a}=(a_1,a_2,a_3)$. Vector

$$\mathbf{p}=\delta\mathcal{L}/\delta\mathbf{j}=\delta\mathcal{L}/\delta\dot{\mathbf{x}}(\mathbf{a}) \tag{10}$$

is the canonical momentum of the liquid element. Generalized vorticity field $\mathbf{\Omega}=\mathrm{curl}\,\mathbf{p}$ is "frozen-in" into the fluid, because it obeys the equation

$$\mathbf{\Omega}_t=\mathrm{curl}[\mathbf{v}\times\mathbf{\Omega}]. \tag{11}$$

In particular, there exists a class of purely potential flows, where $\mathbf{p}=\nabla\varphi$.

It makes sense to consider also different equivalent systems of equations, taking as the basic dynamical variables not $(\rho, \mathbf{j})$, but $(\rho, \mathbf{p})$ or $(n, \mathbf{p})$.

From the methodical point of view, the most preferable is the pair $(\rho, \mathbf{p})$, because in that case the system of equations (4) and (7) acquires the non-canonical Hamiltonian structure:

$$\rho_t = -\nabla \cdot \left(\frac{\delta \mathcal{H}}{\delta \mathbf{p}}\right), \tag{12}$$

$$\mathbf{p}_t = \left[\frac{1}{\rho}\left(\frac{\delta \mathcal{H}}{\delta \mathbf{p}}\right) \times \operatorname{curl} \mathbf{p}\right] - \nabla \frac{\delta \mathcal{H}}{\delta \rho}. \tag{13}$$

The Hamiltonian $\mathcal{H}\{\rho, \mathbf{p}\}$ is the Legendre transform of the Lagrangian on the vector variable $\mathbf{j}$:

$$\mathcal{H}\{\rho, \mathbf{p}\} = \int (\mathbf{j} \cdot \mathbf{p}) d\mathbf{r} - \mathcal{L}, \tag{14}$$

and instead of $\mathbf{j}$ here the solution of equation $\mathbf{p} = \delta \mathcal{L}\{\rho, \mathbf{j}\}/\delta \mathbf{j}$ should be substituted.

Potential flows are described by the single pair of unknown functions ρ and φ, which are canonically conjugate in this case.

3. Equations of hydrodynamics in metrics (3)

Metrics (3) allows us to write the Lagrangian in remarkably compact form

$$\mathcal{L} = -\int \varepsilon(\sqrt{\rho^2 - (\mathbf{j} - \mathbf{U}\rho)^2})d\mathbf{r}. \tag{15}$$

Only the presence of central-symmetric vector field $\mathbf{U}(\mathbf{r})$ distinguishes this Lagrangian from the Lagrangian in the flat spacetime. The corresponding variational derivatives are

$$\delta\mathcal{L}/\delta\mathbf{j} \;=\; \mathbf{J}f(n), \tag{16}$$

$$\delta\mathcal{L}/\delta\rho \;=\; -(\rho + \mathbf{U}\cdot\mathbf{J})f(n), \tag{17}$$

where it has been denoted for brevity

$$\mathbf{J} = \mathbf{j} - \mathbf{U}\rho, \quad n = \sqrt{\rho^2 - \mathbf{J}^2}, \quad f(n) = \varepsilon'(n)/n. \tag{18}$$

From the above, an elegant description can be derived in terms of fields n and $\mathbf{p}$, although the continuity equation looks more complicated:

$$\frac{\partial}{\partial t}(\frac{n}{w}\sqrt{w^2+\mathbf{p}^2})+\nabla\cdot(\frac{n}{w}(\mathbf{p}+\mathbf{U}\sqrt{w^2+\mathbf{p}^2}))=0, \tag{19}$$

where $w=\varepsilon'(n)$ is the relativistic enthalpy per particle. The dynamic equation has the following form:

$$\mathbf{p}_t=[(\mathbf{U}+\frac{\mathbf{p}}{\sqrt{w^2+\mathbf{p}^2}})\times \mathrm{curl}\,\mathbf{p}]-\nabla(\sqrt{w^2+\mathbf{p}^2}+\mathbf{U}\cdot\mathbf{p}). \tag{20}$$

Within system (19)-(20) it is easy to consider potential flows: it is sufficient to put $\mathbf{p}=\nabla\varphi$ and remove the gradient operator in Eq.(20). By acting in this way, we obtain

$$w=\sqrt{(\varphi_t+\mathbf{U}\cdot\nabla\varphi)^2-(\nabla\varphi)^2}. \tag{21}$$

The continuity equation now transforms from being kinematic to dynamic:

$$-\frac{\partial}{\partial t}[(\varphi_t+\mathbf{U}\cdot\nabla\varphi)P'(w)/w]+\nabla\cdot\{[\nabla\varphi-\mathbf{U}(\varphi_t+\mathbf{U}\cdot\nabla\varphi)]P'(w)/w\}=0, \tag{22}$$

where $P(w)$ is the pressure as a function of enthalpy w at fixed entropy. The action functional for this equation is

$$I\{\varphi\}=\int P(\sqrt{(\varphi_t+\mathbf{U}\cdot\nabla\varphi)^2-(\nabla\varphi)^2})d\mathbf{r}dt. \tag{23}$$

4. Properties of the Hamiltonian

Let us return to the problem of Hamiltonian description of flows in variables $(\rho, \mathbf{p})$. If we succeed in analytical solution with respect to J of the scalar equation

$$|\mathbf{p}| = Jf(\sqrt{\rho^2 - J^2}), \tag{24}$$

then it is possible to calculate the Hamiltonian explicitly. In our case we have

$$\mathcal{H}\{\rho, \mathbf{p}\} = \int \rho(\mathbf{U}\cdot\mathbf{p})d\mathbf{r} + \int [\frac{J^2}{\sqrt{\rho^2 - J^2}}\varepsilon'(\sqrt{\rho^2 - J^2}) + \varepsilon(\sqrt{\rho^2 - J^2})]d\mathbf{r}, \tag{25}$$

and in the last integral instead of J the solution of Eq.(24) should be substituted. The last integral is the Legendre transform of the Lagrangian for hydrodynamics in the Minkowskii spacetime. Thus, we have an interesting result: with using metrics (3), the Hamiltonian of a fluid in the presence of a non-rotating black hole is produced by adding the term $\int \rho(\mathbf{U}\cdot\mathbf{p})d\mathbf{r}$ to the Hamiltonian of the same fluid in flat spacetime:

$$\mathcal{H}\{\rho, \mathbf{p}\} = \int [\rho(\mathbf{U}\cdot\mathbf{p}) + H(\rho, |\mathbf{p}|)]d\mathbf{r}. \tag{26}$$

The structural simplicity of expression (26) is due to the good choice of coordinate system. The equations of motion take the form

$$\rho_t \;=\; -\nabla \cdot (\rho \mathbf{U} + H_{\mathbf{p}}), \tag{27}$$

$$\mathbf{p}_t \;=\; [(\mathbf{U} + H_{\mathbf{p}}/\rho) \times \operatorname{curl} \mathbf{p}] - \nabla (H_\rho + \mathbf{U} \cdot \mathbf{p}), \tag{28}$$

It is clear that function $H(\rho, |\mathbf{p}|)$ cannot be arbitrary, because its origin from a member of the specific family of Lagrangians depending on the combination $\sqrt{\rho^2 - J^2}$. It is not difficult to show that each Hamiltonian of a fluid in Minkowskii spacetime should satisfy a simple first-order partial differential equation:

$$H_\rho H_{|\mathbf{p}|} = \rho |\mathbf{p}|. \tag{29}$$

5. First example: the simplest model — stiff matter

In a number of works, an ultra-hard equation of state was considered $\varepsilon = n^2/2$ (stiff matter). In this case $w = n$, and the Hamiltonian $H = (\rho^2 + \mathbf{p}^2)/2$. Eq.(22) for potential flows becomes strictly linear and it coincides with the equation for a massless scalar field. In particular, formation of shock waves is not possible within this model. It is not difficult to find solutions of Eq.(22) describing stationary accretion in the presence of a uniform matter flow at the infinity:

$$\varphi_{\mathrm{a}} = \rho_\infty[-t + 2\sqrt{2Mr} - 4M\ln(1 + \sqrt{2M/r})] + (1 - M/r)(\mathbf{r}\cdot\mathbf{p}_\infty), \qquad (30)$$

where $\rho_\infty = \sqrt{w_\infty^2 + (\mathbf{p}_\infty)^2}$. This expression has no singularity at the gravitational radius $r_g = 2M$, contrary to the same solution but calculated in the Schwarzschild coordinates. Singularity is canceled namely due to the change of time coordinate (1).

As to vortex flows of stiff matter, they are described by a system which is not linear:

$$\rho_t = -\nabla\cdot(\rho\mathbf{U} + \mathbf{p}), \qquad (31)$$

$$\mathbf{p}_t = [(\mathbf{U} + \mathbf{p}/\rho)\times \mathrm{curl}\,\mathbf{p}] - \nabla(\rho + \mathbf{U}\cdot\mathbf{p}). \qquad (32)$$

If we consider the dynamics of relatively weak vortex disturbances at the background flow determined by potential (30), then equation for the vorticity, neglecting density perturbations, can be written as follows:

$$\mathbf{\Omega}_t = \mathrm{curl}\,[(\mathbf{U} + \frac{\nabla\varphi_{\mathrm{a}} + \mathrm{curl}^{-1}\mathbf{\Omega}}{(\rho_\infty - \mathbf{U}\cdot\nabla\varphi_{\mathrm{a}})}) \times \mathbf{\Omega}] \equiv \mathrm{curl}\,[(\mathbf{S} + \mathrm{curl}^{-1}\mathbf{\Omega}) \times \frac{\mathbf{\Omega}}{\bar{\rho}}], \tag{33}$$

where (for simplicity, we have normalized length scale to the gravitational radius r_g)

$$\nabla\varphi_{\mathrm{a}} = \rho_\infty \frac{\mathbf{r}}{\sqrt{r^3}}(1 + \frac{1}{r+\sqrt{r}}) + \mathbf{p}_\infty(1 - \frac{1}{2r}) + \frac{\mathbf{r}}{2r^3}(\mathbf{r}\cdot\mathbf{p}_\infty), \tag{34}$$

$$\bar{\rho}(\mathbf{r}) = \rho_\infty - \mathbf{U}\cdot\nabla\varphi_{\mathrm{a}} = \rho_\infty[1 + \frac{1}{r}(1 + \frac{1}{r+\sqrt{r}})] + \frac{(\mathbf{r}\cdot\mathbf{p}_\infty)}{\sqrt{r^3}}, \tag{35}$$

$$\mathbf{S}(\mathbf{r}) = \bar{\rho}\mathbf{U} + \nabla\varphi_{\mathrm{a}} = -\rho_\infty \frac{\mathbf{r}}{r^3} + \mathbf{p}_\infty(1 - \frac{1}{2r}) - \frac{\mathbf{r}}{2r^3}(\mathbf{r}\cdot\mathbf{p}_\infty). \tag{36}$$

The stream $\mathbf{S}(\mathbf{r})$ is solenoidal. There exists its vector potential, which can be represented as $\mathbf{\Psi} = \mathbf{e}_\phi F(m,z)$, where $m = (x^2+y^2)/2$, and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction, with z axis oriented along $\mathbf{p}_\infty$. An explicit expression is given below:

$$\mathbf{\Psi} = \frac{\rho_\infty \mathbf{e}_\phi z}{\sqrt{2m}\sqrt{2m+z^2}} + |\mathbf{p}_\infty|\mathbf{e}_\phi(\frac{1}{2} - \frac{1}{2\sqrt{2m+z^2}})\sqrt{2m}. \tag{37}$$

Within Eq.(33) one can investigate the motion of frozen-in vortex structures. Consider for example a thin closed vortex filament with circulation Γ. Then it follows from Eq.(33) that in so called local induction approximation (LIA) the dynamics of the filament shape $\mathbf{R}(\xi, t)$ obeys the equation

$$[\mathbf{R}_\xi \times \mathbf{R}_t]\bar{\rho}(\mathbf{R}) = [\mathbf{R}_\xi \times \mathbf{S}(\mathbf{R})] - \Lambda\partial_\xi(\mathbf{R}_\xi/|\mathbf{R}_\xi|). \tag{38}$$

Here ξ is an arbitrary longitudinal parameter, $\Lambda = (\Gamma/4\pi)\ln(L/d)$ is the local induction constant, L is a typical size of vortex filament, d is its small width. The above equation follows from a variational principle with the Lagrangian of the form

$$\mathcal{L}_\Gamma = \Gamma \oint ([\mathbf{R}_\xi \times \mathbf{R}_t] \cdot \mathbf{D}(\mathbf{R}))d\xi - \mathcal{H}_\Gamma\{\mathbf{R}\}, \tag{39}$$

where the vector function $\mathbf{D}(\mathbf{r})$ satisfies the condition

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \bar{\rho}(\mathbf{r}). \tag{40}$$

The local induction Hamiltonian $\mathcal{H}_\Gamma\{\mathbf{R}\}$ in our case looks as follows:

$$\mathcal{H}_\Gamma\{\mathbf{R}\}/\Gamma = \oint \mathbf{\Psi}(\mathbf{R}) \cdot \mathbf{R}_\xi \, d\xi + \Lambda \oint |\mathbf{R}_\xi| d\xi. \tag{41}$$

In the simplest configuration the filament is a coaxial vortex ring with a radius $\sqrt{2m(t)}$ and a center position $z(t)$. Phase trajectories of the vortex in (z,m) plane are determined by level contours of its Hamiltonian

$$H_\Lambda = \frac{\rho_\infty z}{\sqrt{2m+z^2}} + |\mathbf{p}_\infty|(m - \frac{m}{\sqrt{2m+z^2}}) + \Lambda\sqrt{2m}, \tag{42}$$

which is easily calculated with the help of formulas (37) and (41). Equations of motion are almost canonical:

$$\dot{z}\bar{\rho}(m,z) = \partial H_\Lambda/\partial m, \quad -\dot{m}\bar{\rho}(m,z) = \partial H_\Lambda/\partial z. \tag{43}$$

Note that depending on the vorticity direction, parameter Λ can be positive or negative. The requirement of weakness of the vortex is expressed by condition $|\Lambda|/\rho_\infty \lesssim 1$.

6. Second example: ultra-relativistic matter

The Hamiltonian can be calculated in explicit form also for ultra-relativistic matter $\varepsilon = (3/4)n^{4/3}$. We then have the cubic equation

$$|\mathbf{p}|^3(\rho^2 - J^2) = J^3. \tag{44}$$

Solving this equation, we obtain that $\mathbf{J} = H_{\mathbf{p}}$, where

$$H_{\mathbf{p}} = \frac{\mathbf{p}\rho^{2/3}}{(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}} + (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}}}, \tag{45}$$

and in the case of imaginary values of the square root, solutions having minimal argument should be chosen when complex cubic roots are computed.

For calculation of the partial derivative H_ρ, one can use Eq.(29), which gives, together with Eq.(45), that

$$H_\rho = \rho^{\frac{1}{3}}\Big[\Big(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}}\Big)^{\frac{1}{3}} + \Big(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}}\Big)^{\frac{1}{3}}\Big]. \tag{46}$$

The function $H(\rho, |\mathbf{p}|)$ is given by the following expression:

$$H = \frac{|\mathbf{p}|^2 \rho^{2/3}}{(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}} + (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}}} + \frac{(3/4)\rho^{4/3}}{[(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}} + (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{|\mathbf{p}|^6}{27\rho^2}})^{\frac{1}{3}}]^2}. \tag{47}$$

Central-symmetric stationary flows in Painlevé-Gullstrand coordinates are described by the system of two algebraic equations, which follow from Eqs.(19)-(20) (here we have normalized the length scale to r_g, and the enthalpy to its value at the infinity):

$$\frac{n(w)}{w}(p - \sqrt{\frac{(w^2 + p^2)}{r}}) = -\frac{A}{r^2}, \tag{48}$$

$$\sqrt{w^2 + p^2} - p/\sqrt{r} = 1, \tag{49}$$

where p is the radial component of the canonical momentum, A is a positive constant, which is not arbitrary; it should result in a physically acceptable solution (the curve should pass through the so called critical point). For an ultra-relativistic fluid, when $n(w) = w^3$, the system reduces to the cubic equation for p:

$$r^2[(1 + p/\sqrt{r})^2 - p^2][p(1 - 1/r) - 1/\sqrt{r}] = -A. \tag{50}$$

The critical (saddle) point is determined by the conditions of zero values for the first-order partial derivatives of the left hand side of Eq.(50) on variables r and p. Its numerical parameters are the following: $A = \sqrt{27/4} \approx 2.5981$, $p_{\rm cr} = \sqrt{6} - \sqrt{3} \approx 0.71744$, $r_{\rm cr} = 3/2$. An analytical formula for the solution $p(r)$ has the following form:

$$\begin{aligned} p(r) &= \frac{\sqrt{r}}{r-1} \\ &+ \frac{r}{\sqrt{3}(1-r)}\Big[\Big(\frac{27(1-r)}{4r^3} + \sqrt{\frac{27^2(1-r)^2}{16r^6}} - 1\Big)^{\frac{1}{3}} \\ &\qquad + \Big(\frac{27(1-r)}{4r^3} - \sqrt{\frac{27^2(1-r)^2}{16r^6}} - 1\Big)^{\frac{1}{3}}\Big], \end{aligned} \tag{51}$$

and in the case of imaginary square roots that branch should be chosen from the three, which guarantees a smooth dependence $p(r)$ at the entire interval $r > 0$. With the indicated value A, the level contours of the left hand side of Eq.(50) in (r, p) plane consist of three curves, two of them intersecting at the critical point. One from the intersecting curves continues in a smooth manner under the event horizon $r = 1$ [it is the physical solution, and $p(1) = (\sqrt{27/4} - 1)/2 \approx 0.8$], while the two other curves go to infinity as $r \to 1$. What is interesting, as $r \to 0$, the physical solution does not diverge but it tends to a finite value $p(0) = (27/4)^{1/6} \approx 1.38$.

The stationary density profile $\bar{\rho}(r)$ now is expressed by the formula

$$\bar{\rho}(r) = [(1 + p(r)/\sqrt{r})^2 - p^2(r)](1 + p(r)/\sqrt{r}), \tag{52}$$

where the dependence (51) has to be substituted.

Relatively weak vortex disturbances, practically not affecting the density profile $\bar{\rho}$, are described by the following system of equations:

$$\mathbf{\Omega}_t = \mathrm{curl}\,[(- A\frac{\mathbf{r}}{r^3} + H_{\mathbf{pp}}(\mathbf{r})\tilde{\mathbf{p}}) \times \frac{\mathbf{\Omega}}{\bar{\rho}(\mathbf{r})}], \tag{53}$$

$$\nabla \cdot (H_{\mathbf{pp}}\tilde{\mathbf{p}}) = 0, \tag{54}$$

$$\mathrm{curl}\,\tilde{\mathbf{p}} = \mathbf{\Omega}, \tag{55}$$

where $H_{\mathbf{pp}}(\mathbf{r})$ is the matrix of second derivatives of the Hamiltonian (47) on the components of vector $\mathbf{p}$, evaluated at the stationary solution. A more detailed study of system (53)-(55), including derivation of an equation of motion for a thin vortex filament, is planned on the future.