Stability of autoresonance under persistent perturbation

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Outline

1. Statement of the problem
2. Origin of the problem
3. Different approaches
4. Main result
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Summary

A dynamical system under white noise perturbation is considered.
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Theorem of stability of equilibrium is proved.
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Theorem of stability of equilibrium is proved.

The method of parabolic equation is used.

An appropriate barrier function for the Kolmogorov’s equation is the main mathematical achievement.

The result is applied to prove the stability of the autoresonance phenomenon.
Statement of problem

Unperturbed system is ODEq

\[ \frac{d\mathbf{y}}{dT} = \mathbf{a}(\mathbf{y}, T), \quad \mathbf{y} \in \mathbb{R}^n, \quad T > 0. \]  (1)
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\[ \frac{dy}{dT} = a(y, T), \quad y \in \mathbb{R}^n, \quad T > 0. \]  

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\[ dy = a(y, T)dt + \mu B(y, T)dw(T), \quad T > 0; \quad y|_{T=0} = x. \]  \hspace{1cm} (2)
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The problem of stability

Does the trajectory \( y = y_\mu(T; x) \) remain near equilibrium, if the perturbation \( \mu, x \) are small and the matrix is in a ball \( ||B|| < M \)?
Statement of problem

Answer

There is not any Lyapunov’s type stability.
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How to understand the stability under white noise?
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Almost all trajectories drift out the equilibrium.

How to understand the stability under white noise?

There are some Khasminskii’s and Freidlin–Wentzell’s results concerning either dissipative or autonomous systems.
References

Khasmiskii R.
Khasmiskii R.


concerns either the dissipative systems or a special perturbation $B(0, T) \equiv 0$. 
Р. Э. Хасьминский

Устойчивость систем дифференциальных уравнений при случайных возмущениях их параметров
References

Freidlin, M.I., Wentzell, A.D.
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What the problems?
Example

Autoresonance perturbation of the pendulum

\[ \frac{d^2 x}{dt^2} + \sin x = \varepsilon \cdot \cos(t - \alpha t^2), \]
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The problem: Is the capture into resonance stable?

An anzatz on the initial stage:

\[ x(T; \varepsilon) \approx \varepsilon^{1/3} \rho(t) \cos(t + \Psi(t)) \] as \( t = \varepsilon^{2/3}T \).

Autoresonance solutions:

\[ \rho(t) \to \infty, \quad t \to \infty. \]

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Applications

Model systems of autoresonance:
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**Main autoresonance equations**

\[ \frac{d\rho}{dt} = \sin \psi, \quad \rho \left[ \frac{d\psi}{dt} - \rho^2 + \lambda t \right] = \cos \psi. \]
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\frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho^2 - \lambda t.
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\frac{d\rho}{dt} = \sin \psi, \quad \frac{d\psi}{dt} = \rho^2 - \lambda t.
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And so on....
Statement of the autoresonance problem

Specific of the autoresonance equations

The equations are nonlinear and nonautonomous. Almost all systems are nonintegrable. There are not any small parameter in the equations. The main object is an autoresonance solution with increasing amplitude \( \rho(t) \to \infty \) as \( t \to \infty \). There are some solutions of that type. Problem Are stable these solutions with respect to random perturbations?
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Initial stage of autoresonance

Autoresonance solutions
Statement of the autoresonance problem

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There are different solutions with increasing and bounded amplitude as well.
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The well known Khasimskii’s results are not applicable.
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The autoresonance systems are not autonomous
The well known Freidlin’s results are not applicable.
Deterministic stability

Perturbed system

\[ \frac{dy}{dT} = a(y, T) + \mu B(y, T), \quad T > 0; \quad y|_{T=0} = x. \]
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Measure of stability
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## Deterministic stability

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ТЕОРИЯ УСТОЙЧИВОСТИ ДВИЖЕНИЯ

ГОСУДАРСТВЕННОЕ ИЗДАТЕЛЬСТВО ТЕХНИКО-ТЕОРЕТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1952 ЛЕНИНГРАД
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П. Н. КРАСОВСКИЙ

НЕКОТОРЫЕ ЗАДАЧИ ТЕОРИИ УСТОЙЧИВОСТИ ДВИЖЕНИЯ

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МОСКВА 1959
Definitions of stochastic stability

Perturbed system is the Itô’s equation

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Strong stability under white noise can not be at all.
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Weak stability under white noise may be, if the system is dissipative (Khasminskii). It is not the autoresonance case.
Definitions of stochastic stability

Perturbed system

\[ d\mathbf{y} = a(\mathbf{y}, T)dT + \mu B(\mathbf{y}, T)\, dw(T), \ T > 0; \quad \mathbf{y}|_{T=0} = \mathbf{x}. \]

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\[ \mathbb{E}[(\mathbf{y}_\mu(T; \mathbf{x}))^2] = U_\mu(\mathbf{x}, T) \]

Limited weak stability
Definitions of stochastic stability

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Limited weak stability is derived from Freidlin’s results if the system is \textit{autonomous}. It is not the autoresonance case.
Stability under given estimates

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\[ dy = a(y, T) dT + \mu B(y, T) dw(T), \quad T > 0; \quad y|_{T=0} = x. \]

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Limited weak stability (Krasovskii) with given estimates:

\[ \exists \delta = \delta(\varepsilon), \quad \Delta = \Delta(\varepsilon) \]
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\[ \forall \varepsilon > 0, \forall |x| < \delta(\varepsilon), |\mu| < \Delta(\varepsilon) \Rightarrow \sup_{0 < T < \mu^{-2}} E[(y_\mu(T; x))^2] < \varepsilon. \]
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The given estimates are more appropriate at physics, especially at the autoresonance phenomenon. It is not the Freidlin’s case.
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\[ dy = a(y, T) \,dT + \mu B(y, T) \,dw(T), \quad T > 0; \quad y|_{T=0} = x. \]
Main result

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\[ dy = a(y, T) dT + \mu B(y, T) dw(T), \quad T > 0; \quad y|_{T=0} = x. \]

Theorem

Let unperturbed deterministic system has a local Lyapunov function \( U(y, T) \) with the properties

\[ \partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = O(|y|^2), \quad y \to 0; \]
Main result

Perturbed system

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Then the equilibrium is limited weak stable (under white noise) with estimates:

\[ \delta(\varepsilon) = \delta_M \sqrt{\varepsilon}, \quad \Delta(\varepsilon) = \Delta_M \sqrt{\varepsilon}, \quad (\delta_M, \Delta_M = \text{const}). \]
Conclussion

\[ d\mathbf{y} = \mathbf{a}(\mathbf{y}, T) \, dT + \mu \mathbf{B}(\mathbf{y}, T) \, dw(T), \quad T > 0; \quad \mathbf{y}|_{T=0} = \mathbf{x}. \]
Conclusion

\[ d\mathbf{y} = a(\mathbf{y}, T) dT + \mu B(\mathbf{y}, T) d\mathbf{w}(T), \quad T > 0; \quad \mathbf{y}|_{T=0} = \mathbf{x}. \]

Lyapunov function of the unperturbed system is a ground for a barrier function of the Kolmogorov’s equation.
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Lyapunov function of the unperturbed system is a ground for a barrier function of the Kolmogorov’s equation.

The barrier function provides weak stability with the given estimates: \(|\mu|, |\mathbf{x}| = \mathcal{O}(\sqrt{\varepsilon})\) on a long time interval \(0 < T < \mathcal{O}(\mu^{-2})\).
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There is not weak stability up to infinity \(0 < T < \infty\).
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**Open problem**

Is there weak stability on a very long time interval, for example, \(0 < T < \mathcal{O}(\exp(\mu^{-2}))\)?
Example

Autoresonance perturbation of the pendulum

\[ \frac{d^2 x}{dt^2} + \sin x = \varepsilon \cdot \cos(t - \alpha t^2), \]
Example

Autoresonance perturbation of the pendulum

\[ \frac{d^2 x}{dt^2} + \sin x = \varepsilon \cdot \cos(t - \alpha t^2), \quad \alpha \approx \varepsilon^{4/3}, \quad 0 < \varepsilon \ll 1. \]
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Example

Autoresonance perturbation with noise

\[ \frac{d^2 x}{dt^2} + \sin x = \varepsilon \cdot (1 + \mu \dot{w}(t)) \cos(t - \alpha t^2), \quad \alpha \approx \varepsilon^{4/3}, \quad 0 < \varepsilon \ll 1. \]
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Stability of the autoresonance is proved under condition

\[|\mu| \leq \varepsilon^{1/6}.\]
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Hypothesis

\[ |\mu| \leq |\ln \varepsilon|? \]
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Exponential time stability is need to prove this hypothesis.
Example

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Who can suggest an appropriate barrier for the Kolmogorov equation?
Example

Autoresonance perturbation with noise

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\[|\mu| \leq |\ln \varepsilon|?\]

Exponential time stability is need to prove this hypothesis. Who can suggest an appropriate barrier for the Kolmogorov equation?
THANK YOU FOR ATTENTION!
Novelty in technique
Novelty in technique

Perturbed system

\[ dy = a(y, T) dT + \mu B(y, T) dw(T), \quad T > 0; \quad y|_{T=0} = x. \]
Novelty in technique

Perturbed system

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Initial problem for the Kolmogorov’s equation
Novelty in technique

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Initial problem for the Kolmogorov’s equation

\[ \partial_t u - a(x, T-t) \partial_x u - \mu^2 \sum_{i,j=1}^{n} b_{i,j}(x, t-T) \partial_{x_i} \partial_{x_j} u = 0, \]
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\[ dy = a(y, T) dT + \mu B(y, T) dw(T), \ T > 0; \ y|_{T=0} = x. \]

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\]

\[ u(x, t; T, \mu)|_{t=0} = |x|^2, \quad x \in \mathbb{R}^n, \quad \{b_{i,j}\} = \frac{1}{2} BB^*(x, T-t). \]
Novelty in technique

Perturbed system

\[ \frac{dy}{dT} = a(y, T) dT + \mu B(y, T) dw(T), \quad T > 0; \quad y|_{T=0} = x. \]

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Connection with the random trajectories
Novelty in technique

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\[ dy = a(y, T) dT + \mu B(y, T) d\mathbf{w}(T), \quad T > 0; \quad y|_{T=0} = \mathbf{x}. \]

Initial problem for the Kolmogorov’s equation
\[ \partial_t u - a(\mathbf{x}, T - t) \partial_x u - \mu^2 \sum_{i,j=1}^{n} b_{i,j}(\mathbf{x}, t - T) \partial_{x_i} \partial_{x_j} u = 0, \]
\[ u(\mathbf{x}, t; T, \mu)|_{t=0} = |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^n, \quad \{b_{i,j}\} = \frac{1}{2} \mathbf{B}\mathbf{B}^*(\mathbf{x}, T - t). \]

Connection with the random trajectories
\[ \mathbb{E}[(y_\mu(T; \mathbf{x}))^2] = u(\mathbf{x}, t; T, \mu)|_{t=T}. \]
Novelty in technique

Perturbed system

\[ \frac{dy}{dT} = a(y, T)dT + \mu B(y, T)dw(T), \quad T > 0; \quad y|_{T=0} = x. \]

Initial problem for the Kolmogorov’s equation

\[ \begin{align*}
\partial_t u - a(x, T - t)\partial_x u - \mu^2 \sum_{i,j=1}^{n} b_{i,j}(x, t - T)\partial_{x_i}\partial_{x_j}u &= 0, \\
u(x, t; T, \mu)|_{t=0} &= |x|^2, \quad x \in \mathbb{R}^n, \quad \{b_{i,j}\} = \frac{1}{2}BB^*(x, T - t).\end{align*} \]

Connection with the random trajectories

\[ \mathbb{E}[(y_{\mu}(T; x))^2] = u(x, t; T, \mu)|_{t=T}. \]
<table>
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<th>Origin of the problem</th>
<th>Different approaches</th>
<th>Main result</th>
<th>Conclusion</th>
<th>Novelty</th>
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**Novelty in mathematics**

Initial problem for the parabolic equation:

\[
\partial_t u - a(x, T-t) \partial_x u - \mu^2 \sum_{i,j=1}^{n} b_{i,j}(x,t-T) \partial_x i \partial_x j u = 0, \\
0 < t < T, \\
u(x,t; T, \mu) |_{t=0} = |x|^2, x \in \mathbb{R}^n; (\forall T > 0).
\]

**Barrier function**

\[
u(x,t; T, \mu) < V(x,t; T, \mu)
\]

Problem: how to construct an appropriate barrier

\[
V(x,t; T, \mu) = O(|x|^2 + \mu^2), x \to 0, \mu \to 0.
\]
Novelty in mathematics

Initial problem for the parabolic equation

\[
\partial_t u - a(x, T - t) \partial_x u - \mu^2 \sum_{i,j=1}^n b_{i,j}(x, t - T) \partial_{x_i} \partial_{x_j} u = 0, \quad 0 < t < T,
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Novelty in mathematics

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**Barrier function**
Novelty in mathematics

Initial problem for the parabolic equation

\[ \frac{\partial u}{\partial t} - a(x, T-t) \frac{\partial u}{\partial x} - \mu^2 \sum_{i,j=1}^{n} b_{i,j}(x, t-T) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad 0 < t < T, \]

\[ u(x, t; T, \mu)|_{t=0} = |x|^2, \quad x \in \mathbb{R}^n; \quad (\forall T > 0). \]

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Barrier function

\[ u(x, t; T, \mu) < V(x, t; T, \mu) \]

Problem: how to construct an appropriate barrier

\[ V(x, t; T, \mu) = \mathcal{O}(|x|^2 + \mu^2), \quad x \to 0, \quad \mu \to 0. \]
Barrier skill

\[ U(y, T) \]

\[ \partial_T U + a(y, T) \partial_y U \leq -\gamma U; \]

\[ U(y, T) = O(|y|^2), y \to 0; \]

\[ V_0(x, t; T, \mu) = U(x, T-t) \exp(-\alpha_0 t) + \mu^2 M\left[1 - \exp(-\alpha_0 t)\right], \]

\[ V_2(x, t; T, \mu) = U(x, T-t) \exp(\alpha t), \alpha, \alpha_0 = \text{const} > 0, \]

\[ V_1(x, t; T, \mu) = m\mu^2 \exp(\alpha t + U - \rho \mu^2), m, \rho = \text{const} > 0. \]
Barrier skill

Lyapunov function $U(y, T)$ with the local properties
$$\partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = O(|y|^2), \quad y \to 0;$$
Lyapunov function $U(y, T)$ with the local properties

$$\partial_T U + a(y, T)\partial_y U \leq -\gamma U; \quad U(y, T) = \mathcal{O}(|y|^2), \quad y \to 0;$$

**Global part of barrier**

- Far from equilibrium:
  $$V_2(x, t; T, \mu) = U(x, T-t) \exp(\alpha t), \quad \alpha, \alpha_0 = \text{const} > 0,$$

- Local part of barrier:
  $$V_1(x, t; T, \mu) = m \mu^2 \exp(\alpha t + U - \rho \mu^2), \quad m, \rho = \text{const} > 0.$$
Barrier skill

Lyapunov function $U(y, T)$ with the local properties
\[ \partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = O(|y|^2), \quad y \to 0; \]

Global part of barrier

\[ V_0(x, t; T, \mu) = U(x, T - t) \exp(-\alpha_0 t) + \mu^2 M[1 - \exp(-\alpha_0 t)], \]
Barrier skill

Lyapunov function \( U(y, T) \) with the local properties
\[
\partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = O(|y|^2), \quad y \to 0;
\]

**Global part of barrier**

\[
V_0(x, t; T, \mu) = U(x, T - t) \exp(-\alpha_0 t) + \mu^2 M[1 - \exp(-\alpha_0 t)],
\]

**Far from equilibrium**
Barrier skill

Lyapunov function $U(y, T)$ with the local properties
\[ \partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = O(|y|^2), \quad y \to 0; \]

**Global part of barrier**

\[ V_0(x, t; T, \mu) = U(x, T - t) \exp(-\alpha_0 t) + \mu^2 M[1 - \exp(-\alpha_0 t)], \]

**Far from equilibrium**

\[ V_2(x, t; T, \mu) = U(x, T - t) \exp(\alpha t), \quad \alpha, \alpha_0 = \text{const} > 0, \]
Lyapunov function $U(y, T)$ with the local properties:

$$\partial_T U + a(y, T)\partial_y U \leq -\gamma U; \quad U(y, T) = \mathcal{O}(|y|^2), \quad y \to 0;$$

### Global part of barrier

$$V_0(x, t; T, \mu) = U(x, T - t) \exp(-\alpha_0 t) + \mu^2 M[1 - \exp(-\alpha_0 t)],$$

### Far from equilibrium

$$V_2(x, t; T, \mu) = U(x, T - t) \exp(\alpha t), \quad \alpha, \alpha_0 = \text{const} > 0,$$

### Local part of barrier
Barrier skill

Lyapunov function $U(y, T)$ with the local properties
\[ \partial_T U + a(y, T) \partial_y U \leq -\gamma U; \quad U(y, T) = \mathcal{O}(|y|^2), \ y \to 0; \]

**Global part of barrier**

\[ V_0(x, t; T, \mu) = U(x, T - t) \exp(-\alpha_0 t) + \mu^2 M[1 - \exp(-\alpha_0 t)], \]

**Far from equilibrium**

\[ V_2(x, t; T, \mu) = U(x, T - t) \exp(\alpha t), \ \alpha, \alpha_0 = \text{const} > 0, \]

**Local part of barrier**

\[ V_1(x, t; T, \mu) = m \mu^2 \exp \left( \alpha t + \frac{U - \rho}{\mu^2} \right), \ m, \rho = \text{const} > 0 \]