Resonant conrol of solitons

S.V. Batalov, A.G. Shagalov

Institute of Metal Physics, Ekaterinburg, Russia

Workshop Solitons, Collapses and Turbulence, 2014

Introduction

Autoresonant excitation of solitons

KdV: I.Aranson, B.Meerson, T.Tajima, *Phys.Rev.A* 45 (1992) 7500
NLS: L.Friedland, A.Shagalov, *Phys.Rev.Lett.* 81 (1998) 4357

Autoresonant control (in particular, amplification) of solitons

KdV: R.Grimshow, E.Pelinovsky, P.Sakov,
Studies in Applied Mathematics 97 (1996) 235
SG: E.Maslov, L.Kalykin, A.Shagalov, *Theor.Math.Phys.* 152 (2007) 356
NLS: S.Batalov, E.Maslov, A.Shagalov, Sov. Phys. JEPT 108 (2009) 890

Problems of the autoresonant control of solitons:

(1) narrow range of the driving frequences
(2) strong restrictions on the phase of the driving

Resonant perturbation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = \varepsilon e^{i\psi(t)}, \quad 0 \le \varepsilon \ll 1, \qquad (NLS)$$

with a slowly varying frequency: $\Omega(t) = \psi_t(t), \quad \Omega_t \sim \varepsilon$

Soliton solution ($\varepsilon = 0$): $u_s = \frac{ae^{i\theta}}{cosh[a(x-\xi_0)]}, \quad \theta = \omega t + \theta_0,$

soliton amplitude: a

soliton frequency: $\omega = a^2/2$

Resonance of the soliton and the driving:

$$\boxed{\omega \approx \Omega(t)}$$

Equations for soliton parameters

$$\delta_t = \Delta\Omega - \frac{\varepsilon\pi a}{\Omega}\cos\delta$$
$$a_t = -\frac{\varepsilon\pi a^2}{2\Omega}\sin\delta$$

S. Batalov, E. Maslov, A. Shagalov, JETP **108***, 1021 (2009)*

$$\delta = \theta - \psi$$ — phase shift between the soliton and the driving

$$\Delta\Omega(t) = a^2/2 - \Omega(t)$$ — difference in frequencies of the soliton and the driving

suppose: $\Omega(t) = \Omega(0) - \alpha t\ ,\qquad \alpha \sim \varepsilon,\qquad \Delta\Omega \sim \sqrt{\varepsilon}$

introduce "slow" time: $\tau = \sqrt{\varepsilon}\, t$.

Equations for soliton parameters

$$\begin{aligned}\delta(\tau) &= \delta_0(\tau) + \sqrt{\varepsilon}\delta_1(\tau) + \varepsilon\delta_2(\tau) + \ldots \\ a(\tau) &= a_0 + \sqrt{\varepsilon}a_1(\tau) + \varepsilon a_2(\tau) + \ldots\end{aligned}$$

$$a(0) = a_0$$

$$\Omega = \underbrace{\Omega_0 + \sqrt{\varepsilon}\Omega_1}_{\Omega(0)} - \sqrt{\varepsilon}\,\beta\tau\;, \quad \Omega_0 = a_0^2/2\;, \quad \beta = \alpha/\varepsilon \sim 1$$

one finds in the lowest order

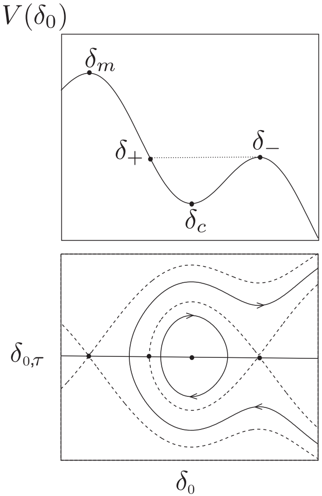
$$\begin{aligned}\delta_{0,\tau} &= a_0 a_1 - \Omega_1 + \beta\tau\;, \\ a_{1,\tau} &= -\pi\sin\delta_0\end{aligned}$$

equation for the phase difference:

$$\boxed{\delta_{0,\tau\tau} = -\pi a_0\sin\delta_0 + \beta}$$

Scattering on resonance

$$V(\delta_o) = -\pi a_o \cos\delta_o - \beta\delta_o, \qquad \lambda = \frac{\beta}{\pi a_o}$$

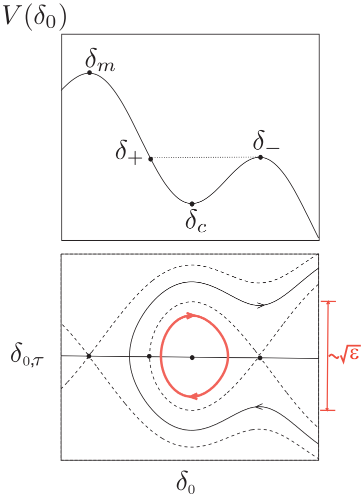


$0 < \lambda < 1 \implies$ threshold:

$$\boxed{\varepsilon > \varepsilon_{th} = \frac{\alpha}{\pi a_o}}$$

Scattering on resonance

$$V(\delta_o) = -\pi a_o \cos \delta_o - \beta \delta_o, \qquad \lambda = \frac{\beta}{\pi a_o}$$



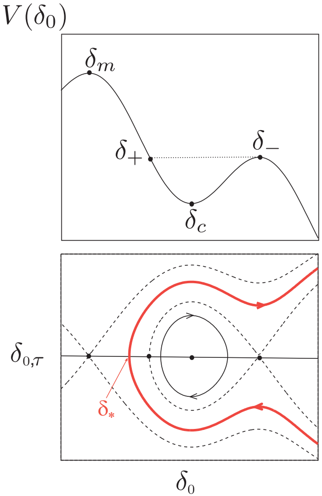
$0 < \lambda < 1 \implies$ threshold:

$$\boxed{\varepsilon > \varepsilon_{th} = \frac{\alpha}{\pi a_o}}$$

$$\Delta\Omega(o) \sim \sqrt{\varepsilon}\,, \qquad \delta_+ < \delta(o) < \delta_-$$

Scattering on resonance

$$V(\delta_o) = -\pi a_o \cos\delta_o - \beta\delta_o, \qquad \lambda = \frac{\beta}{\pi a_o}$$



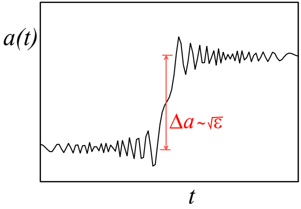
$0 < \lambda < 1 \implies$ threshold:

$$\boxed{\varepsilon > \varepsilon_{th} = \frac{\alpha}{\pi a_o}}$$

Resonance: $\delta_{0,\tau} \approx 0$

Scattering on resonance

Normal point: $\delta_0(0) = \delta_*$ $\quad (\delta_m < \delta_* < \delta_+)$



$$\Delta a = -\pi\sqrt{\varepsilon}\int_{-\infty}^{\infty} \sin\delta_0(\tau)d\tau$$

$$\delta_0(\tau) \approx \delta^* - C\tau^2/2,$$

$$C = \pi a_0 \sin\delta^* - \beta$$

$$(C \to 0, \quad \delta_* \to \delta_m)$$

$$(\Delta a)_0 = -\frac{\sqrt{\varepsilon}\pi^{3/2}(\mathrm{sign}(\alpha)\cos\delta^* + \sin\delta^*)}{\sqrt{|C|}}$$

similar approach: B.V.Chirikiv (1959), A.I.Neishtadt (1975)

Scattering on resonance

Singular point: $\delta_* \approx \delta_+$

near the saddle point $\delta_0 = \delta_-$ $(\delta_{0,\tau} > 0)$ suppose

$$\delta_0 = \delta_- + \xi(\tau),\ \xi(0) = 0.$$

$$\xi(\tau) = \frac{\delta_{0,\tau}(0)}{2\kappa}\left(e^{\kappa\tau} - e^{-\kappa\tau}\right), \quad \kappa^2 = \pi a_0 \sqrt{1-\lambda^2}.$$

$$(\Delta a)_1 = \sqrt{\varepsilon}\,\frac{2\pi\lambda}{\kappa}\,\mathrm{K}_0\left(\frac{\delta_{0,\tau}(0)}{\kappa}\right),$$

Singular point: $\delta_* \approx \delta_m$

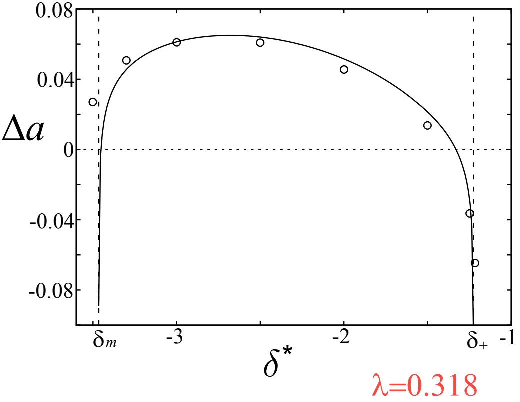
$$(\Delta a)_2 = \sqrt{\varepsilon}\,\frac{\pi^2}{\sqrt{A}}\left[\sin\left(\delta^* + \frac{C}{A}\right)\mathrm{N}_0\left(\frac{|C|}{A}\right) - \mathrm{sign}(\alpha)\cos\left(\delta^* + \frac{C}{A}\right)\mathrm{J}_0\left(\frac{|C|}{A}\right)\right],$$

$$A = -\pi a_0 \cos\delta^*.$$

Scattering on resonance

$$\delta_* = -\pi/2: \qquad (\Delta a)_2 = (\Delta a)_0$$

$$\Delta a = \begin{cases} 2(\Delta a)_1 + (\Delta a)_2, & \delta_m < \delta^* < -\pi/2, \\ 2(\Delta a)_1 + (\Delta a)_0, & -\pi/2 < \delta^* < \delta_+, \end{cases}$$



Numerical simulation of soliton amplification in NLS

Multiple scattering for an impulse-like ("saw-tooth") modulation of the driving frequency:

$$\Omega(t) = \Omega_{\text{max}} - \alpha[t - (n-1)T],$$
$$(n-1)\,T \leq t \leq n\,T, \quad n = 1, 2, 3, ...$$

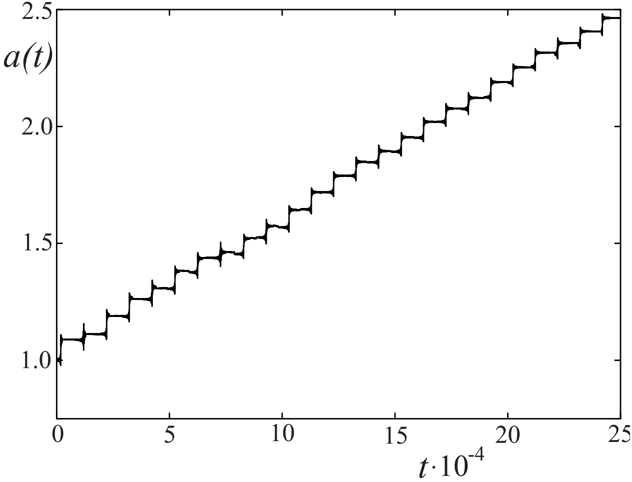
n - the number of the impulse
T - its period.

In a single impulse the frequency is changed in the range from Ω_{max} to $\Omega_{\text{min}} = \Omega_{\text{max}} - \alpha T$

Numerical simulation of soliton amplification in NLS

$a(0) = 1$; $\alpha = 0.00025$, $\varepsilon = 0.0005$, $\Omega_{\text{min}} = 0.4$, $\Omega_{\text{max}} = 3$, $T = 1.04 \cdot 10^4$.

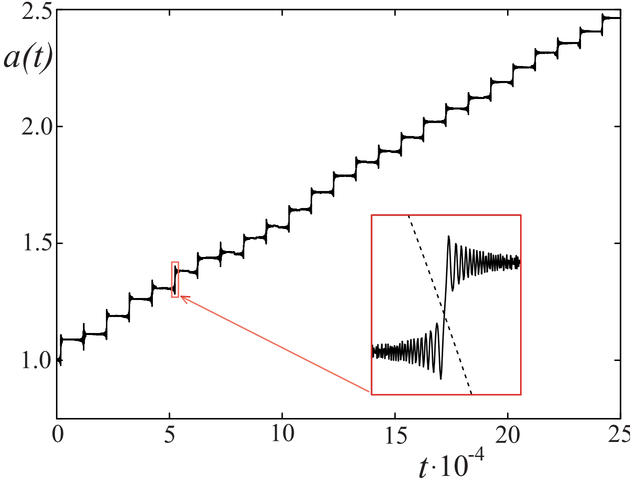
$\lambda = 0.16$



Numerical simulation of soliton amplification in NLS

$a(0) = 1$; $\alpha = 0.00025$, $\varepsilon = 0.0005$, $\Omega_{\min} = 0.4$, $\Omega_{\max} = 3$, $T = 1.04 \cdot 10^4$.

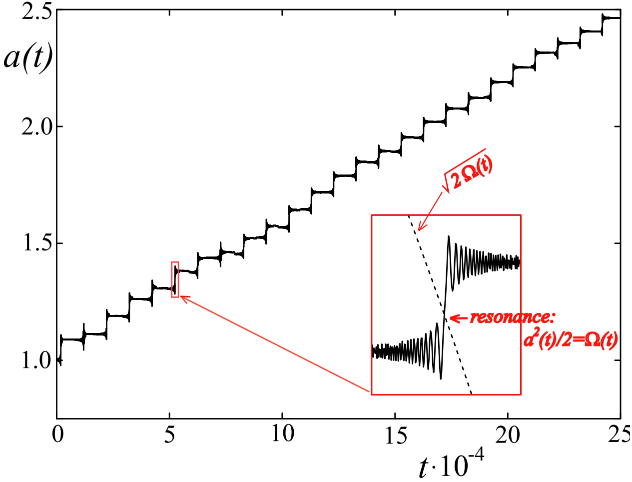
$\lambda = 0.16$



Numerical simulation of soliton amplification in NLS

$a(0) = 1$; $\alpha = 0.00025$, $\varepsilon = 0.0005$, $\Omega_{\min} = 0.4$, $\Omega_{\max} = 3$, $T = 1.04 \cdot 10^4$.

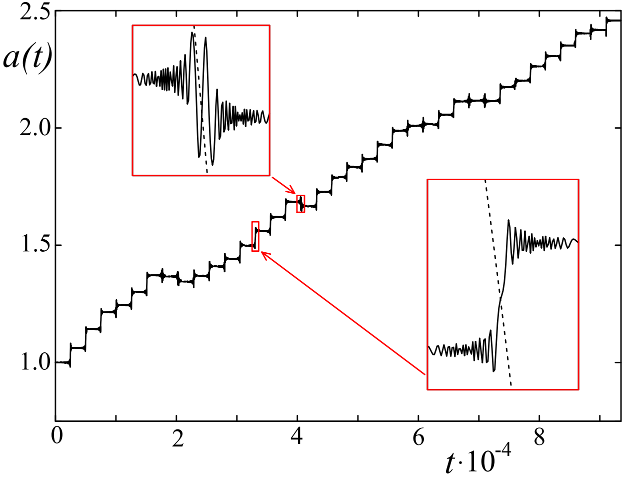
$\lambda = 0.16$



Numerical simulation of soliton amplification in NLS

$a(0) = 1$; $\alpha = 0.001$, $\varepsilon = 0.0005$, $\Omega_{\text{min}} = 0.4$, $\Omega_{\text{max}} = 3$, $T = 2.6 \cdot 10^3$.

$\lambda = 0.64$



Control of the soliton in NLS

$a(0) = 1$; $\varepsilon = 0.0005$, $\Omega_{\text{min}} = 0.4$, $\Omega_{\text{max}} = 3$, $T = 1.04 \cdot 10^4$.