KPII: Cauchy–Jost function, Darboux transformations and totally nonnegative matrices

M. Boiti, F. Pempinelli and A. Pogrebkov

VII-th International Conference
SOLITONS, COLLAPSES AND TURBULENCE:
Achievements, Developments and Perspectives (SCT-14)

CHERNOGOLOVKA, August 4–8, 2014

Zakharov's 75 anniversary

Multisoliton solutions of the KPII equation: $(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}$ are known to be parametrized by means of totally nonnegative matrices

Notation. Lax operator

$$\mathcal{L}(x,\partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x)$$

Jost solutions of $\mathcal{L}$ and its dual $\mathcal{L}'$:

$$\mathcal{L}\,\varphi(x,\lambda) = 0, \qquad \mathcal{L}'\,\psi(x,\lambda) = 0, \qquad \lambda \in \mathbb{C}$$

are normalized by condition

$$\lim_{\lambda\to\infty} e^{-\ell(\lambda)x}\,\varphi(x,\lambda) = \lim_{\lambda\to\infty} e^{\ell(\lambda)x}\psi(x,\lambda) = 1$$

where

$$\ell(\lambda)x = \lambda\,x_1 + \lambda^2\,x_2 + \lambda^3\,x_3 + \dots$$

Cauchy–Jost function (Cauchy–Baker–Akhiezer kernel, see Grinevich–Orlov, 1997) is defined as

$$F(x,\lambda,\lambda') = \int_{(\lambda-\lambda')_{\mathrm{Re}}\infty}^{x_1} dy_1 \psi(y,\lambda)\,\varphi(y,\lambda'), \qquad y_i = x_i, \quad i \geq 2$$

where factor $(\lambda - \lambda')_{\mathrm{Re}}$ in the limit of the integral denotes sign of infinity. Just by definition we have that F has pole with unity residual at $\lambda' = \lambda$

$$F(x,\lambda,\lambda') = \frac{1}{\lambda' - \lambda} + O(1), \quad \lambda' \sim \lambda$$

and obey asymptotic relations

$$\lim_{\lambda'\to\infty} e^{-\ell(\lambda')x}\,\lambda'\,F(x,\lambda,\lambda') = \psi(x,\lambda), \qquad \lim_{\lambda\to\infty} e^{\ell(\lambda)x}\lambda F(x,\lambda,\lambda') = -\,\varphi(x,\lambda')$$

In the case where Hirota bilinear identity is valid we have that

$$\oint_C \frac{d\,\lambda''}{2\pi i} F(x,\lambda,\lambda'')F(y,\lambda'',\lambda') = F(y,\lambda,\lambda') - F(x,\lambda,\lambda')$$

for λ' and λ inside the contour C.

Cauchy–Jost function (Cauchy–Baker–Akhiezer kernel, see Grinevich–Orlov, 1997) is defined as

$$F(x,\lambda,\lambda') = \int_{(\lambda-\lambda')_{\mathrm{Re}}\infty}^{x_1} dy_1 \psi(y,\lambda)\,\varphi(y,\lambda'), \qquad y_i = x_i, \quad i \geq 2$$

where factor $(\lambda-\lambda')_{\mathrm{Re}}$ in the limit of the integral denotes sign of infinity. Just by definition we have that F has pole with unity residual at $\lambda' = \lambda$

$$F(x,\lambda,\lambda') = \frac{1}{\lambda'-\lambda} + O(1), \quad \lambda' \sim \lambda$$

and obey asymptotic relations

$$\lim_{\lambda'\to\infty} e^{-\ell(\lambda')x}\,\lambda'\,F(x,\lambda,\lambda') = \psi(x,\lambda), \qquad \lim_{\lambda\to\infty} e^{\ell(\lambda)x}\lambda F(x,\lambda,\lambda') = -\,\varphi(x,\lambda')$$

In the case where Hirota bilinear identity is valid we have that

$$\oint_C \frac{d\,\lambda''}{2\pi i} F(x,\lambda,\lambda'')F(y,\lambda'',\lambda') = F(y,\lambda,\lambda') - F(x,\lambda,\lambda')$$

for λ' and λ inside the contour C. In solitonic case we prove also that

$$\partial_{x_k} F(x,\lambda,\lambda') = -\oint_C \frac{d\,\lambda''}{2\pi i} F(x,\lambda,\lambda'')\lambda''^k F(x,\lambda'',\lambda'), \quad k = 0, 1, \ldots$$

(N_a, N_b)**-soliton solutions** are given in terms of the tau-function:

$$u(x) = -2\partial_{x_1}^2 \log \tau(x) \equiv -2\partial_{x_1}^2 \log \tau'(x)$$

i.e., in terms of determinants

$$\tau(x) = \det\bigl(\mathcal{V}\, e^{\ell x}\, \mathcal{D}\bigr), \quad \tau'(x) = \det\left(\mathcal{D}'\, e^{-\ell x} \gamma\, \mathcal{V}'\right), \quad \tau(x) = \mathrm{const}\left(\prod_{n=1}^{\mathcal{N}} e^{\ell_n x}\right) \tau'(x)$$

By means of the Binet–Cauchy formula:

$$\tau(x) = \sum_{1 \le n_1, \ldots < n_{N_b} \le \mathcal{N}} \mathcal{D}(\{n_i\})\, \mathcal{V}(\{n_i\}) \prod_{l=1}^{N_b} e^{\ell_{n_l} x}$$

where $\mathcal{D}(\{n_i\})$, $\mathcal{V}(\{n_i\})$ are maximal minors of matrices $\mathcal{D}$ and $\mathcal{V}$.

Here $N_a, N_b \geq 1$, $\mathcal{N} = N_a + N_b$, $\mathcal{N} \geq 2$. $\mathcal{N}$ real parameters: $\kappa_1 < \kappa_2 < \ldots < \kappa_{\mathcal{N}}$, Let $\ell(\lambda)x = \lambda\, x_1 + \lambda^2\, x_2 + \ldots$, $\ell_n x = \kappa_n x_1 + \kappa_n^2 x_2 + \ldots$ and

$$\mathcal{N} \times \mathcal{N}\text{-matrix}: \qquad e^{\ell x} = \operatorname{diag}\{e^{\ell_n x}\}_{n=1}^{\mathcal{N}}$$

$$\text{real constant } \mathcal{N} \times N_b\text{-matrix}\, \mathcal{D}, \qquad N_a \times \mathcal{N}\text{-matrix}\, \mathcal{D}', \qquad \mathcal{D}'\,\mathcal{D} = 0$$

Two incomplete Vandermonde matrices:

$$\mathcal{V} = \begin{pmatrix} 1 & \ldots & 1 \\ \kappa_1 & \ldots & \kappa_{\mathcal{N}} \\ \vdots & & \vdots \\ \kappa_1^{N_b-1} & \ldots & \kappa_{\mathcal{N}}^{N_b-1} \end{pmatrix}, \qquad \mathcal{V}' = \begin{pmatrix} 1 & \ldots & \kappa_1^{N_a-1} \\ \vdots & & \vdots \\ 1 & \ldots & \kappa_{\mathcal{N}}^{N_a-1} \end{pmatrix},$$

$$\mathcal{V}\, A^{-1}\, \mathcal{V}' = 0, \qquad A = \operatorname{diag}\{a_n\}_{n=1}^{\mathcal{N}}, \qquad a_n = \prod_{\substack{n'=1 \\ n' \neq n}}^{\mathcal{N}} (\kappa_n - \kappa_{n'}), \qquad a_n = A(\kappa_n)$$

$$A(\lambda) = \prod_{n=1}^{\mathcal{N}} (\lambda - \kappa_n),$$

Jost solutions are renormalized here in order to get

$$e^{-\ell(\lambda)x}\,\varphi(x,\lambda) \text{ polynomial in } \lambda\text{: } e^{-\ell(\lambda)x}\,\varphi(x,\lambda) = \lambda^{N_b} + \dots$$
$$e^{\ell(\lambda)x}\psi(x,\lambda) \text{ polynomial in } \lambda\text{: } e^{\ell(\lambda)x}\psi(x,\lambda) = \lambda^{N_a} + \dots$$

Discrete values of the Jost solutions:

$$\varphi_n(x) = \varphi(x,\kappa_n), \qquad \psi_n(x) = \psi(x,\kappa_n), \quad n = 1,\dots,\mathcal{N}\,.$$

obey

$$\bigl(\varphi_1(x),\dots,\varphi_{\mathcal{N}}(x)\bigr)\,\mathcal{D} = 0, \qquad \mathcal{D}'\bigl(\psi_1(x),\dots,\psi_{\mathcal{N}}(x)\bigr)^{\mathrm{T}} = 0$$

Conditions of analyticity imposed on the Jost solutions are equivalent to relations

$$\bigl(\varphi_1(x),\dots,\varphi_{\mathcal{N}}(x)\bigr)e^{-\ell x}A^{-1}\,\mathcal{V}' = \bigl(\underbrace{0,\dots,0,1}_{N_a}\bigr),$$
$$\mathcal{V}A^{-1}e^{\ell x}\bigl(\psi_1(x),\dots,\psi_{\mathcal{N}}(x)\bigr)^{\mathrm{T}} = \bigl(\underbrace{0,\dots,0,1}_{N_b}\bigr)^{\mathrm{T}}.$$

that are defining the Jost solutions themselves.

Cauchy–Jost function.

$$F(x,\lambda,\lambda') = \int_{(\lambda-\lambda')_{\mathrm{Re}}\infty}^{x_1} dy_1 \psi(y,\lambda)\,\varphi(y,\lambda'), \qquad y_2 = x_2, \ldots$$

This function obeys

$$F_{x_1}(x,\lambda,\lambda') = \psi(x,\lambda)\,\varphi(x,\lambda'),$$
$$F_{x_2}(x,\lambda,\lambda') = \psi(x,\lambda)\,\varphi_{x_1}(x,\lambda') - \psi_{x_1}(x,\lambda)\,\varphi(x,\lambda')$$

$$F(x,\lambda,\lambda') = \frac{A(\lambda)}{\lambda'-\lambda} + O(1), \quad \lambda' \sim \lambda,$$

$$e^{(\ell(\lambda)-\ell(\lambda'))x} F(x,\lambda,\lambda') \equiv \int_{(\lambda-\lambda')_{\mathrm{Re}}\infty}^{x_1} dy_1 e^{(\lambda-\lambda')(x_1-y_1)} \left(e^{\ell(\lambda)y} \psi(y,\lambda) e^{-\ell(\lambda')y}\,\varphi(y,\lambda')\right)$$

$$e^{(\ell(\lambda)-\ell(\lambda'))x} F(x,\lambda,\lambda') = \frac{A(\lambda')}{\lambda'-\lambda} + A(\lambda)A(\lambda') \sum_{m,n=1}^{\mathcal{N}} \frac{f_{mn}(x)}{(\lambda-\kappa_m)(\lambda'-\kappa_n)}$$

Asymptotics for large λ and λ':

$$e^{(\ell(\lambda)-\ell(\lambda'))x}F(x,\lambda,\lambda') = \frac{\lambda^{N_a-1}\,{\lambda'}^{N_b-1}}{\lambda'-\lambda}\Big(\lambda - \frac{1}{2}\sum_{m=1}^{\mathcal{N}}\kappa_m\Big)\Big(\lambda' - \frac{1}{2}\sum_{m=1}^{\mathcal{N}}\kappa_m\Big) - \\ - \frac{1}{2}\int u(x)dx_1 + \ldots$$

Values at points κ's:'

$$e^{(\ell_m-\ell(\lambda'))x}F(x,\kappa_m,\lambda') = \frac{A(\lambda')}{\lambda'-\kappa_m} + a_m A(\lambda')\sum_{n=1}^{\mathcal{N}}\frac{f_{mn}(x)}{\lambda'-\kappa_n}$$

$$e^{(\ell(\lambda)-\ell_n)x}F(x,\lambda,\kappa_n) = A(\lambda)a_n\sum_{m=1}^{\mathcal{N}}\frac{f_{mn}(x)}{\lambda'-\kappa_n}$$

are polynomials with the leading behavior:

$$e^{(\ell_m-\ell(\lambda'))x}F(x,\kappa_m,\lambda') = {\lambda'}^{N_b-1}\psi(x,\kappa_m) + \ldots,$$

$$e^{(\ell(\lambda)-\ell_n)x}F(x,\lambda,\kappa_n) = -\lambda^{N_a-1}\psi(x,\kappa_n) + \ldots$$

Finally,

$$\lim_{\lambda'\to\kappa_n} e^{(\ell_m-\ell(\lambda'))x}F(x,\kappa_m,\lambda') = a_m\delta_{mn} + a_m f_{mn}(x)a_n$$
$$\lim_{\lambda\to\kappa_m} e^{(\ell(\lambda)-\ell_n)x}F(x,\lambda,\kappa_n) = a_m f_{mn}(x)a_n$$

Matrix $f(x)$ obeys properties

$$f(x)e^{\ell x}A\,\mathcal{D} = 0, \qquad f(x)\,\mathcal{V}' = -A^{-1}e^{-\ell x}\,\mathcal{V}'$$

that are defining this matrix:

$$f(x) = -A^{-1}\,\mathcal{V}'(\mathcal{D}'\,e^{-\ell x}A^{-1}\,\mathcal{V}')^{-1}\,\mathcal{D}'\,e^{-\ell x}$$
$$f(x)A = -I + e^{\ell x}\,\mathcal{D}(\mathcal{V}\,e^{\ell x}\,\mathcal{D})^{-1}\,\mathcal{V}$$

This proves that $-f(x)A$ is the orthogonal projector: $f(x)Af(x) = -f(x)$. All standard objects of the the theory can be given in terms of the matrix $f(x)$:

$$\bigl(\varphi_1(x),\dots,\varphi_{\mathcal{N}}(x)\bigr) = -\bigl(\kappa_1^{N_b},\dots,\kappa_{\mathcal{N}}^{N_b}\bigr)f(x)e^{\ell x}A,$$
$$\bigl(\psi_1(x),\dots,\psi_{\mathcal{N}}(x)\bigr)^{\mathrm{T}} = (e^{-\ell x} + e^{-\ell x}Af(x))\bigl(\kappa_1^{N_a},\dots,\kappa_{\mathcal{N}}^{N_a}\bigr)^{\mathrm{T}},$$
$$u(x) = -2\bigl(\kappa_1^{N_b},\dots,\kappa_{\mathcal{N}}^{N_b}\bigr)f_{x_1}(x)\bigl(\kappa_1^{N_a},\dots,\kappa_{\mathcal{N}}^{N_a}\bigr)^{\mathrm{T}}$$

By means of the exact formulas for time derivatives of $f(x)$ we have

$$\partial_{x_m}f(x) = -f(x)\kappa^m - f(x)A\kappa^m f(x), \qquad \kappa = \mathrm{diag}\{\kappa_1,\dots,\kappa_{\mathcal{N}}\}$$

Analyticity properties of the Cauchy–Jost kernel enables to write:

$$\oint_C \frac{d\,\lambda''}{2\pi i A(\lambda'')} F(x,\lambda,\lambda'') F(y,\lambda'',\lambda') = F(y,\lambda,\lambda') - F(x,\lambda,\lambda')$$

$$\partial_{x_k} F(x,\lambda,\lambda') = -\oint_C \frac{d\,\lambda''}{2\pi i A(\lambda'')} F(x,\lambda,\lambda'') \lambda''^k F(x,\lambda'',\lambda'), \quad k = 0,1,\ldots$$

Darboux transformation: $N_a \to N_a + 1$, $N_b \to N_b$

$$\widetilde{F}(x,\lambda,\lambda') = (\lambda - \kappa_{\mathcal{N}+1})\left[F(x,\lambda,\lambda') - \frac{F(x,\lambda,\kappa_{\mathcal{N}+1})\sum_{l=1}^{\mathcal{N}} v_l F(x,\kappa_l,\lambda')}{\sum_{l=1}^{\mathcal{N}} v_l F(x,\kappa_l,\kappa_{\mathcal{N}+1}) + 1}\right]$$

Correspondingly, this solution is parametrized by the matrices

$$\widetilde{\mathcal{D}} = \begin{pmatrix} \mathcal{D} \\ vA\,\mathcal{D} \end{pmatrix}, \qquad \widetilde{\mathcal{D}}' = \begin{pmatrix} & 0 \\ \mathcal{D}' & \vdots \\ & 0 \\ -vA & 1 \end{pmatrix}, \qquad v = (v_1, \dots, v_{\mathcal{N}})$$

New tau-function is given as

$$\widetilde{\tau}(x) = \tau(x)\left(\sum_l v_l F(x,\kappa_l,\kappa_{\mathcal{N}+1}) + 1\right)$$

Darboux transformation: $N_a \to N_a,\ N_b \to N_b + 1$

$$\widetilde{F}(x,\lambda,\lambda') = (\lambda' - \kappa_{\mathcal{N}+1})\left[F(x,\lambda,\lambda') - \frac{\sum_{l=1}^{\mathcal{N}} v_l F(x,\lambda,\kappa_l) F(x,\kappa_{\mathcal{N}+1},\lambda')}{\sum_{l=1}^{\mathcal{N}} v_l F(x,\kappa_{\mathcal{N}+1},\kappa_l) + 1}\right]$$

Correspondingly, this solution is parametrized by the matrices

$$\widetilde{\mathcal{D}} = \begin{pmatrix} \mathcal{D}, & v \\ 0, & 1 \end{pmatrix},$$

$$\widetilde{\mathcal{D}}' = \left(\mathcal{D}', -\mathcal{D}'\, v\right),$$

$$v = (v_1, \ldots, v_{\mathcal{N}})^{\mathrm{T}}.$$